

Spectral Transfer, Interaction, and No-Leakage for Thin Yang–Mills Observable Representations: Physical Zero-Mode Architecture, Hilbert-Gerbe Spectral Stability, Incision-Relative Mass-Gap Formation, and Haag-Type Free-Net Non-Equivalence

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This manuscript develops the spectral closure of a five-paper construction of thin Yang–Mills observable systems on incised four-dimensional Minkowski spacetime. It establishes common spectral, form, and Gauss–Ward cores; Hilbert-gerbe Glue–Mosco–Riesz stability; sharp seam covariance; exact zero-mode no-leakage; and a certified positive physical hyperbolic threshold. A Poincaré-mobile class of controlled codimension-two incisions is constructed together with remote-incision excision for the complete regional observable, state, ground-sector, form, Gauss–Ward, surgery, and spectral packets. Normal equalizer descent over incisions avoiding each finite carrier yields an ordinary regional net with strict Poincaré action while every coordinate realization remains incised. The selected state descends to a locally normal invariant state with strongly continuous GNS representation. Positive energy, full Lorentz covariance, and forward-cone support then imply invariant-mass separation: the vacuum is the unique zero-mass sector and no spectrum occurs between zero and the certified threshold on the fully certified branches. The scalar-twisted represented object retains its nontrivial coefficient-line associator without chart-, corridor-, or incision-induced spectral leakage. Independently, an exact fourth-order Wick-defect calculation gives nonzero certified intervals for the reconstructed $SU(3)$ branches and a strict $SU(2)$ witness, excluding every admissible packet-germ-preserving quasi-free or generalized-free comparator. Sector-II and correspondence-valued conclusions retain their declared realization conditions. The result does not by itself establish nontrivial scattering, an area law, or an unqualified Clay Millennium Prize resolution.

Keywords: Yang–Mills; algebraic quantum field theory; spectral transfer; no leakage; invariant mass; interaction; Haag-type non-equivalence

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I. INTRODUCTION AND EXACT CLAIM

This long-form technical manuscript develops the fifth paper in the Yang–Mills reconstruction series^{1–4}. It begins from the definitive Paper-IV handoff and constructs the physical spectral architecture through displayed branchwise hypotheses: invariant state selection, represented implementation, joint translation generators, orbit disintegration, support transfer, Hilbert-gerbe spectral stability, exact zero-mode identification, and invariant-mass formation. The historical gauge-field and observable-net background is the Yang–Mills and Haag–Kastler framework^{5,6}; the external Millennium Problem boundary is the Jaffe–Witten formulation⁷.

The gauge-group scope is

$$G = \tilde{G}/\Gamma, \quad \tilde{G} = \prod_i G_i^{\text{sc}},$$

with G compact, connected, and semisimple. Sector-II and non-scalar correspondence-valued statements remain conditional exactly where physical realization data are absent.

The ordinary thin observable net has a global observable representation on each selected background. The represented physical object depends on branch type: it is globally strict in Sector I, corridorwise strict and relatively twisted in Sector II, and chart–corridor coefficient-line gerbe valued in scalar Sector III. No global Hilbert-space strictification is imposed on a twisted represented branch.

Definitive incision-relative certification verdict

The final covariance problem is resolved without postulating a Yang–Mills theory at empty incision. The incision is constitutive local reconstruction data. Phase V.IX.E selects a Poincaré-mobile class of controlled codimension-two incisions and proves that every finite regional/support/collar/Cauchy packet can be represented in an incised coordinate whose collar avoids that packet. Finite unions give controlled common refinements. Support locality, normal packet naturality, and corrected SLSP restart uniqueness give exact remote-incision excision for the regional observable, state, complete-ground, form, Gauss–Ward, surgery, and reference-weight data.

The normal equalizer over all incisions avoiding a region defines the incision-relative ordinary observable algebra. Poincaré transformations move a region and one avoiding

incision together and descend to a strict action on the ordinary shadow. No $C_0(\mathcal{P}/\text{Stab}(\Sigma))$ orbit algebra, background-position PVM, or empty-incision coordinate is introduced.

For every fully certified compact connected semisimple exact-dilation Sector-I or scalar Sector-III branch over this mobile-incision realization, the descended state has a strongly continuous Poincaré GNS implementation with

$$\text{supp } E_{G,\xi} \subseteq \{0\} \cup \{p \in \overline{V}_+ : p^2 \geq \kappa_*\},$$

$$E_{G,\xi}(\{0\}) = E_{M_{G,\xi}^2}(\{0\}) = P_{\Omega,G,\xi,0}, \quad E_{M_{G,\xi}^2}((0, \kappa_*)) = 0.$$

The strict or scalar-twisted represented lift retains its exact coherence invariant and creates neither new zero modes nor sub-threshold mass support. The certified normalized mass-square lower bound remains κ_* . Sector-II and correspondence-valued statements retain their separately declared realization conditions.

Architectural correction

The primitive bridge is not a generally valid inequality of the form $M^2 \geq cK^{\text{thin}}$ at every momentum. The definitive chain is

$$P_0^2 = K^{\text{thin}} \geq \kappa_*(I - P_\Omega),$$

combined with forward-cone support, Lorentz-orbit disintegration, elimination of nonzero null sectors, and exact ground triviality. The invariant conclusion

$$M^2 \geq \kappa_*(I - P_\Omega)$$

is derived only after the incision-relative Poincaré theorem.

Exact nonclaims

The manuscript does not infer rank-one vacuum uniqueness without a rank certificate, full Haag duality, unconditional Sector-II physical non-vacuity, an unconditional non-scalar correspondence mass gap, asymptotic completeness, confinement by an area law, or an unqualified Clay-resolution claim. The internal physical hyperbolic identity alone does not

prove interaction; the separate theorem in [section XV](#) establishes a nonzero physical Wilson-germ Wick defect, a nonzero response to the exact compact-group magnetic term, and Haag-type non-equivalence in the explicitly declared packet-decorated category. No nonidentity scattering matrix is inferred. The conjectural interpretation chapter is downstream of the certified and conditional theorem boundaries.

Reader orientation. Paper V is a sequence of upgrades, not one long spectral estimate. The Paper-IV certificate first supplies a thin form and complete-ground sector. Paper V then selects an invariant state, constructs its represented Poincaré system and joint translation PVM, transfers positive-thickness support to the forward cone, proves finite-diagram Glue–Mosco–Riesz stability, identifies the physical hyperbolic generator, eliminates zero-mode and future-null leakage, and only then states the invariant-mass threshold. The Wilson-germ interaction theorem is a second, independent route.

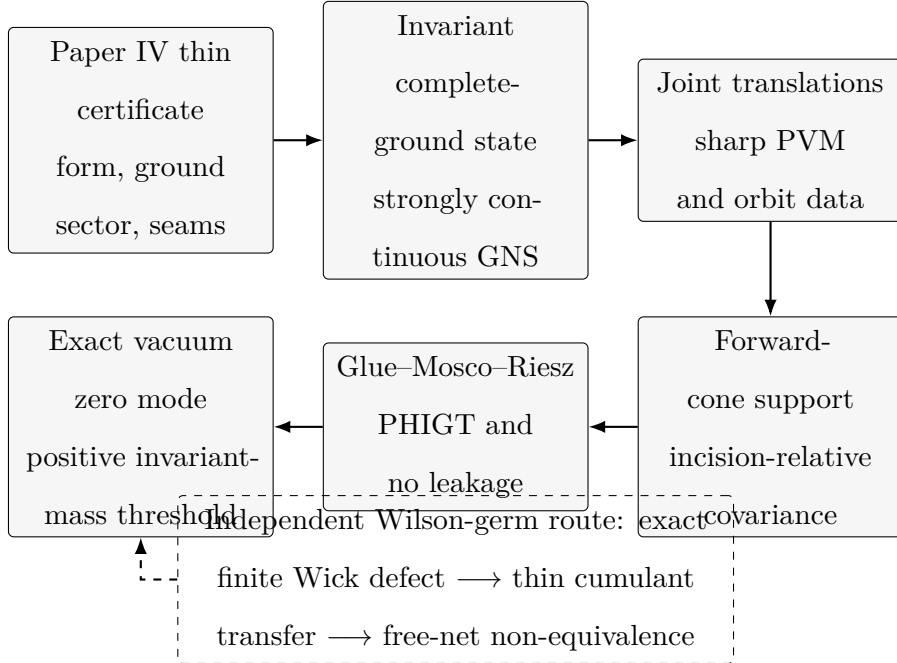


FIG. 1. The logical architecture of Paper V. The interaction certificate is independent of the spectral mass-gap proof and meets it only in the final combined theorem.

II. PAPER IV CERTIFICATE, GAUGE SCOPE, AND SPECTRAL RESTART DATUM

Reader orientation. This section is a logical restart rather than a compressed reproof of Paper IV. It records exactly which thin objects and retained positive-thickness certificates may enter the spectral argument, and it excludes every conclusion that Paper V must still construct: invariant state selection, strong implementation, joint generators, forward-cone support, invariant mass, no leakage, and the final gap.

A. Gauge and branch scope

Fix

$$G = \tilde{G}/\Gamma, \quad \tilde{G} = \prod_i G_i^{\text{sc}}, \quad \Gamma \leq Z(\tilde{G}),$$

with

$$P_G = \{\lambda \in P : \chi_\lambda|_\Gamma = 1\}, \quad P_G/Q \cong \widehat{Z(G)}.$$

A branch label is (G, ξ, S) , with $S \in \{\text{I, II, III, Corr}\}$. The definitive invariants are

$$\begin{aligned} \text{I} : \quad & \kappa_\xi^{\text{fib}} = 0, \quad \hat{\kappa}_\xi^{\text{cs}} = 0 \text{ plus admissible strictification,} \\ \text{II} : \quad & \kappa_\xi^{\text{fib}} = 0, \quad \hat{\kappa}_\xi^{\text{cs}} \neq 0, \\ \text{III} : \quad & \kappa_\xi^{\text{fib}} = [\Omega_\xi] \neq 0, \quad [\Omega_\xi] \in H^3(BA_\xi; U(1)), \\ \text{Corr} : \quad & \kappa_\xi^{\text{Corr}} \neq [1]. \end{aligned}$$

Declared root-datum Sector-I and scalar cyclic/finite-abelian Sector-III branches are certified. Sector-II non-vacuity is branch specific. Correspondence-valued physical non-vacuity remains conditional.

B. Exact Paper IV to Paper V datum

Definition II.1 (Definitive handoff certificate). For every branch let

$$\begin{aligned} \mathfrak{A}_{\text{IV} \rightarrow \text{V}}^{G, \xi, S} = & \left(\mathcal{A}_{G, \xi, 0}^W, \hat{\Omega}_{G, \xi, 0}^b, \mathcal{S}_{G, \xi, 0}^{\text{st, as}}, \mathfrak{K}_{G, \xi}^{\text{Orl}}, \mathfrak{W}_{G, \xi}^S, \right. \\ & \bar{q}_{G, \xi, 0}, P_{\Omega, G, \xi, 0}, \kappa_*, \text{Restart}_{G, \xi}, \text{AC}_{1:10}^{G, \xi}, \mathbf{H}_{0:8}^{G, \xi}, \\ & \left. \kappa_\xi^{\text{fib}}, \hat{\kappa}_\xi^{\text{cs}}, \kappa_\xi^{\text{Corr}}, [\chi_{G, \xi, 0}^\partial]_{\text{cs}} \right), \end{aligned}$$

with inapplicable entries omitted.

Paper V may use without reproof the following certified Paper-IV exports⁴:

- the ordinary thin Haag–Kastler net and branch-appropriate represented object;
- inherited Čech-descended surgery and Orlicz/reference-logarithm control;
- the integrated Gauss–Ward, collar, current, and boundary package;
- the complete-ground projection and state face; and
- the comparison–Mosco form and TT1–TT2, with

$$\ker \bar{q}_{G,\xi,0} = \text{Ran } P_{\Omega,G,\xi,0}, \quad \bar{q}_{G,\xi,0} \geq \kappa_*(\mathbf{1} - P_{\Omega,G,\xi,0}).$$

It may not assume a global vacuum representation, strong implementers, translation generators, the spectrum condition, invariant mass, a form-to-mass inequality, physical sharp spectral covariance, no-leakage, vacuum uniqueness, or a mass gap.

Theorem II.2 (Certified Paper-IV-to-Paper-V intake). *For every branch in the declared Paper-IV scope, the complete admissible intake of Paper V is the pair*

$$\mathfrak{H}_{\text{IV} \rightarrow \text{V}}^{G,\xi,S} = \left(\mathfrak{A}_{\text{IV} \rightarrow \text{V}}^{G,\xi,S}, \mathfrak{C}_{\text{spec,up}}^{G,\xi,S} \right).$$

Here $\mathfrak{A}_{\text{IV} \rightarrow \text{V}}^{G,\xi,S}$ is the thin reconstruction certificate proved in Paper IV, while $\mathfrak{C}_{\text{spec,up}}^{G,\xi,S}$ is the explicitly retained upstream spectral certificate carried in the Paper-IV handoff shell for obligations that Paper IV intentionally defers. The latter contains only the positive-thickness stationary spectral and hyperbolic comparison data needed to start spectral transfer; it contains no thin spectrum condition, invariant-mass operator, no-leakage conclusion, or mass-gap statement.

Consequently every load-bearing Paper-V argument factors through the declared Paper-IV handoff. References to Paper III inside Paper V identify the provenance of a retained upstream certificate; they do not bypass Paper IV, re-import the positive-thickness reconstruction as an independent premise, or transfer any Paper-III conclusion directly to the final thin spectrum.

Proof. The first component is exactly the definitive Paper-IV handoff certificate above. Paper IV proves the thin net, represented branch object, expectation graph, Gauss–Ward and collar package, closed thin form, exact complete-ground kernel, restart data, and uniform off-ground threshold. Its ownership firewall expressly defers full translations, joint spectrum,

the forward-cone theorem, invariant mass, and no-leakage to Paper V. The only earlier-stage datum still needed by the spectral-transfer argument is the certified positive-thickness stationary spectral packet. Recording that packet as a provenance-tagged component of the Paper-IV certification shell preserves the ownership boundary: Paper IV certifies and hands forward the admissible input package, while Paper V alone proves all thin spectral conclusions. The clause-by-clause provenance ledger supplied with this release records the same factorization. $\square$

Scope firewall II.3 (No dependency bypass). Paper V does not independently assume OJ, RJ, SJ, GJ, or M, does not reconstruct a positive-thickness branch, and does not appeal directly to Papers I–III for any thin-limit conclusion. Any citation to an earlier paper after this intake theorem is either historical provenance or identifies a retained upstream certificate named in $\mathfrak{C}_{\text{spec},\text{up}}$. All thin observable, represented, surgery, form, complete-ground, and restart data enter through Paper IV.

C. The branch-appropriate restart category

Definition II.4 (Spectral comparison target). The category $\mathfrak{C}_{\text{spec}}^{G,\xi,S}$ has the following type.

- (i) Sector I: global represented systems after certified strictification, with forms, ground projections, and future joint spectral measures.
- (ii) Scalar Sector III: one global ordinary observable representation plus a chart–corridor family of represented Hilbert objects and coefficient-line intertwiners preserving multiplication and associator.
- (iii) Sector II: the same corridorwise architecture, conditional on a physical scale-obstructed realization.
- (iv) Correspondence: ground-pointed normal correspondences with selected Markov shadows and spectral correspondences only where separately defined.

Definition II.5 (Algebraic–analytic restart). Set

$$\mathfrak{P}_{\text{restart,aa}}^{G,\xi,S} = \left(\mathfrak{A}_{\text{IV} \rightarrow \text{V}}^{G,\xi,S}, \alpha_0^{G,\xi}, \mathfrak{F}_{\Omega,0}^{G,\xi}, \mathfrak{C}_{\text{spec}}^{G,\xi,S} \right),$$

where $\alpha_0^{G,\xi}$ is the thin spacetime automorphism action and $\mathfrak{F}_{\Omega,0}^{G,\xi}$ is the complete-ground state face, not a chosen state.

A face is sufficient to state a selection problem but is not sufficient to construct canonical GNS implementers.

D. The exact ground-state boundary

Proposition II.6 (Projection and face, not selected state). *The authoritative package constructs and transports the complete-ground projection and associated convex state face. It does not prove that this face is a singleton, choose an invariant extreme point, or provide a globally selected Poincaré-invariant state.*

Proof. Paper III and Paper IV use the complete-ground projection as the invariant object and explicitly permit higher rank. A normal state then needs additional density-matrix or weight data. Background covariance transports the projection, face, Gauss–Ward data, surgery, and forms, but explicitly excludes a strongly continuous Poincaré representation and a global vacuum vector. The definitive Paper IV firewall retains this exclusion. $\square$

Remark II.7 (No hidden averaging). No averaging over the full Poincaré group is inserted. Such an argument would require additional invariant-mean, compactness, normality, and continuity hypotheses not contained in the source package. The invariant state must come from a physical limiting or dynamical theorem.

III. INVARIANT COMPLETE-GROUND STATE SELECTION

Reader orientation. The complete-ground projection determines a normal state face, but not a preferred invariant state. This section uses smooth recovery packets, compact-uniform matrix-coefficient convergence, and support-saturating barycentric selection to choose an invariant state whose support is exactly the full complete-ground sector. Rank-one vacuum uniqueness is neither assumed nor required.

A. Input packet and branch typing

Fix a compact connected semisimple global form $G = \tilde{G}/\Gamma$, a declared branch label ξ , and a sector $S \in \{\text{I, II, III}\}$. Let r run through the selected positive-thickness cofinal system, $r \downarrow 0$. Denote the ordinary thick and thin regional algebras by

$$\mathfrak{A}_r(O), \quad \mathfrak{A}_0(O),$$

and their Poincaré actions by $\alpha_g^{(r)}$ and $\alpha_g^{(0)}$.

Hypothesis III.1 (V.I.A positive-thickness invariant-state packet). For every fixed finite regional-chart-corridor-Cauchy-background diagram, the following data are available on one cofinal tail.

(P1) **Coherent observable recovery.** There is a norm-dense Poincaré-invariant local $*$ -algebra $\mathfrak{A}_{0,\text{rec}}$ and, for every $A \in \mathfrak{A}_{0,\text{rec}} \cap \mathfrak{A}_0(O)$, a uniformly bounded coherent packet $A_r \in \mathfrak{A}_r(O_r)$. Products, adjoints, units, regional restrictions, and the declared spacetime action are recovered in the Paper-IV comparison topology. The action is point-norm continuous on the recovery algebra, uniformly on compact group sets for each fixed coherent packet.

(P2) **Exact invariant ground sections.** There is a convex family of coherent normal-state sections

$$\omega^j = \{\omega_{\Omega,r;O}^j\}_{r,O}, \quad j \in J,$$

such that

$$\omega_{\Omega,r}^j \circ \alpha_g^{(r)} = \omega_{\Omega,r}^j, \quad \omega_{\Omega,r}^j(P_{\Omega,r}) = 1,$$

for every allowed g , and the sections are compatible with the positive-thickness regional, Cauchy, background, chart, and scale arrows in the branch-appropriate sense.

(P3) **Local-normal tightness and uniqueness.** After transport to every fixed thin region, the restrictions of the sections are relatively compact in the local predual weak topology, every cluster point is normal, and moments on $\mathfrak{A}_{0,\text{rec}}$ determine the cluster point uniquely. Normal-state extension tightness holds along regional exhaustion.

- (P4) **Pointwise covariant moment convergence.** For coherent packets A_r, B_r and each fixed $g \in \mathcal{P}_+^\uparrow$,

$$\omega_{\Omega,r}^j(A_r^* \alpha_g^{(r)}(B_r)) \longrightarrow \omega_{\Omega,0}^j(A_0^* \alpha_g^{(0)}(B_0))$$

whenever the limiting functional is defined by (P3). The convergence is independent of the permitted packet and corridor choices.

- (P5) **Complete-ground recovery and support-tightness.** The Paper-IV complete-ground projections satisfy the certified projection recovery and are compatible under regional inclusion. There is one countable subfamily $J_0 \subset J$ such that, for every member O_n of a selected countable cofinal regional exhaustion and every nonzero

$$0 \neq x_0 \in P_{\Omega,0}(O_n) \mathfrak{A}_0(O_n) P_{\Omega,0}(O_n), \quad x_0 \geq 0,$$

there are a coherent positive recovery packet x_r in the moving complete-ground corners and an index $j = j(x_0) \in J_0$ satisfying

$$\liminf_{r \downarrow 0} \omega_{\Omega,r}^j(x_r) > 0.$$

This is a positive-thickness detection estimate; it does not assume the support of any limiting state.

- (P6) **Prepared Gauss–Ward and surgery topology.** Every section lies in the source-defined Gauss–Ward-prepared complete-ground face; graph-controlled Ward moments and state surgery pass through the selected predual comparison topology. Positive-thickness surgery fixes these sections.
- (P7) **Sector-correct seams.** Sector I sections obey the chosen total strictification. Sector II sections are strict on each fixed corridor and obey the relative corridor seams. Scalar Sector III sections obey the chart–corridor coefficient-line seams. All observable and normal-state shadows compose strictly.

Remark III.2 (Why support-tightness is separate). Condition (P5) is not replaceable by ordinary weak-* compactness. A sequence of faithful density matrices may converge to a state with smaller support. The displayed liminf estimate is verified, for example, by a finite-multiplicity normalized ground trace with uniform rank control, or by a countable invariant family carrying uniform detection estimates on a positive recovery cone. No rank-one assertion is required, and exact thin support is a conclusion below rather than an input.

Remark III.3 (Source allocation). Observable and normal-state tightness, rooted finite-diagram lifting, complete-ground face preservation, projection recovery, and prepared Gauss–Ward/surgery convergence are Paper-III/Paper-IV inputs. The present phase uses the positive-thickness invariant state sections as the new spectral-restart input and proves their invariant thin assembly.

B. Uniformly smooth Poincaré packets

Let dg be a left Haar measure on $\mathcal{P}_+^\uparrow$. For $f \in C_c^\infty(\mathcal{P}_+^\uparrow)$ and $A \in \mathfrak{A}_r$, define

$$\alpha_f^{(r)}(A) = \int_{\mathcal{P}_+^\uparrow} f(h) \alpha_h^{(r)}(A) dh.$$

The integral is a norm Bochner integral on the declared point-norm continuous observable action.

Definition III.4 (Common smooth recovery algebra). Let $\mathfrak{A}_0^{\text{sm}}$ be the $*$ -algebra generated by elements $\alpha_f^{(0)}(A)$ with $A \in \mathfrak{A}_{0,\text{rec}}$ and $f \in C_c^\infty(\mathcal{P}_+^\uparrow)$. A coherent lift is

$$\left(\alpha_f^{(0)}(A) \right)_r := \alpha_f^{(r)}(A_r).$$

Lemma III.5 (Density and covariance of the smooth algebra). *The algebra $\mathfrak{A}_0^{\text{sm}}$ is norm dense in the thin quasilocal algebra and invariant under $\alpha^{(0)}$. The displayed lifts are uniformly bounded and coherent.*

Proof. Choose a standard approximate identity $f_n \in C_c^\infty(\mathcal{P}_+^\uparrow)$ with $\int f_n = 1$ and supports shrinking to the identity. Point-norm continuity gives $\alpha_{f_n}^{(0)}(A) \rightarrow A$ for every recovered local observable, hence density after taking the local union and norm closure. For $g \in \mathcal{P}_+^\uparrow$,

$$\alpha_g^{(r)} \alpha_f^{(r)}(A) = \alpha_{L_g f}^{(r)}(A), \quad (L_g f)(k) = f(g^{-1}k),$$

which proves invariance. Uniform boundedness follows from

$$\|\alpha_f^{(r)}(A_r)\| \leq \|f\|_1 \|A_r\|.$$

Coherence follows by integrating the source comparison identities and using dominated convergence. $\square$

Lemma III.6 (Uniform translation modulus). *For a coherent smoothed packet $B_r = \alpha_f^{(r)}(C_r)$,*

$$\|\alpha_g^{(r)}(B_r) - \alpha_h^{(r)}(B_r)\| \leq \sup_r \|C_r\| \|L_g f - L_h f\|_1.$$

The right side is independent of r and tends to zero as $h \rightarrow g$, locally uniformly in g .

Proof. Using the covariance identity from [Theorem III.5](#),

$$\alpha_g^{(r)}(B_r) - \alpha_h^{(r)}(B_r) = \alpha_{L_g f - L_h f}^{(r)}(C_r).$$

The Bochner integral norm estimate gives the inequality. Translation is continuous in $L^1(\mathcal{P}_+^\uparrow)$, and continuity is uniform on compact sets by compactness of the parameter set. $\square$

C. Compact-uniform matrix-coefficient convergence

For one invariant section ω^j and coherent smooth packets A_r, B_r , set

$$F_r^j(g) = \omega_{\Omega, r}^j \left(A_r^* \alpha_g^{(r)}(B_r) \right).$$

Lemma III.7 (Uniform equicontinuity). *For every compact $K \subset \mathcal{P}_+^\uparrow$, the family $\{F_r^j\}_r$ is uniformly bounded and equicontinuous on K .*

Proof. Uniform boundedness is immediate:

$$|F_r^j(g)| \leq \|A_r\| \|B_r\|.$$

If $B_r = \alpha_f^{(r)}(C_r)$, then

$$|F_r^j(g) - F_r^j(h)| \leq \|A_r\| \|\alpha_g^{(r)}(B_r) - \alpha_h^{(r)}(B_r)\|.$$

Apply [Theorem III.6](#). Products and linear combinations inherit a common modulus on each fixed finite smooth packet. $\square$

Theorem III.8 (Compact-uniform Poincaré coefficient transfer). *For every compact $K \subset \mathcal{P}_+^\uparrow$ and coherent smooth packets A_r, B_r ,*

$$\boxed{\sup_{g \in K} \left| \omega_{\Omega, r}^j \left(A_r^* \alpha_g^{(r)}(B_r) \right) - \omega_{\Omega, 0}^j \left(A_0^* \alpha_g^{(0)}(B_0) \right) \right| \longrightarrow 0.}$$

Proof. By (P4), convergence holds at each fixed g . By [Theorem III.7](#), the thick coefficients are equicontinuous with one modulus on K . The same modulus passes to the pointwise limit. Given $\delta > 0$, choose a finite set $g_1, \dots, g_N \in K$ so that every $g \in K$ lies in a modulus neighborhood of some g_i and both the thick and limiting coefficient changes are less than $\delta/3$. Pointwise convergence at the finitely many g_i gives, on a cofinal tail,

$$|F_r^j(g_i) - F_0^j(g_i)| < \delta/3$$

for all i . The triangle inequality gives $\sup_K |F_r^j - F_0^j| < \delta$. $\square$

Remark III.9 (Topology). The theorem is compact-uniform in the Poincaré variable, predual-normal in every fixed regional algebra, and directed-uniform on each fixed finite source diagram. It is not norm convergence of states on the full quasilocal algebra.

D. Construction of the local thin invariant states

Proposition III.10 (Unique locally normal limit of one section). *Each positive-thickness invariant section ω^j defines a unique compatible family of normal thin regional states $\omega_{\Omega,0;O}^j$ satisfying:*

- (i) $\omega_{\Omega,0;O}^j \circ \alpha_g^{(0)} = \omega_{\Omega,0;O}^j$ whenever the transformed observable is regionalized in the declared diagram;
- (ii) $\omega_{\Omega,0;O}^j(P_{\Omega,0}(O)) = 1$;
- (iii) the Poincaré matrix coefficients are continuous on $\mathfrak{A}_0^{\text{sm}}$;
- (iv) regional restrictions and Cauchy transports agree exactly;
- (v) the state is Gauss–Ward admissible and fixed by thin surgery.

Proof. Fix O . Local-normal tightness gives a normal cluster point in $\mathfrak{A}_0(O)_*$. Moments on the dense recovery algebra are prescribed by (P4), so uniqueness in (P3) makes the whole directed section converge to one normal state.

For $B_0 \in \mathfrak{A}_0^{\text{sm}}$ with coherent lift B_r , exact positive-thickness invariance gives

$$\omega_{\Omega,r}^j(\alpha_g^{(r)}(B_r)) = \omega_{\Omega,r}^j(B_r).$$

Pass to the limit using [Theorem III.8](#) with $A = \mathbf{1}$ and the covariance recovery in (P1). This proves invariance on $\mathfrak{A}_0^{\text{sm}}$, hence on the local algebra by norm density.

Projection recovery and $\omega_{\Omega,r}^j(P_{\Omega,r}) = 1$ give

$$\omega_{\Omega,0;O}^j(P_{\Omega,0}(O)) = 1.$$

Continuity of matrix coefficients is the compact-uniform limit theorem. Regional, Cauchy, and background compatibility follows by passing the source state-seam identities or vanishing defects through the unique predual limit.

For a prepared Ward insertion $\delta(A)$, the positive-thickness complete-ground Ward identity is zero. Graph-predual convergence therefore gives the thin Ward identity. Finally, either pass

$$\mathcal{S}_{r,*}^{\text{st,as}} \omega_{\Omega,r}^j = \omega_{\Omega,r}^j$$

through the corrected surgery graph, or invoke the Paper-IV theorem that thin surgery fixes the complete-ground face. Both give $\mathcal{S}_{0,*}^{\text{st,as}} \omega_{\Omega,0}^j = \omega_{\Omega,0}^j$. $\square$

E. Support-saturating barycentric selection

A single cluster state may occupy only a proper subprojection of a higher-rank complete-ground sector. The following construction is therefore load bearing.

Construction III.11 (Invariant complete-ground barycenter). Choose the countable support-tight family from (P5), relabelled $j \geq 1$, and weights $c_j > 0$ with $\sum_j c_j = 1$. Define regionally

$$\omega_{\Omega,0;O} = \sum_{j=1}^{\infty} c_j \omega_{\Omega,0;O}^j.$$

Lemma III.12 (Support of a countable positive barycenter). *For normal positive functionals ϕ_j and strictly positive c_j ,*

$$s\left(\sum_j c_j \phi_j\right) = \bigvee_j s(\phi_j).$$

Proof. Let e be a projection. Since all terms are nonnegative,

$$\sum_j c_j \phi_j(e) = 0 \iff \phi_j(e) = 0 \text{ for every } j.$$

Thus the largest projection annihilated by the sum is the meet of the largest projections annihilated by all ϕ_j . Taking orthocomplements gives the stated join of supports. $\square$

Theorem III.13 (Exact complete-ground support). *The barycentric state in [Theorem III.11](#) is normal on every regional algebra and satisfies*

$$s(\omega_{\Omega,0;O}) = P_{\Omega,0}(O).$$

It is Poincaré invariant, Gauss–Ward admissible, surgery fixed, and has continuous matrix coefficients on $\mathfrak{A}_0^{\text{sm}}$.

Proof. Normality follows because the predual is closed under absolutely convergent sums. Each summand is supported below $P_{\Omega,0}(O)$ by [Theorem III.10](#). Let $0 \neq x_0 \geq 0$ belong to the complete-ground corner of an exhaustion region O_n . By (P5), choose j and a coherent positive recovery packet x_r with positive liminf expectation. The local predual limit then gives

$$\omega_{\Omega,0;O_n}^j(x_0) = \lim_{r \downarrow 0} \omega_{\Omega,r;O_n}^j(x_r) > 0,$$

so the barycenter satisfies $\omega_{\Omega,0;O_n}(x_0) \geq c_j \omega_{\Omega,0;O_n}^j(x_0) > 0$. It is therefore faithful on the entire complete-ground corner, and its support equals $P_{\Omega,0}(O_n)$. Equivalently, [Theorem III.12](#) identifies its support with the join of the detected section supports. For an arbitrary admissible $O \subset O_n$, a nonzero positive element of the complete-ground corner of O remains nonzero in the compatible corner of O_n ; faithfulness there makes the restricted expectation strictly positive. Thus the restriction has support $P_{\Omega,0}(O)$. Invariance, Ward admissibility, and surgery fixedness are preserved by convex combinations.

For coefficient continuity, each summand obeys

$$|\omega^j(A^* \alpha_g(B))| \leq \|A\| \|B\|.$$

The weighted series therefore converges uniformly in g , and a uniform limit of continuous functions is continuous. The same dominated-tail argument combines [Theorem III.8](#) over j . □

Scope firewall [III.14](#) (Multiplicity firewall). The equality $s(\omega_{\Omega,0;O}) = P_{\Omega,0}(O)$ is not a rank-one statement. It says that the selected invariant state is faithful on the complete-ground corner. Vacuum multiplicity remains SP11.

F. Global quasilocal assembly

Theorem III.15 (Global ordinary invariant state). *The regional barycentric states assemble uniquely to a state $\omega_{\Omega,0}$ on the ordinary thin quasilocal C^* -algebra. It is locally normal, Poincaré invariant, and its regional support is the certified complete-ground projection.*

Proof. For $O_1 \subset O_2$, regional compatibility of every summand gives

$$\omega_{\Omega,0;O_2}|_{\mathfrak{A}_0(O_1)} = \omega_{\Omega,0;O_1}.$$

Hence define $\omega_{\Omega,0}(A) = \omega_{\Omega,0;O}(A)$ for $A \in \mathfrak{A}_0(O)$. This is well defined on the algebraic local union, positive, and norm one, so it extends uniquely to the quasilocal norm closure. Local normality and support are regional statements already proved. Poincaré invariance follows first on the invariant local smooth algebra and then by norm density. $\square$

G. Strict and twisted branch cases

The state construction on the ordinary observable algebra is common to all three scalar sectors. The represented interpretation is not.

1. Sector I: totally strict

Theorem III.16 (Sector-I invariant-state selection). *Let the branch be a fully certified Sector-I branch with*

$$\kappa_{\xi}^{\text{fib}} = 0, \quad \widehat{\kappa}_{\xi}^{\text{cs}} = 0,$$

and an effective total strictification. Under Theorem III.1, the state $\omega_{\Omega,0}$ of Theorem III.15 satisfies all VSR clauses. The represented complete-ground state family descends through the chosen strictification to one global represented state system.

Proof. The ordinary state theorem gives local normality, invariance, continuity, support, and prepared-domain compatibility. Total strictification identifies the chart, corridor, ground, and state-seam objects by honest global intertwiners. Pushing the barycentric family through these intertwiners gives one global represented state. Since the strictification is part of the certified Sector-I datum, no new coherence class is introduced. $\square$

Remark III.17. A nontrivial boundary-character class may remain. Sector-I strictification of the vacuum coefficient system does not trivialize collar-charge affine descent.

2. Sector II: fiberwise strict and relatively twisted

Theorem III.18 (Sector-II invariant-state selection). *Let the branch be a physically verified Sector-II branch:*

$$\kappa_{\xi}^{\text{fib}} = 0, \quad \widehat{\kappa}_{\xi}^{\text{cs}} \neq 0.$$

Assume [Theorem III.1](#) uniformly on every fixed finite corridor diagram. Then:

- (i) *there is one global locally normal Poincaré-invariant state $\omega_{\Omega,0}$ on the ordinary thin observable algebra;*
- (ii) *on every fixed corridor, the represented complete-ground state system is strictifiable and satisfies the same invariant-state theorem;*
- (iii) *across corridor germs, the represented states remain relatively coefficient-line twisted and retain $\widehat{\kappa}_{\xi}^{\text{cs}} \neq 0$;*
- (iv) *no global represented Hilbert-space state is inferred.*

Proof. The ordinary observable and normal-state shadows of the corridor transitions compose strictly, so [Theorem III.15](#) is independent of the chosen corridor. On a fixed corridor, the fiber class vanishes and the certified local strictification applies. For a corridor transition $a : \mathfrak{c} \rightarrow \mathfrak{d}$, state compatibility has the form

$$\omega_r^{\mathfrak{d}} = (\text{Ad } V_{a,r})_* \omega_r^{\mathfrak{c}}$$

on represented observables. Passing through the compact-uniform and predual limits gives the corresponding thin seam. The relative tensorator is retained by the represented system, but any scalar external phase cancels under Ad. A total strictification would contradict the nonzero relative class and is not used. $\square$

Scope firewall III.19 (Sector-II physical boundary). The theorem is unconditional for a branch only after its full physical chart-scale realization and positive-thickness invariant-state packet are certified. Paper V does not infer universal Sector-II non-vacuity from the abstract transfer architecture.

3. Sector III: fixed-corridor gerbe twist

Theorem III.20 (Scalar Sector-III invariant-state selection). *Let the branch be a fully certified scalar Sector-III branch with*

$$\kappa_\xi^{\text{fib}} = [\Omega_\xi] \neq 0.$$

Under [Theorem III.1](#):

- (i) *the ordinary thin observable algebra carries the global state $\omega_{\Omega,0}$ of [Theorem III.15](#);*
- (ii) *every chart–corridor represented object carries a locally normal invariant complete-ground state with exact support $P_{\Omega,\alpha,\mathfrak{c},0}$;*
- (iii) *on overlaps,*

$$\omega_{\alpha,\mathfrak{c},0} = (\text{Ad } V_{\alpha\beta}^\mathfrak{c})_* \omega_{\beta,\mathfrak{c},0};$$

- (iv) *the scalar associator $\Omega_{\alpha\beta\gamma}^\mathfrak{c}$ acts trivially on the state shadow but remains nontrivial in the coefficient-line represented lift;*
- (v) *no single global represented Hilbert space is constructed.*

Proof. At positive thickness the overlap relation is

$$V_{\alpha\beta,r}^\mathfrak{c} V_{\beta\gamma,r}^\mathfrak{c} = \Omega_{\alpha\beta\gamma,r}^\mathfrak{c} V_{\alpha\gamma,r}^\mathfrak{c}$$

with scalar Ω . Therefore

$$\text{Ad}(V_{\alpha\beta,r}^\mathfrak{c} V_{\beta\gamma,r}^\mathfrak{c}) = \text{Ad}(V_{\alpha\gamma,r}^\mathfrak{c}).$$

The observable and state shadows satisfy an ordinary cocycle law. The predual seam convergence and [Theorem III.8](#) pass this identity to zero thickness, giving a global ordinary state and chart–corridor local represented states. Paper-IV sector-conservative specialization retains $[\Omega_\xi] \neq 0$, so the represented family cannot be replaced by one global strict Hilbert realization. $\square$

Remark III.21 (Strict versus twisted terminology). Sector I is totally strict after effective descent. Sector II is strict on each fixed corridor but twisted in the corridor direction. Sector III is already twisted on at least one fixed corridor. The ordinary observable state shadow is global in all three cases.

H. The VSR theorem

Interpretive lead-in. The preceding constructions have separately supplied compact-uniform covariance, local normality, exact complete-ground support, global quasilocal assembly, and the strict or twisted branch interpretation. The next theorem packages those ingredients into the state that all later spectral constructions use.

Theorem III.22 (Poincaré-Invariant Complete-Ground State Selection Theorem — VSR).

Assume the definitive Paper-IV handoff and [Theorem III.1](#).

- (i) *Every fully certified Sector-I branch admits a locally normal state $\omega_{\Omega,0}^{G,\xi}$ on the ordinary thin quasilocal algebra satisfying VSR1–VSR6, together with its global represented strict lift.*
- (ii) *Every physically verified Sector-II branch admits the same global ordinary state; its represented invariant-state family is strict corridorwise and relatively twisted across corridors.*
- (iii) *Every fully certified scalar Sector-III branch admits the same global ordinary state; its represented invariant-state family remains chart-corridor coefficient-line banded and retains $[\Omega_\xi] \neq 0$.*

- (iv) *The state satisfies, for every compact $K \subset \mathcal{P}_+^\uparrow$ and coherent smooth packet,*

$$\sup_{g \in K} \left| \omega_{\Omega,r}(A_r^* \alpha_g^{(r)}(B_r)) - \omega_{\Omega,0}(A_0^* \alpha_g^{(0)}(B_0)) \right| \rightarrow 0.$$

- (v) *Regionally,*

$$s(\omega_{\Omega,0;O}) = P_{\Omega,0}(O),$$

without a rank-one assumption.

- (vi) *The state is fixed by inherited surgery and satisfies the declared complete-ground interior Ward identities. This does not prove that surgery commutes with future physical Poincaré spectral projections.*

Proof. The compact-uniform estimate is [Theorem III.8](#); the local state and prepared-domain properties are [Theorem III.10](#); exact complete-ground support is [Theorem III.13](#); and global assembly is [Theorem III.15](#). The three represented interpretations are [Theorems III.16](#), [III.18](#) and [III.20](#). These are precisely the six VSR clauses isolated in Phase V.0. $\square$

I. GNS implementation and SP3 closure

The representation-theoretic passage from an invariant state to its cyclic Hilbert-space realization uses the standard GNS construction^{8,9}; the covariance, continuity, and branchwise compatibility asserted below are established from the preceding Paper-V hypotheses.

Corollary III.23 (Strongly continuous ordinary Poincaré

implementation). *Let $(\mathcal{H}_{G,\xi}^{\text{obs}}, \pi_{G,\xi}^{\text{obs}}, \Omega_{G,\xi}^{\text{obs}})$ be the GNS triple of $\omega_{\Omega,0}^{G,\xi}$. Then*

$$U_{G,\xi}^{\text{obs}}(g)\pi_{G,\xi}^{\text{obs}}(A)\Omega_{G,\xi}^{\text{obs}} = \pi_{G,\xi}^{\text{obs}}(\alpha_g^{(0)}(A))\Omega_{G,\xi}^{\text{obs}}$$

defines a strongly continuous unitary representation of $\mathcal{P}_+^\uparrow$, fixes the GNS vector, and implements the thin observable action.

Proof. Invariance makes the displayed map isometric and well defined on the dense GNS cyclic domain; the inverse is the map for g^{-1} and the group law follows from the automorphism law. For $A \in \mathfrak{A}_0^{\text{sm}}$,

$$\|U^{\text{obs}}(g)\pi^{\text{obs}}(A)\Omega - U^{\text{obs}}(h)\pi^{\text{obs}}(A)\Omega\|^2 = 2\omega(A^*A) - 2\text{Re}\omega(A^*\alpha_{g^{-1}h}(A)).$$

The right side tends to zero by the coefficient continuity proved above. Density and unitarity extend strong continuity to all vectors. $\square$

Scope firewall III.24 (What SP3 does not provide). The GNS implementation gives strongly continuous spacetime symmetries. It does not yet construct the joint translation spectral measure, prove the forward-cone condition, or compare the Paper-IV stiffness form with invariant mass.

IV. SECTOR-APPROPRIATE REPRESENTED COVARIANCE AND SURGERY NATURALITY

A. Branch-indexed represented input

Fix a compact connected semisimple global form $G = \tilde{G}/\Gamma$, a branch label ξ , and a scalar sector $S \in \{\text{I, II, III}\}$. Let $r \downarrow 0$ denote the selected positive-thickness tail. A represented

local object is indexed by

$$\lambda = \begin{cases} *, & S = \text{I}, \\ \mathfrak{c}, & S = \text{II}, \\ (\alpha, \mathfrak{c}), & S = \text{III}, \end{cases}$$

where $\mathfrak{c}$ is a scale corridor and α is a chart index. The thick and thin represented local von Neumann algebras are denoted

$$\mathcal{M}_{r,\lambda}(O), \quad \mathcal{M}_{0,\lambda}(O),$$

with Poincaré actions $\beta_{r,\lambda,g}$ and $\beta_{0,\lambda,g}$.

Hypothesis IV.1 (V.I.B represented covariance packet). On every fixed finite regional-chart-corridor-Cauchy diagram, the definitive Paper-III/Paper-IV pass-through and Phase V.I.A provide the following.

- (R1) **Point-norm represented actions.** Each $g \mapsto \beta_{r,\lambda,g}$ and $g \mapsto \beta_{0,\lambda,g}$ is an automorphic Poincaré action, point-norm continuous on a norm-dense represented recovery $*$ -algebra and normal on the local von Neumann closure.
- (R2) **Invariant represented complete-ground states.** There are normal states $\omega_{r,\lambda}$ and their Phase-V.I.A limits $\omega_{0,\lambda}$ such that

$$\omega_{r,\lambda} \circ \beta_{r,\lambda,g} = \omega_{r,\lambda}, \quad \omega_{0,\lambda} \circ \beta_{0,\lambda,g} = \omega_{0,\lambda},$$

and their supports are the certified represented complete-ground projections $P_{\Omega,r,\lambda}$ and $P_{\Omega,0,\lambda}$.

- (R3) **Compact-uniform represented coefficient transfer.** For coherent represented smooth packets $A_{r,\lambda}, B_{r,\lambda}$ and every compact $K \subset \mathcal{P}_+^\uparrow$,

$$\sup_{g \in K} \left| \omega_{r,\lambda} \left(A_{r,\lambda}^* \beta_{r,\lambda,g} (B_{r,\lambda}) \right) - \omega_{0,\lambda} \left(A_{0,\lambda}^* \beta_{0,\lambda,g} (B_{0,\lambda}) \right) \right| \longrightarrow 0.$$

The estimate is uniform over the finitely many objects and arrows of the fixed diagram.

- (R4) **Equivariant represented seams.** Every allowed seam $a : \lambda \rightarrow \mu$ has a normal represented equivalence $\theta_{a,r} : \mathcal{M}_{r,\lambda} \rightarrow \mathcal{M}_{r,\mu}$ and a thin limit $\theta_{a,0}$ satisfying

$$\theta_{a,r} \beta_{r,\lambda,g} = \beta_{r,\mu,g} \theta_{a,r}, \quad \theta_{a,0} \beta_{0,\lambda,g} = \beta_{0,\mu,g} \theta_{a,0},$$

as exact identities on the declared regionalized action domain. The states obey

$$\omega_{r,\mu} \circ \theta_{a,r} = \omega_{r,\lambda}, \quad \omega_{0,\mu} \circ \theta_{a,0} = \omega_{0,\lambda}.$$

(R5) **Branch coherence.** Sector I seams strictify globally. Sector II seams strictify inside each fixed corridor and retain the nonzero relative chart–scale class across corridor germs. Scalar Sector III seams are implemented by coefficient-line unitaries and obey the certified scalar associator law. Observable and normal-state shadows compose strictly.

(R6) **Regional and Cauchy naturality.** Represented restrictions, Cauchy evolutions, complete-ground projections, and Gauss–Ward graph domains commute with $\beta_{r,\lambda,g}$ and $\beta_{0,\lambda,g}$ whenever both sides are defined in the covariant regional diagram.

Remark IV.2 (Why (R4) is not a global strictification). In Sectors II and III the equations in (R4) are arrowwise. They say that every legitimate seam is Poincaré equivariant. They do not say that all seams can be simultaneously trivialized or that all $\mathcal{M}_{0,\lambda}$ act on one Hilbert space.

B. Local represented GNS implementation

For each branch object λ , let

$$(\mathcal{H}_\lambda, \pi_\lambda, \Omega_\lambda)$$

be the GNS triple of $\omega_{0,\lambda}$ on the represented local or quasilocal algebra appropriate to λ .

Construction IV.3 (Local Poincaré implementer). On the cyclic domain define

$$U_\lambda(g)\pi_\lambda(A)\Omega_\lambda := \pi_\lambda(\beta_{0,\lambda,g}(A))\Omega_\lambda.$$

Lemma IV.4 (Well-defined unitarity and group law). *The map in [Theorem IV.3](#) extends uniquely to a unitary $U_\lambda(g)$, and*

$$U_\lambda(g)U_\lambda(h) = U_\lambda(gh), \quad U_\lambda(g)\Omega_\lambda = \Omega_\lambda.$$

It implements the represented action:

$$U_\lambda(g)\pi_\lambda(A)U_\lambda(g)^* = \pi_\lambda(\beta_{0,\lambda,g}(A)).$$

Proof. State invariance gives

$$\|\pi_\lambda(\beta_g(A))\Omega_\lambda\|^2 = \omega_{0,\lambda}(\beta_g(A^*A)) = \omega_{0,\lambda}(A^*A),$$

so the definition is well posed and isometric on the cyclic domain. The inverse is the map for g^{-1} , hence the extension is unitary. The group law follows on the dense cyclic domain from the automorphism law and extends by continuity. The implementation identity follows first on cyclic vectors and then by density. $\square$

Lemma IV.5 (Strong continuity). *The representation $g \mapsto U_\lambda(g)$ is strongly continuous.*

Proof. For a represented smooth element A and $g, h \in \mathcal{P}_+^\uparrow$,

$$\begin{aligned} & \|U_\lambda(g)\pi_\lambda(A)\Omega_\lambda - U_\lambda(h)\pi_\lambda(A)\Omega_\lambda\|^2 \\ &= 2\omega_{0,\lambda}(A^*A) - 2\operatorname{Re}\omega_{0,\lambda}(A^*\beta_{0,\lambda,g^{-1}h}(A)). \end{aligned}$$

The right side tends to zero by compact-uniform represented coefficient continuity. Smooth cyclic vectors are dense by Haar smoothing and the norm density of the recovery algebra. Uniform boundedness of the unitaries extends continuity to all vectors. $\square$

Proposition IV.6 (Complete-ground covariance). *For every branch object λ ,*

$$\beta_{0,\lambda,g}(P_{\Omega,0,\lambda}) = P_{\Omega,0,\lambda}$$

in the local normal carrier, and therefore

$$U_\lambda(g)\pi_\lambda(P_{\Omega,0,\lambda}) = \pi_\lambda(P_{\Omega,0,\lambda})U_\lambda(g).$$

Proof. For a normal positive functional ϕ and a normal automorphism β ,

$$s(\phi \circ \beta^{-1}) = \beta(s(\phi)).$$

Apply this to the invariant state $\omega_{0,\lambda}$, whose support is $P_{\Omega,0,\lambda}$ by Phase V.I.A. The implementation identity then gives the commutation relation. $\square$

Scope firewall IV.7 (Support is not rank one). The invariant projection in [Theorem IV.6](#) may have arbitrary finite or infinite multiplicity permitted by the source package. No vacuum-uniqueness claim is made.

C. Equivariant GNS seams

Let $a : \lambda \rightarrow \mu$ be an allowed represented seam. The state relation $\omega_{0,\mu} \circ \theta_{a,0} = \omega_{0,\lambda}$ gives a canonical GNS isometry.

Construction IV.8 (Canonical seam implementer). Define on cyclic vectors

$$W_a \pi_\lambda(A) \Omega_\lambda := \pi_\mu(\theta_{a,0}(A)) \Omega_\mu.$$

When $\theta_{a,0}$ is an equivalence onto the relevant represented object, W_a is unitary.

Lemma IV.9 (Poincaré intertwining of seams). *Every canonical seam implementer satisfies*

$$\boxed{W_a U_\lambda(g) = U_\mu(g) W_a.}$$

It also intertwines the represented complete-ground projections.

Proof. On a cyclic vector,

$$\begin{aligned} W_a U_\lambda(g) \pi_\lambda(A) \Omega_\lambda &= \pi_\mu(\theta_a \beta_{\lambda,g}(A)) \Omega_\mu \\ &= \pi_\mu(\beta_{\mu,g} \theta_a(A)) \Omega_\mu \\ &= U_\mu(g) W_a \pi_\lambda(A) \Omega_\lambda. \end{aligned}$$

Density gives the unitary identity. Complete-ground intertwining follows from the certified seam relation for the projections, or equivalently from support transport of the seam-related states. $\square$

Proposition IV.10 (Strict shadow and retained coefficient law). *For composable seams a, b , the canonical GNS shadows satisfy*

$$W_b W_a = W_{ba}$$

whenever the induced algebra seam composes strictly. In scalar-twisted represented sectors, the physical coefficient-line implementers may instead satisfy

$$V_b V_a = \Omega(b, a) V_{ba}, \quad \Omega(b, a) \in U(1),$$

while

$$\text{Ad}(V_b V_a) = \text{Ad}(V_{ba}).$$

Both V_a and its canonical GNS shadow intertwine the local Poincaré implementers.

Proof. The first identity follows by evaluating both sides on cyclic vectors and using strict composition of the algebra/state shadow. For a scalar coefficient line, multiplication by $\Omega(b, a)$ disappears under adjoint action. The Poincaré intertwining follows from [Theorem IV.9](#); multiplying an intertwiner by a scalar does not alter the relation. $\square$

D. Sector I: global strict represented lift

Theorem IV.11 (Sector-I global represented covariance theorem). *Let (G, ξ, I) be a fully certified Sector-I branch:*

$$\kappa_\xi^{\text{fib}} = 0, \quad \hat{\kappa}_\xi^{\text{cs}} = 0,$$

with the Paper-IV admissible total strictification. Then the local represented GNS systems assemble to

$$\boxed{(\mathcal{H}_{G,\xi}^I, \pi_{G,\xi}^I, U_{G,\xi}^I, P_{\Omega,G,\xi}^I)}$$

such that:

- (i) $U_{G,\xi}^I$ is a strongly continuous unitary representation of $\mathcal{P}_+^\uparrow$;
- (ii) it implements the global represented thin action;
- (iii) $P_{\Omega,G,\xi}^I$ is invariant;
- (iv) regional restrictions, Cauchy evolution, time-slice maps, and the Gauss–Ward graph domains are Poincaré covariant;
- (v) the strictification introduces no new scalar or correspondence coherence class.

Proof. The certified total strictification identifies all represented chart and corridor objects through a coherent family of honest unitary seams. By [Theorem IV.9](#), the local Poincaré implementers agree under every identification. Hence they descend to one unitary $U_{G,\xi}^I(g)$ on the strictified Hilbert space. Strong continuity is local on a dense union of strictified smooth cyclic subspaces and follows from [Theorem IV.5](#); unitarity then extends it globally. The implementation and complete-ground covariance identities pass through the same identifications. Regional, Cauchy, and Gauss–Ward covariance is exactly (R6) after strict descent. $\square$

Remark IV.12 (Boundary data). Sector-I strictification concerns the represented vacuum coefficient system. It does not force a nontrivial retained collar or boundary-character class to vanish.

E. Sector II: corridorwise strict and relatively twisted lift

Theorem IV.13 (Sector-II corridorwise represented covariance theorem). *Let (G, ξ, Π) be a physically verified Sector-II branch:*

$$\kappa_{\xi}^{\text{fib}} = 0, \quad \hat{\kappa}_{\xi}^{\text{cs}} \neq 0.$$

Then:

- (i) *for every fixed scale corridor $\mathfrak{c}$, the represented chart system strictifies to a strongly continuous covariant quadruple*

$$(\mathcal{H}_{\mathfrak{c}}^{\Pi}, \pi_{\mathfrak{c}}^{\Pi}, U_{\mathfrak{c}}^{\Pi}, P_{\Omega, \mathfrak{c}}^{\Pi});$$

- (ii) *every allowed corridor transition $a : \mathfrak{c} \rightarrow \mathfrak{d}$ has a relative coefficient-line or tensorator intertwiner V_a satisfying*

$$V_a U_{\mathfrak{c}}^{\Pi}(g) = U_{\mathfrak{d}}^{\Pi}(g) V_a;$$

- (iii) *the ordinary observable shadow remains the single global Poincaré representation constructed in Phase V.I.A;*
- (iv) *the nonzero class $\hat{\kappa}_{\xi}^{\text{cs}}$ is retained, and no global represented Hilbert-space descent is inferred.*

Proof. Inside one fixed corridor, the fiber class vanishes and the certified corridor strictification applies. The proof of [Theorem IV.11](#) therefore gives the corridorwise quadruple. Across corridors, [Theorem IV.9](#) gives Poincaré intertwining for each relative seam. The seam composition law retains the nontrivial chart-scale tensorator by (R5). Since the observable and state shadows compose strictly, their GNS representation is global; since the represented relative class is nonzero, a total represented strictification is neither proved nor used. $\square$

Scope firewall IV.14 (Sector-II non-vacuity). The theorem is unconditional only for a Sector-II branch whose physical represented realization, invariant state packet, and relative seams are

independently certified. The abstract analytic architecture alone does not establish physical non-vacuity for every compact semisimple group.

F. Sector III: chart–corridor coefficient-line lift

For a scalar Sector-III branch, write the chart–corridor object as $\lambda = (\alpha, \mathfrak{c})$. Let

$$L_{\alpha\beta}^{\mathfrak{c}}$$

be the one-dimensional coefficient line on an overlap and

$$V_{\alpha\beta}^{\mathfrak{c}} : L_{\alpha\beta}^{\mathfrak{c}} \otimes \mathcal{H}_{\beta}^{\mathfrak{c}} \longrightarrow \mathcal{H}_{\alpha}^{\mathfrak{c}}$$

the physical represented intertwiner. The coefficient lines carry the trivial Poincaré action unless a source-certified equivariant line action is explicitly present; in the latter case the identity below includes that line action.

Theorem IV.15 (Sector-III banded represented covariance theorem). *Let (G, ξ, III) be a fully certified scalar Sector-III branch with*

$$\kappa_{\xi}^{\text{fib}} = [\Omega_{\xi}] \neq 0.$$

Then there is a chart–corridor-banded family

$$\boxed{\left\{ (\mathcal{H}_{\alpha, G, \xi}^{\mathfrak{c}}, \pi_{\alpha, G, \xi}^{\mathfrak{c}}, U_{\alpha, G, \xi}^{\mathfrak{c}}, P_{\Omega, \alpha, G, \xi}^{\mathfrak{c}}) \right\}_{\alpha, \mathfrak{c}}}$$

with the following properties.

- (i) *Every $U_{\alpha, G, \xi}^{\mathfrak{c}}$ is a strongly continuous unitary Poincaré representation implementing the local represented action.*
- (ii) *Every coefficient-line seam is Poincaré equivariant:*

$$\boxed{V_{\alpha\beta}^{\mathfrak{c}}(\mathbf{1}_{L_{\alpha\beta}^{\mathfrak{c}}} \otimes U_{\beta}^{\mathfrak{c}}(g)) = U_{\alpha}^{\mathfrak{c}}(g)V_{\alpha\beta}^{\mathfrak{c}}.}$$

- (iii) *On a composable triple, the physical intertwiners obey the certified twisted associativity law*

$$V_{\alpha\beta}^{\mathfrak{c}}(\mathbf{1} \otimes V_{\beta\gamma}^{\mathfrak{c}}) = \Omega_{\alpha\beta\gamma}^{\mathfrak{c}} V_{\alpha\gamma}^{\mathfrak{c}} \mu_{\alpha\beta\gamma}^{\mathfrak{c}}, \quad \Omega_{\alpha\beta\gamma}^{\mathfrak{c}} \in U(1),$$

with the source coefficient-line multiplication $\mu_{\alpha\beta\gamma}^{\mathfrak{c}}$.

- (iv) *The scalar associator disappears under adjoint action but remains in the represented coefficient system.*
- (v) *The complete-ground family is Poincaré invariant and seam compatible.*
- (vi) *No global represented Hilbert space is constructed.*

Proof. Local implementation and strong continuity are [Theorems IV.4](#) and [IV.5](#). The overlap relation is [Theorem IV.9](#), written with the coefficient line displayed. The twisted associativity law is the certified Sector-III seam law in (R5); it survives the thin comparison because the seam convergence is performed on one rooted finite chart–corridor diagram. Since Ω is scalar,

$$\text{Ad}(V_{\alpha\beta}(\mathbf{1} \otimes V_{\beta\gamma})) = \text{Ad}(V_{\alpha\gamma}\mu_{\alpha\beta\gamma}),$$

so the algebra and state shadows compose strictly. The nonzero class is retained by sector-conservative specialization, which forbids replacing the banded family by one strict global represented object. Ground covariance follows from [Theorem IV.6](#) and seam compatibility from [Theorem IV.9](#). $\square$

Remark IV.16 (Same observable covariance, inequivalent represented coherence). The ordinary observable Poincaré representation can be the same strict AQFT object in Sectors I and III while the represented lifts remain inequivalent. Equality of ordinary covariance does not trivialize $[\Omega_\xi]$.

G. Poincaré-natural entropy surgery

We now prove spacetime covariance of inherited surgery independently of the represented implementation theorem. The argument is formulated on normal states because the source surgery is a state-side entropy selector and need not be linear.

For a branch object λ , define the state action

$$(g \cdot \phi)(A) := \phi(\beta_{0,\lambda,g^{-1}}(A)).$$

Let

$$\mathcal{D}_{0,\lambda}^{\text{prep}} \subset (\mathcal{M}_{0,\lambda})_*^+$$

be the selected physical Orlicz/Gauss–Ward-prepared state domain, and similarly at positive thickness.

Hypothesis IV.17 (Primitive surgery-naturality packet). For each $r \geq 0$ and branch object λ , the surgery selector $\mathcal{S}_{r,\lambda,*}^{\text{st,as}}$ is defined by a single-valued canonical selection from a variational graph

$$\Gamma_{r,\lambda} \subset \mathcal{D}_{r,\lambda}^{\text{prep}} \times \mathcal{D}_{r,\lambda}^{\text{prep}}.$$

The graph is generated by the following source data.

(S1) **Covariant domain and constraint fibers.** $g \cdot \mathcal{D}_{r,\lambda}^{\text{prep}} = \mathcal{D}_{r,\lambda}^{\text{prep}}$ and

$$\mathcal{C}_{r,\lambda}(g \cdot \phi) = g \cdot \mathcal{C}_{r,\lambda}(\phi).$$

The constraints include the declared regional, normalization, complete-ground, Orlicz, Gauss-graph, interior Ward, collar quotient, and retained boundary-character conditions.

(S2) **Covariant reference logarithms.** The source reference weights and logarithms satisfy the appropriate transported identity

$$\beta_{r,\lambda,g}(\log \rho_{r,\lambda}) = \log \rho_{r,g\lambda}$$

inside a covariant background orbit, or are invariant when $g\lambda = \lambda$.

(S3) **Invariant objective.** The entropy/penalty functional obeys

$$\mathcal{J}_{r,g\lambda}(g \cdot \psi \mid g \cdot \phi) = \mathcal{J}_{r,\lambda}(\psi \mid \phi).$$

This uses automorphism invariance of relative entropy and covariance of every source penalty term.

(S4) **Canonical single-valued selection.** The selected minimizer is unique, or the source selector is characterized by a Poincaré-natural rule that is single valued on the declared domain.

(S5) **Comparison-graph convergence.** If $\phi_r \rightarrow \phi_0$ in the selected local predual/Orlicz graph topology, then

$$\mathcal{S}_{r,\lambda,*}^{\text{st,as}}(\phi_r) \rightarrow \mathcal{S}_{0,\lambda,*}^{\text{st,as}}(\phi_0),$$

uniformly for g in compact subsets after transport through any fixed finite regional-chart-corridor diagram.

(S6) **Seam naturality.** For an allowed seam $a : \lambda \rightarrow \mu$, the variational data are transported by $\theta_{a,r}$, including constraint fibers, reference logarithms, objective, and canonical selector.

Remark IV.18 (What is genuinely new in the audit). Paper IV already proves that surgery preserves the prepared Gauss–Ward face and fixes the selected complete-ground state family. Those statements alone do not imply (S1)–(S4). Phase V.I.B requires and verifies Poincaré naturality of the actual defining variational data.

Lemma IV.19 (Finite-thickness covariance of the surgery graph). *Under [Theorem IV.17](#), for every $r > 0$,*

$$(\phi, \psi) \in \Gamma_{r,\lambda} \iff (g \cdot \phi, g \cdot \psi) \in \Gamma_{r,g\lambda}.$$

Consequently,

$$\boxed{\mathcal{S}_{r,g\lambda,*}^{\text{st,as}}(g \cdot \phi) = g \cdot \mathcal{S}_{r,\lambda,*}^{\text{st,as}}(\phi).}$$

Proof. Let ψ be the selected minimizer in $\mathcal{C}_{r,\lambda}(\phi)$. By (S1), $g \cdot \psi$ belongs to $\mathcal{C}_{r,g\lambda}(g \cdot \phi)$. If η is any competitor in that transformed fiber, then $g^{-1} \cdot \eta$ is a competitor in the original fiber. By (S2)–(S3),

$$\mathcal{J}_{r,g\lambda}(\eta \mid g \cdot \phi) = \mathcal{J}_{r,\lambda}(g^{-1} \cdot \eta \mid \phi) \geq \mathcal{J}_{r,\lambda}(\psi \mid \phi) = \mathcal{J}_{r,g\lambda}(g \cdot \psi \mid g \cdot \phi).$$

Thus $g \cdot \psi$ is a selected minimizer. Single-valued canonicity in (S4) gives the equality. Applying the same argument to g^{-1} gives equivalence of graph membership. $\square$

Theorem IV.20 (Thin Poincaré covariance of inherited surgery). *For every scalar branch object and every $\phi \in \mathcal{D}_{0,\lambda}^{\text{prep}}$,*

$$\boxed{\mathcal{S}_{0,g\lambda,*}^{\text{st,as}}(g \cdot \phi) = g \cdot \mathcal{S}_{0,\lambda,*}^{\text{st,as}}(\phi).}$$

If g preserves the branch object, this is

$$\boxed{\beta_{0,\lambda,g,*} \mathcal{S}_{0,\lambda,*}^{\text{st,as}} = \mathcal{S}_{0,\lambda,*}^{\text{st,as}} \beta_{0,\lambda,g,*}.$$

The convergence proving the identity is compact-uniform in g on fixed finite prepared diagrams.

Proof. Choose a coherent positive-thickness recovery section $\phi_r \rightarrow \phi_0$ in the prepared topology. By [Theorem IV.19](#),

$$\mathcal{S}_{r,g\lambda,*}^{\text{st,as}}(g \cdot \phi_r) = g \cdot \mathcal{S}_{r,\lambda,*}^{\text{st,as}}(\phi_r)$$

for every $r > 0$. The covariant comparison maps send $g \cdot \phi_r$ to $g \cdot \phi_0$. Apply (S5) to the left and right sides. Both converge in the same Hausdorff local predual/Orlicz graph topology, uniformly for g in a compact set. Equality of the limits gives the theorem. $\square$

Corollary IV.21 (Seam covariance of surgery). *For every allowed seam $a : \lambda \rightarrow \mu$,*

$$\boxed{(\theta_{a,0})_* \mathcal{S}_{0,\lambda,*}^{\text{st,as}} = \mathcal{S}_{0,\mu,*}^{\text{st,as}} (\theta_{a,0})_*}$$

on the prepared state domain. In scalar Sector III, the coefficient-line phase has no effect on this state-side identity, while its represented associator remains nontrivial.

Proof. Repeat the proof of [Theorem IV.19](#) with the seam transport in (S6) in place of the Poincaré action, then pass to the thin graph limit. Scalar phases disappear on normal-state pushforwards by adjoint action. $\square$

Corollary IV.22 (Complete-ground fixed face and no covariance overclaim). *The selected invariant complete-ground state family is fixed by surgery and Poincaré stable. Surgery therefore neither moves that state face nor violates its Gauss–Ward preparation. This phase does not prove that surgery commutes with a future sharp spectral projection of M^2 .*

Proof. Fixedness is the Paper-IV/Phase-V.I.A state theorem. Poincaré stability is [Theorem IV.6](#); surgery covariance is [Theorem IV.20](#). Sharp spectral projections have not yet been constructed, so no such commutation statement is available. $\square$

H. Gauss–Ward, collar, and regional compatibility

Theorem IV.23 (Prepared-domain covariance theorem). *For every branch object λ , the represented Poincaré action and inherited surgery preserve the declared prepared structure:*

- (i) *the Gauss graph domain and its graph norm class;*
- (ii) *the interior Ward identity on its insertion domain;*
- (iii) *the collar quotient and retained current class;*
- (iv) *the branch boundary-character datum;*
- (v) *regional restriction and Cauchy evolution;*

(vi) *the complete-ground projection and state face.*

The statement is covariance of the declared data, not numerical invariance of every charge expectation.

Proof. Poincaré preservation follows from (R6) and normal extension to the graph closures. Surgery preservation is the corrected Paper-IV Gauss–Ward/surgery theorem on the prepared domain. The compatibility of the two operations follows from [Theorem IV.20](#): applying a Poincaré transport before or after surgery gives the same prepared state, so the Ward and collar identities transported by either route agree. Boundary-character data are transported in their declared affine or coefficient-line class; no claim is made that all charge moments are fixed pointwise. $\square$

I. Main reconstruction theorem

Theorem IV.24 (Sector-Appropriate Covariant Representation Reconstruction and Surgery Theorem). *Assume the definitive Paper-IV handoff, Phase V.I.A, and the packets in [Theorems IV.1](#) and [IV.17](#). Then:*

- (1) *Every fully certified Sector-I branch has one global represented strongly continuous Poincaré system*

$$(\mathcal{H}_{G,\xi}^I, \pi_{G,\xi}^I, U_{G,\xi}^I, P_{\Omega,G,\xi}^I).$$

- (2) *Every physically verified Sector-II branch has strongly continuous represented systems on each fixed corridor, joined by Poincaré-equivariant relative seams retaining $\widehat{\kappa}_\xi^{\text{CS}} \neq 0$.*
- (3) *Every fully certified scalar Sector-III branch has a chart–corridor-banded strongly continuous represented family with coefficient-line intertwiners conjugating the local Poincaré actions and retaining $[\Omega_\xi] \neq 0$.*
- (4) *The complete-ground projections are Poincaré invariant in every branch object.*
- (5) *Inherited entropy surgery is Poincaré covariant on the selected physical Orlicz/Gauss–Ward-prepared state domain and is compatible with every represented seam.*
- (6) *Regional, Cauchy, time-slice, Gauss–Ward, collar, current, and boundary-character structures remain compatible with the represented action and surgery.*

(7) *No global Sector-II or Sector-III physical Hilbert-space strictification is inferred.*

(8) *No translation spectrum, spectrum condition, sharp spectral projection, invariant-mass operator, or mass gap is asserted.*

Proof. Items (1)–(3) are [Theorems IV.11, IV.13 and IV.15](#). Item (4) is [Theorem IV.6](#). Item (5) is [Theorems IV.20 and IV.21](#). Item (6) is [Theorem IV.23](#). Items (7)–(8) are the sector and spectral firewalls already enforced in the component theorems. $\square$

What has now been established. Each branch now carries the correct strongly continuous represented Poincaré action: global in Sector I, corridorwise in Sector II, and coefficient-line banded in Sector III. The observable shadow is strict, but the represented seams retain their branch coherence. No spectral measure has yet been constructed; that is the next upgrade.

V. JOINT TRANSLATION GENERATORS, SPECTRAL MEASURES, AND COMMON CORES

Reader orientation. Strongly continuous translations provide four commuting one-parameter groups, but the later arguments require one sharp joint spectral system and a common core compatible with the Paper-IV form, Gauss–Ward data, regional arrows, and branch seams. This section constructs that joint PVM before imposing positivity or a mass threshold.

A. Branch-indexed translation input

Fix $G = \tilde{G}/\Gamma$ compact, connected, and semisimple, a selected branch ξ , and a scalar sector $S \in \{I, II, III\}$. A represented object is indexed by

$$\lambda = \begin{cases} *, & S = I, \\ \mathfrak{c}, & S = II, \\ (\alpha, \mathfrak{c}), & S = III. \end{cases}$$

Write $(\mathcal{H}_\lambda, \pi_\lambda, U_\lambda)$ for the Phase-V.I.B represented Poincaré system. The ordinary global system is treated by the same argument with the label $\lambda = \text{obs}$.

Hypothesis V.1 (Definitive prepared translation packet). The certified Paper-IV-to-Paper-V intake of [Theorem II.2](#) and Phases V.I.A–V.I.B provide the following on every fixed finite regional-chart-corridor-Cauchy diagram.

- (T1) **Strongly continuous Poincaré systems.** Each U_λ is a strongly continuous unitary representation of $\mathcal{P}_+^\uparrow$ and satisfies the semidirect-product relation

$$U_\lambda(g)U_\lambda(a)U_\lambda(g)^* = U_\lambda(\Lambda_g a), \quad a \in \mathbb{R}^{3,1},$$

where Λ_g is the Lorentz part of g .

- (T2) **Exact branch seams.** Every allowed represented seam $a : \lambda \rightarrow \mu$ is implemented by a unitary, an isometry, or a coefficient-line unitary W_a satisfying

$$W_a(\mathbf{1} \otimes U_\lambda(b)) = U_\mu(b)W_a, \quad b \in \mathbb{R}^{3,1},$$

with the identity factor suppressed in strict branches. Sector-I seams strictify globally; Sector-II seams retain the relative chart-scale class; scalar Sector-III seams obey the certified coefficient-line associator.

- (T3) **Recovered prepared packet core.** For each λ there is one linear core $\mathcal{D}_{\text{IV},\lambda}^{\text{prep}}$ containing the coherent represented local Wilson packet vectors and satisfying:

- (a) it is dense in $\mathcal{H}_\lambda$;
- (b) it is a form core for the Paper-IV closed form $\bar{q}_{0,\lambda}$;
- (c) it is graph dense in every declared finite support-stage Gauss graph space and every Ward insertion domain;
- (d) it is preserved by $P_{\Omega,\lambda}$, regional/Cauchy comparison arrows, and the represented branch seams;
- (e) the Poincaré action maps it into itself after covariant transport of the regional, gauge-parameter, and insertion labels.

- (T4) **Prepared graph continuity.** Equip $\mathcal{D}_{\text{IV},\lambda}^{\text{prep}}$ with the projective locally convex topology generated by the Hilbert norm, the thin-form norm, and the finite support-stage Gauss–Ward graph seminorms. The translation action is continuous in this

topology and locally equicontinuous: for each defining seminorm p and compact $K \subset \mathbb{R}^{3,1}$, there exist a defining seminorm p' and $C_{p,K} < \infty$ such that

$$\sup_{a \in K} p(U_\lambda(a)\Psi) \leq C_{p,K} p'(\Psi).$$

(T5) **Ground reduction.** The complete-ground projection is translation invariant in the reducing sense:

$$U_\lambda(a)P_{\Omega,\lambda} = P_{\Omega,\lambda}U_\lambda(a).$$

(T6) **Prepared state stability.** Translation of a prepared normal state remains in the selected Orlicz/Gauss–Ward surgery domain, and the Phase-V.I.B surgery covariance identity holds there.

Proposition V.2 (Prepared graph continuity is inherited, not new spectral input). *Clause (T4) follows from the definitive comparison package: exact positive-thickness Poincaré transport of the form and Gauss–Ward graph families, compact-uniform finite-diagram graph bounds, common-core recovery, and TT1–TT2 pass to the thin limit. No translation generator or spectral measure is used in that passage.*

Proof. At positive thickness, each Poincaré arrow transports the relevant regional form, graph map, gauge parameter, and Ward insertion by an exact bounded map between the corresponding graph completions. On a compact set K and a fixed finite diagram, the source package gives one finite maximum of the graph-transport constants. Coherent recovery identifies the transported thick packet with the translated thin packet. The liminf half of comparison–Mosco controls the limiting graph norm, while TT2 supplies graph-dense recovery and the reverse approximation. The same argument applies to each finite support-stage Gauss–Ward seminorm because its graph convergence is part of the corrected common-core package. Translation continuity on the algebraic packet core passes by density and local equicontinuity to the projective graph topology. No unbounded generator has entered the proof. $\square$

B. Joint spectral measure and strongly commuting generators

We first suppress the branch label and consider one strongly continuous translation representation

$$U : \mathbb{R}^{3,1} \longrightarrow \mathcal{U}(\mathcal{H}).$$

For $\mu = 0, 1, 2, 3$, let e_μ be the standard coordinate vector and write $U_\mu(t) = U(te_\mu)$.

The standard one-parameter generator theorem and joint spectral/functional calculus used in this construction are recalled from^{10–12}; the common-core and branch-covariance conclusions are Paper-V results.

Theorem V.3 (Joint translation spectral theorem). *There exists a unique projection-valued measure*

$$E : \mathfrak{B}(\mathbb{R}^4) \longrightarrow \text{Proj}(\mathcal{H})$$

such that

$$U(a) = \int_{\mathbb{R}^4} e^{i\langle p, a \rangle} dE(p), \quad a \in \mathbb{R}^{3,1}.$$

The coordinate operators

$$P_\mu := \int_{\mathbb{R}^4} p_\mu dE(p)$$

are self-adjoint, strongly commute, and satisfy

$$U(te_\mu) = e^{itP_\mu}.$$

Their domains are

$$\text{Dom}(P_\mu) = \left\{ \Psi : \int p_\mu^2 d\langle \Psi, E(p)\Psi \rangle < \infty \right\}.$$

More generally, every real Borel function F defines a self-adjoint joint functional-calculus operator

$$F(P) = \int F(p) dE(p)$$

with its maximal spectral domain.

Proof. Each one-parameter subgroup U_μ has a self-adjoint Stone generator P_μ . Since the translation group is abelian, the unitary groups e^{itP_μ} commute pairwise for all parameters. Hence the generators strongly commute, so their individual spectral measures admit a unique product joint spectral measure. The displayed Fourier representation follows first on coordinate translations and then on all of $\mathbb{R}^4$ by the group law. Uniqueness follows from uniqueness of Fourier transforms of finite complex measures applied to every matrix coefficient $\langle \Phi, U(a)\Psi \rangle$. $\square$

Definition V.4 (Joint spectrum). The translation spectrum is the closed support

$$\text{sp } U := \text{supp } E$$

of the joint projection-valued measure. This definition is branch-objectwise in Sectors II and III.

Definition V.5 (Invariant-mass operator before positivity). Define the real polynomial

$$m^2(p) := p_0^2 - \sum_{j=1}^3 p_j^2$$

and the self-adjoint operator

$$M^2 := m^2(P) = P_0^2 - \sum_{j=1}^3 P_j^2$$

by joint functional calculus, with domain

$$\text{Dom}(M^2) = \left\{ \Psi : \int |m^2(p)|^2 d\langle \Psi, E(p)\Psi \rangle < \infty \right\}.$$

Its spectral measure is

$$E_{M^2}(C) = E\left((m^2)^{-1}(C)\right), \quad C \subset \mathbb{R} \text{ Borel}.$$

Remark V.6 (No square-root form yet). Before the forward-cone theorem, M^2 need not be nonnegative. The canonical nonnegative quantity presently available is $|M^2|$, not M^2 . Phase V.IV may compare K^{thin} to M^2 only after SP5 has established nonnegativity on the physical spectrum.

C. Lorentz covariance of the sharp joint spectral measure

For a Lorentz transformation Λ , define the dual momentum transformation $\Lambda^\vee$ by

$$\langle \Lambda^\vee p, a \rangle := \langle p, \Lambda a \rangle.$$

This convention removes index-placement ambiguity.

Theorem V.7 (Sharp Poincaré covariance of the joint spectral measure). *Let $g \in \mathcal{P}_+^\uparrow$ have Lorentz part Λ_g . Then for every Borel set $B \subset \mathbb{R}^4$,*

$$U(g)E(B)U(g)^* = E\left((\Lambda_g^\vee)^{-1}B\right).$$

Consequently,

$$U(g)M^2U(g)^* = M^2, \quad U(g)E_{M^2}(C)U(g)^* = E_{M^2}(C)$$

for every Borel $C \subset \mathbb{R}$.

Proof. By the semidirect-product law,

$$U(g)U(a)U(g)^* = U(\Lambda_g a).$$

Using [Theorem V.3](#), the left side is the Fourier transform of the projection-valued measure $B \mapsto U(g)E(B)U(g)^*$, while the right side is

$$\int e^{i\langle p, \Lambda_g a \rangle} dE(p) = \int e^{i\langle \Lambda_g^\vee p, a \rangle} dE(p),$$

the Fourier transform of the pushforward measure $B \mapsto E((\Lambda_g^\vee)^{-1}B)$. Uniqueness of the joint spectral measure gives the first identity for all Borel sets, not merely continuity sets. Since $m^2(\Lambda_g^\vee p) = m^2(p)$, the joint functional calculus gives the remaining identities. $\square$

Corollary V.8 (Covariance of coordinate generators). *For every $g \in \mathcal{P}_+^\uparrow$, the four generators transform as the dual Lorentz four-vector determined by $\Lambda_g^\vee$, with equality of transported domains. The operator M^2 is a Poincaré scalar.*

D. Translation convolution and the common spectral core

For $f \in L^1(\mathbb{R}^4)$, define the bounded convolution operator

$$\mathcal{U}(f) := \int_{\mathbb{R}^4} f(a)U(a) da, \quad \|\mathcal{U}(f)\| \leq \|f\|_1,$$

where the integral is taken in the strong Bochner sense.

Lemma V.9 (Differentiation of translation convolution). *If $f \in C_c^\infty(\mathbb{R}^4)$, then $\mathcal{U}(f)\mathcal{H}$ lies in the common C^∞ -domain of the four generators and, for every multi-index α ,*

$$P^\alpha \mathcal{U}(f) = i^{|\alpha|} \mathcal{U}(\partial^\alpha f)$$

as a bounded operator from $\mathcal{H}$ into $\mathcal{H}$. In particular,

$$M^2 \mathcal{U}(f) = -\mathcal{U}(\square_a f), \quad \square_a := \partial_0^2 - \sum_{j=1}^3 \partial_j^2.$$

Proof. For one coordinate, strong differentiation gives $\partial_{a^\mu} U(a) = iP_\mu U(a)$ on the generator domain. Integration by parts against compactly supported f yields

$$P_\mu \mathcal{U}(f) = i\mathcal{U}(\partial_\mu f).$$

The right side is bounded on all of $\mathcal{H}$, which proves that $\mathcal{U}(f)\mathcal{H} \subset \text{Dom}(P_\mu)$. Iteration proves the multi-index identity. The formula for M^2 follows from the joint polynomial definition. $\square$

Definition V.10 (Common invariant spectral core). For each branch object λ , define

$$\mathcal{C}_\lambda^{\text{spec}} := \text{span} \left\{ \mathcal{U}_\lambda(f) \Psi : f \in C_c^\infty(\mathbb{R}^4), \Psi \in \mathcal{D}_{\text{IV}, \lambda}^{\text{prep}} \right\}.$$

The ordinary global core is denoted $\mathcal{C}_{\text{obs}}^{\text{spec}}$.

Lemma V.11 (Density and Poincaré invariance). *The space $\mathcal{C}_\lambda^{\text{spec}}$ is dense in $\mathcal{H}_\lambda$ and invariant under the full represented Poincaré action. It is also invariant under $P_{\Omega, \lambda}$ and every exact branch seam.*

Proof. Let $(f_n) \subset C_c^\infty(\mathbb{R}^4)$ be a nonnegative approximate identity with $\int f_n = 1$ and supports shrinking to 0. Strong continuity gives $\mathcal{U}(f_n) \Psi \rightarrow \Psi$ for every $\Psi \in \mathcal{H}$. Since $\mathcal{D}_{\text{IV}}^{\text{prep}}$ is Hilbert dense, convolution of vectors from $\mathcal{D}_{\text{IV}}^{\text{prep}}$ is dense.

For a translation b ,

$$U(b) \mathcal{U}(f) = \mathcal{U}(L_b f), \quad (L_b f)(a) = f(a - b),$$

and $L_b f \in C_c^\infty(\mathbb{R}^4)$. For a Lorentz transformation g , the semidirect-product law and change of variables give

$$U(g) \mathcal{U}(f) U(g)^* = \mathcal{U}(f^g)$$

for another compactly supported smooth function f^g obtained from the linear change of variables. Clause (T3e) transports $\mathcal{D}_{\text{IV}}^{\text{prep}}$, proving full Poincaré invariance. Ground invariance follows from (T5). Seam invariance follows by moving the seam through the Bochner integral using (T2). $\square$

Theorem V.12 (Common operator-core theorem). *The space $\mathcal{C}_\lambda^{\text{spec}}$ is a common invariant operator core for*

$$P_{0, \lambda}, P_{1, \lambda}, P_{2, \lambda}, P_{3, \lambda}, M_\lambda^2,$$

and for every real polynomial in the jointly commuting generators.

Proof. By [Theorem V.9](#), $\mathcal{C}^{\text{spec}}$ is contained in the domain of every generator polynomial. Let $F(P)$ be such a polynomial and let $\Psi \in \text{Dom}(F(P))$. Because $F(P)$ commutes with every translation,

$$F(P) \mathcal{U}(f_n) \Psi = \mathcal{U}(f_n) F(P) \Psi.$$

Thus $\mathcal{U}(f_n)\Psi \rightarrow \Psi$ in the graph norm of $F(P)$. It remains to replace the unsmoothed seed Ψ by vectors in $\mathcal{D}_{\text{IV}}^{\text{prep}}$. Fix n and choose $\Psi_k \in \mathcal{D}_{\text{IV}}^{\text{prep}}$ with $\Psi_k \rightarrow \Psi$ in Hilbert norm. The differentiation formula expresses

$$F(P)\mathcal{U}(f_n)(\Psi_k - \Psi)$$

as a finite sum of bounded convolution operators with derivatives of f_n applied to $\Psi_k - \Psi$. Hence it converges to zero in norm. A diagonal choice gives approximation in the graph norm of $F(P)$ by elements of $\mathcal{C}^{\text{spec}}$. $\square$

E. Compatibility with the Paper-IV form and Gauss–Ward graphs

Let

$$\|\Psi\|_q^2 := \|\Psi\|^2 + \bar{q}_0[\Psi]$$

be the form norm. For a finite support-stage Gauss–Ward packet $\mathcal{K}$, write $p_{\mathcal{K}}$ for any of the defining graph seminorms in [Theorem V.1](#).

Lemma V.13 (Graph-topology convolution estimate). *For every $f \in C_c^\infty(\mathbb{R}^4)$ and every defining prepared seminorm p , there exist a defining seminorm p' and a finite constant*

$$C_{p,f} := \int_{\text{supp } f} |f(a)| C_{p,\text{supp } f} da$$

such that

$$p(\mathcal{U}(f)\Psi) \leq C_{p,f} p'(\Psi), \quad \Psi \in \mathcal{D}_{\text{IV}}^{\text{prep}}.$$

If (f_n) is an approximate identity with shrinking support, then

$$\mathcal{U}(f_n)\Psi \longrightarrow \Psi$$

in every prepared graph seminorm.

Proof. The local equicontinuity estimate in (T4) makes the Bochner integral converge in each graph completion and yields the displayed inequality by Minkowski’s integral inequality. Translation continuity in the projective graph topology and shrinking support imply convergence of the approximate identity in each seminorm. $\square$

Theorem V.14 (Paper-IV form-core compatibility). *The inclusion*

$$\mathcal{C}_\lambda^{\text{spec}} \subset \text{Dom}(\bar{q}_{0,\lambda})$$

holds, and $\mathcal{C}_\lambda^{\text{spec}}$ is a form core for $\bar{q}_{0,\lambda}$. More precisely, every $\Psi \in \text{Dom}(\bar{q}_{0,\lambda})$ admits $\Psi_n \in \mathcal{C}_\lambda^{\text{spec}}$ such that

$$\|\Psi_n - \Psi\|^2 + \bar{q}_{0,\lambda}[\Psi_n - \Psi] \longrightarrow 0.$$

The result is compatible with regional exhaustion and branch seams.

Proof. Take the form norm as one of the prepared seminorms in [Theorem V.13](#). Convolution preserves the form domain. Since $\mathcal{D}_{\text{IV}}^{\text{prep}}$ is a form core, first approximate Ψ in form norm by $\Phi_k \in \mathcal{D}_{\text{IV}}^{\text{prep}}$ and then smooth Φ_k by an approximate identity. The graph-topology convergence gives form-norm convergence. A diagonal sequence lies in $\mathcal{C}^{\text{spec}}$ and converges to Ψ . Regional and seam compatibility follow because the corresponding arrows preserve $\mathcal{D}_{\text{IV}}^{\text{prep}}$ and intertwine translations before convolution. $\square$

Theorem V.15 (Gauss–Ward common-core compatibility). *For every finite support-stage Gauss graph, Ward insertion domain, collar quotient test packet, and retained current or boundary-character graph packet, $\mathcal{C}_\lambda^{\text{spec}}$ is graph dense in the corresponding prepared domain. Translation convolution obeys the covariant graph identities obtained by transporting the gauge parameters and insertion labels. No self-adjoint global Gauss operator is inferred in a merely graph-form branch.*

Proof. Apply [Theorem V.13](#) to the relevant finite support-stage graph seminorms. The covariance law in (T3e) transports each gauge parameter or insertion along the translation before integration; the projective prepared topology includes all members of the finite translated packet needed for the estimate. Approximate identities recover the original vector in the graph topology. The conclusion is therefore graph-space density and compatibility, not an upgrade from a graph form to a self-adjoint operator. $\square$

Corollary V.16 (Complete-ground compatibility). *The projection $P_{\Omega,\lambda}$ preserves $\mathcal{C}_\lambda^{\text{spec}}$ and commutes with every $P_{\mu,\lambda}$, every joint spectral projection $E_\lambda(B)$, and M_λ^2 .*

Proof. Clause (T5) says that P_Ω commutes with the entire translation representation. It therefore commutes with the unique joint spectral measure and all joint functional-calculus operators. Preservation of the core follows from [Theorem V.11](#). $\square$

Remark V.17 (Exact limitation of [Theorem V.16](#)). The spectral measure decomposes as

$$E(B) = P_\Omega E(B) P_\Omega + (\mathbf{1} - P_\Omega) E(B) (\mathbf{1} - P_\Omega).$$

The first summand need not be concentrated at $p = 0$ on the information currently available. This is the precise remaining issue in the vacuum inclusion half of SP7.

F. All-Borel spectral intertwining under seams

Theorem V.18 (Spectral intertwining theorem). *Let $W : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a unitary satisfying*

$$WU_1(a) = U_2(a)W \quad \text{for all } a \in \mathbb{R}^{3,1}.$$

Then for every Borel $B \subset \mathbb{R}^4$,

$$WE_1(B) = E_2(B)W.$$

Moreover,

$$W \operatorname{Dom}(P_{\mu,1}) = \operatorname{Dom}(P_{\mu,2}), \quad WP_{\mu,1} = P_{\mu,2}W,$$

and

$$W \operatorname{Dom}(M_1^2) = \operatorname{Dom}(M_2^2), \quad WM_1^2 = M_2^2W.$$

The same conclusion holds for an isometry onto its closed range, with the spectral measure on that range.

Proof. The Fourier transform of the projection-valued measure $B \mapsto WE_1(B)W^*$ is

$$WU_1(a)W^* = U_2(a)$$

on the range of W . Uniqueness of the joint spectral measure gives the all-Borel identity. The generator and mass identities follow from joint functional calculus, including equality of the transported maximal domains. $\square$

Corollary V.19 (Spectral support is seam invariant). *Unitary branch seams preserve the joint translation support and the M^2 support exactly. This is an arrowwise equivalence statement. It is not yet the global assembly theorem for products, central quotients, direct sums, direct integrals, or correspondence fusion required by SP9.*

G. Sector I: global strict spectral system

Theorem V.20 (Sector-I global joint-generator theorem). *For every fully certified Sector-I branch, the global represented Poincaré system admits:*

- (i) *one joint spectral measure $E_{G,\xi}^I$ on $\mathbb{R}^4$;*
- (ii) *four strongly commuting self-adjoint generators $P_{\mu,G,\xi}^I$;*
- (iii) *one self-adjoint operator $(M^2)_{G,\xi}^I$;*
- (iv) *one Poincaré-invariant common spectral core $\mathcal{C}_{G,\xi}^{\text{spec},I}$ compatible with the thin form, Gauss–Ward graphs, complete-ground projection, regional inclusions, and Cauchy evolution;*
- (v) *sharp Poincaré covariance of $E_{G,\xi}^I$.*

All global strictification seams intertwine the entire joint functional calculus.

Proof. Apply [Theorems V.3, V.7, V.12, V.14, V.15](#) and [V.18](#) to the global Sector-I represented system constructed in Phase V.I.B. □

H. Sector II: corridorwise spectral systems

Theorem V.21 (Sector-II corridorwise joint-generator theorem). *Let (G, ξ, Π) be a physically verified Sector-II branch. For every corridor $\mathfrak{c}$ there exist*

$$(E_{\mathfrak{c}}^{\Pi}, P_{\mu,\mathfrak{c}}^{\Pi}, (M^2)_{\mathfrak{c}}^{\Pi}, \mathcal{C}_{\mathfrak{c}}^{\text{spec},\Pi}).$$

Every allowed relative corridor seam $W_{\mathfrak{c}\mathfrak{d}}$ satisfies

$$W_{\mathfrak{c}\mathfrak{d}} E_{\mathfrak{c}}^{\Pi}(B) = E_{\mathfrak{d}}^{\Pi}(B) W_{\mathfrak{c}\mathfrak{d}}$$

for all Borel B , and similarly intertwines all generators and M^2 . The nonzero relative chart–scale class $\widehat{\kappa}_{\xi}^{\text{cs}}$ remains encoded in the composition of corridor seams; it is not erased by equality of spectral supports.

Proof. Construct the spectral system corridorwise. Phase V.I.B gives exact translation intertwining of every relative seam, so [Theorem V.18](#) applies. The spectral PVM is unique objectwise, but uniqueness does not choose a simultaneous trivialization of the relative tensorators. Therefore the obstruction class remains. □

Scope firewall V.22 (Sector-II physical qualification). The theorem is unconditional only for a Sector-II branch whose physical represented corridor systems and relative seams are independently certified. The abstract branch architecture alone remains insufficient for universal physical non-vacuity.

I. Sector III: coefficient-line-banded spectral systems

For a scalar Sector-III seam, write

$$V_{\alpha\beta}^{\mathfrak{c}} : \mathcal{H}_{\beta}^{\mathfrak{c}} \hat{\otimes} L_{\alpha\beta}^{\mathfrak{c}} \longrightarrow \mathcal{H}_{\alpha}^{\mathfrak{c}},$$

where the one-dimensional coefficient line carries the trivial translation action.

Theorem V.23 (Sector-III coefficient-line joint-generator theorem). *For every fully certified scalar Sector-III branch and every chart-corridor object $(\alpha, \mathfrak{c})$, there exist*

$$E_{\alpha}^{\mathfrak{c}}, \quad P_{\mu, \alpha}^{\mathfrak{c}}, \quad (M^2)_{\alpha}^{\mathfrak{c}}, \quad \mathcal{C}_{\alpha}^{\text{spec}, \mathfrak{c}}.$$

For every Borel $B \subset \mathbb{R}^4$,

$$V_{\alpha\beta}^{\mathfrak{c}}(\mathbf{1}_{L_{\alpha\beta}^{\mathfrak{c}}} \otimes E_{\beta}^{\mathfrak{c}}(B)) = E_{\alpha}^{\mathfrak{c}}(B)V_{\alpha\beta}^{\mathfrak{c}}.$$

The same identity holds for every joint Borel function, in particular the four momentum generators and M^2 . On triple overlaps, the two spectral intertwiners differ by the certified scalar associator

$$\Omega_{\alpha\beta\gamma}^{\mathfrak{c}} \in U(1),$$

which commutes with every spectral projection. Thus spectral support is chart independent while the represented associator class remains nontrivial. No global Sector-III Hilbert-space representation is constructed.

Proof. Apply [Theorem V.3](#) objectwise. The coefficient line has trivial translation action, so the Phase-V.I.B overlap equation is exactly the hypothesis of [Theorem V.18](#) after displaying the identity factor. The scalar associator commutes with the spectral calculus, so both parenthesized triple intertwiners induce the same conjugation of every spectral projection while remaining distinct coefficient-line two-cells by the factor $\Omega_{\alpha\beta\gamma}^{\mathfrak{c}}$. This proves chart-independent support without strictifying the gerbe. $\square$

Remark V.24 (Ordinary and represented spectra). The ordinary observable representation is one global strict spectral system. The Sector-III represented system is a banded family of spectrally equivalent local objects. Equality of their support sets does not imply unitary equivalence of the represented branches or triviality of $[\Omega_\xi]$.

J. Regional, Cauchy, and prepared-state compatibility

Theorem V.25 (Spectral naturality for exact comparison arrows). *Every bounded exact regional, Cauchy, chart, corridor, or background comparison arrow that intertwines the Phase-V.I.B translation actions also intertwines the joint PVMs and their Borel functional calculi. The common spectral cores are mapped into the corresponding common spectral cores, and are onto whenever the comparison arrow is unitary.*

Proof. The PVM statement is [Theorem V.18](#). Core compatibility follows by moving the arrow through translation convolution and using preservation of the recovered prepared packet core. $\square$

Proposition V.26 (Prepared surgery compatibility without spectral commutation). *Let φ lie in the selected normal-state surgery domain. Translation covariance gives*

$$\mathcal{S}_*^{\text{st,as}}(a \cdot \varphi) = a \cdot \mathcal{S}_*^{\text{st,as}}(\varphi).$$

Every Poincaré-smoothed observable and every normal functional generated on the prepared spectral-core test domain remains an admissible test before and after surgery, and its translated expectation obeys the displayed covariance law. All Gauss–Ward/collar structural constraints therefore remain defined. No identity between $\mathcal{S}_*^{\text{st,as}}$ and $E(B)$ follows unless a separate affine-linear or represented surgery lift is constructed.

Proof. Translation convolution keeps the observable/vector tests in the prepared projective graph domain, and surgery preserves the corresponding normal-state domain by (T6). Phase V.I.B supplies the covariance identity and Gauss–Ward/collar preservation. A nonlinear state selector cannot be composed with a Hilbert-space projection as though it were a bounded operator, so no spectral commutator is defined at this level. $\square$

K. Main theorem and obligation verdict

Theorem V.27 (Branchwise Joint-Generator and Common-Core Theorem). *Assume the definitive Paper-IV handoff and Phases V.I.A–V.I.B. Then:*

- (1) *The global ordinary observable translation representation has a unique joint PVM $E_{G,\xi}^{\text{obs}}$ and four strongly commuting self-adjoint generators $P_{\mu,G,\xi}^{\text{obs}}$.*
- (2) *The self-adjoint Borel functional-calculus operator*

$$(M^2)_{G,\xi}^{\text{obs}} = (P_0^{\text{obs}})^2 - \sum_{j=1}^3 (P_j^{\text{obs}})^2$$

exists without a positivity claim.

- (3) *There is one dense Poincaré-invariant common spectral core $\mathcal{C}_{G,\xi}^{\text{spec,obs}}$ that is a common operator core for the momentum generators and M^2 , a form core for the Paper-IV thin form, and graph-core compatible with the Gauss–Ward and insertion domains.*
- (4) *The complete-ground projection reduces the joint spectral measure but is not yet identified with the zero-momentum or zero-mass projection.*
- (5) *Every fully certified Sector-I branch has the corresponding global represented spectral system.*
- (6) *Every physically verified Sector-II branch has corridorwise spectral systems joined by all-Borel relative spectral intertwiners retaining $\hat{\kappa}_\xi^{\text{cs}}$.*
- (7) *Every fully certified scalar Sector-III branch has chart–corridor spectral systems joined by coefficient-line all-Borel spectral intertwiners retaining $[\Omega_\xi] \neq 0$.*
- (8) *Sharp Poincaré covariance and exact comparison-arrow covariance of the joint PVM are proved.*
- (9) *The spectrum condition, positivity of M^2 , form-to-mass comparison, surgery/spectral commutation, no-leakage, and mass gap remain unproved.*

Proof. Items (1)–(4) follow from [Theorems V.3, V.5, V.12 and V.14 to V.16](#). Items (5)–(7) are [Theorems V.20, V.21 and V.23](#). Item (8) is [Theorems V.7, V.18 and V.25](#). Item (9) is enforced by the firewalls and [Theorem V.26](#). $\square$

Proposition V.28 (SP ledger after Phase V.II). *After this phase:*

- (1) *SP1–SP3 remain closed with their previous branch qualifications.*
- (2) *SP4 is closed for the global ordinary system, the global Sector-I represented system, every physically verified Sector-II corridor system, and every certified scalar Sector-III chart–corridor system.*
- (3) *SP8 is closed for sharp Poincaré covariance, complete-ground reduction, and exact spectral intertwining under represented seams and exact comparison arrows. The surgery/spectral-restriction clause remains open.*
- (4) *The arrowwise spectral part of SP9 is derived, but product, quotient, direct-sum, direct-integral, and correspondence-fusion stability remain open.*
- (5) *SP5–SP7 and SP10–SP12 remain open.*
- (6) *The Paper-IV threshold κ_* is unchanged. No comparison constant $c_{G,\xi}$ and no positive mass threshold have been defined.*

Scope firewall V.29 (Phase V.II outcome).

Phase V.II is closed for fully certified Sector-I and scalar Sector-III branches, and branch-conditionally for physically verified Sector-II branches. Joint translation generators, joint projection-valued measures, the self-adjoint invariant-mass polynomial, and one common invariant spectral core now exist in their correct global, corridorwise, or coefficient-line-banded categories. No support restriction or positive mass conclusion has been used. The next exact phase is V.III: prove the thin forward-cone spectrum condition independently of the Paper-IV stiffness threshold.

VI. LORENTZ-ORBIT DISINTEGRATION AND SECTOR-CORRECT INDUCTION

A. Poincaré spectral input and standardness

Fix a branch object λ . It is one of

$$\lambda = \text{obs}, \quad \lambda = * \text{ (S = I)}, \quad \lambda = \mathfrak{c} \text{ (S = II)}, \quad \lambda = (\alpha, \mathfrak{c}) \text{ (S = III)}.$$

Write $(\mathcal{H}_\lambda, U_\lambda, E_\lambda)$ for the strongly continuous Poincaré representation and joint translation PVM constructed in Phases V.I.B–V.II.

Hypothesis VI.1 (Phase-V.II spectral input). For every valid branch object λ :

(I1) U_λ is a strongly continuous unitary representation of $\mathbb{R}^{3,1} \rtimes \mathcal{L}$.

(I2)

$$U_\lambda(a, 1) = \int_{\mathbb{R}^4} e^{i\langle p, a \rangle} dE_\lambda(p).$$

(I3) Lorentz covariance holds for every Borel set:

$$U_\lambda(0, \ell)E_\lambda(B)U_\lambda(0, \ell)^* = E_\lambda(\Lambda_\ell B).$$

(I4) Every legitimate branch seam $W : \mathcal{H}_\lambda \rightarrow \mathcal{H}_\mu$, with the coefficient-line identity factor displayed where required, intertwines the full Poincaré representations and therefore all joint spectral projections.

(I5) The complete-ground projection $P_{\Omega, \lambda}$ commutes with U_λ and E_λ but is not yet identified with a zero-orbit or zero-mass projection.

(I6) The selected recovery presentation is countably generated on each ordinary, corridor, or chart–corridor object. Equivalently, the relevant cyclic GNS/represented Hilbert spaces are separable. If a nonseparable enlargement is retained, all statements below apply to its canonical orthogonal sum of separable cyclic Poincaré subrepresentations.

Lemma VI.2 (Reduction to separable cyclic summands). *Let U be any strongly continuous unitary representation of the second-countable group $\mathcal{P}_+^\uparrow$. Then $\mathcal{H}$ is an orthogonal Hilbert direct sum of closed cyclic reducing subspaces for U . Every such summand is separable and reduces the translation PVM E . Consequently the orbit-disintegration theorem may be proved on separable summands and then assembled by orthogonal direct sum.*

Proof. A nondegenerate representation of the full group C^* -algebra is an orthogonal direct sum of cyclic representations. Since $\mathcal{P}_+^\uparrow$ is second countable, its full group C^* -algebra is separable, so every cyclic representation space is separable. A closed subspace reducing U reduces the von Neumann algebra generated by the translation unitaries and therefore reduces their joint spectral measure. Apply the separable theorem on each summand. $\square$

B. Explicit Lorentz-orbit stratification of momentum space

Use the metric convention

$$p^2 = p_0^2 - |\mathbf{p}|^2.$$

Define the standard Borel orbit-label space

$$\mathfrak{T}_{\text{orb}} := \{\mathbf{0}\} \sqcup \left(\{t^+, t^-\} \times (0, \infty) \right) \sqcup \{n^+, n^-\} \sqcup \left(\{s\} \times (0, \infty) \right).$$

Define $\vartheta : \mathbb{R}^4 \rightarrow \mathfrak{T}_{\text{orb}}$ by

$$\vartheta(p) = \begin{cases} \mathbf{0}, & p = 0, \\ (t^+, \sqrt{p^2}), & p^2 > 0, p_0 > 0, \\ (t^-, \sqrt{p^2}), & p^2 > 0, p_0 < 0, \\ n^+, & p^2 = 0, p_0 > 0, \\ n^-, & p^2 = 0, p_0 < 0, \\ (s, \sqrt{-p^2}), & p^2 < 0. \end{cases}$$

Choose representatives

$$\begin{aligned} p_{\mathbf{0}} &= (0, 0, 0, 0), \\ p_{t^+, m} &= (m, 0, 0, 0), \quad p_{t^-, m} = (-m, 0, 0, 0), \\ p_{n^+} &= (1, 0, 0, 1), \quad p_{n^-} = (-1, 0, 0, 1), \\ p_{s, \mu} &= (0, 0, 0, \mu). \end{aligned}$$

For $\tau \in \mathfrak{T}_{\text{orb}}$, let

$$\mathcal{O}_\tau := \mathcal{L} \cdot p_\tau, \quad \mathcal{L}_\tau := \{\ell \in \mathcal{L} : \Lambda_\ell p_\tau = p_\tau\}.$$

Lemma VI.3 (Orbit classifier). *The map ϑ is Borel, Lorentz invariant, and its fibers are exactly the $\mathcal{L}$ -orbits in $\mathbb{R}^4$. Hence $\mathfrak{T}_{\text{orb}}$ is a concrete countably separated standard Borel model for the orbit quotient.*

Proof. Borel measurability follows from the displayed inequalities and continuous functions $p \mapsto p^2$ and $p \mapsto p_0$. Proper orthochronous Lorentz transformations preserve p^2 and the time orientation of nonspacelike nonzero vectors, so ϑ is invariant. The standard normal-form calculation sends every timelike, nonzero null, or spacelike vector to the displayed representative with the same label. Conversely distinct labels differ in Lorentz-invariant norm, time orientation, or zero/nonzero status and cannot lie in the same orbit. $\square$

Remark VI.4 (Little groups). For the ordinary connected Lorentz group the stabilizers are, up to the chosen covering group,

$$\mathcal{L}_{\mathbf{t}^\pm, m} \simeq SO(3), \quad \mathcal{L}_{\mathbf{n}^\pm} \simeq E(2), \quad \mathcal{L}_{\mathbf{s}, \mu} \simeq SO^+(1, 2), \quad \mathcal{L}_{\mathbf{0}} = \mathcal{L}.$$

The theorem uses only the stabilizer definition, so it is insensitive to whether the representation is written on $SO^+(1, 3)$ or its fixed connected cover.

C. The orbit-label PVM and central reduction

Definition VI.5 (Orbit-label projection-valued measure). For $C \in \mathfrak{B}(\mathfrak{T}_{\text{orb}})$ define

$$F_\lambda(C) := E_\lambda(\vartheta^{-1}(C)).$$

Proposition VI.6 (Full Poincaré reduction by orbit label). *For every branch object λ , F_λ is a PVM on $\mathfrak{T}_{\text{orb}}$ and*

$$[F_\lambda(C), U_\lambda(g)] = 0 \quad (g \in \mathcal{P}_+^\uparrow, C \in \mathfrak{B}(\mathfrak{T}_{\text{orb}})).$$

Moreover

$$F_\lambda(\{\mathbf{0}\}) = E_\lambda(\{0\}),$$

and

$$E_{M_\lambda^2}(J) = F_\lambda(q^{-1}(J)),$$

where

$$q(\tau) := p_\tau^2 = \begin{cases} 0, & \tau \in \{\mathbf{0}, \mathbf{n}^+, \mathbf{n}^-\}, \\ m^2, & \tau = (\mathbf{t}^\pm, m), \\ -\mu^2, & \tau = (\mathbf{s}, \mu). \end{cases}$$

Proof. The first assertion is functoriality of a PVM under inverse image. Translation unitaries commute with all $E_\lambda(B)$. Lorentz covariance and $\vartheta(\Lambda p) = \vartheta(p)$ give

$$U(0, \ell)F(C)U(0, \ell)^* = E(\Lambda\vartheta^{-1}(C)) = E(\vartheta^{-1}(C)).$$

The zero-orbit identity is immediate. The last identity follows from the joint functional calculus because $p^2 = q(\vartheta(p))$. $\square$

Corollary VI.7 (Zero momentum versus zero mass). *Before the spectrum condition,*

$$E_\lambda(\{0\}) = F_\lambda(\{\mathbf{0}\}),$$

whereas

$$E_{M_\lambda^2}(\{0\}) = F_\lambda(\{\mathbf{0}, \mathbf{n}^+, \mathbf{n}^-\}).$$

After the forward-cone condition, the past-null term disappears but the future nonzero null orbit remains:

$$E_{M_\lambda^2}(\{0\}) = F_\lambda(\{\mathbf{0}, \mathbf{n}^+\}).$$

Thus the future no-leakage theorem must prove both

$$F_\lambda(\{\mathbf{0}\}) = P_{\Omega, \lambda} \quad \text{and} \quad F_\lambda(\{\mathbf{n}^+\}) = 0.$$

Scope firewall VI.8 (Ground reduction is not orbit identification). Since $P_{\Omega, \lambda}$ commutes with U_λ , it also commutes with F_λ and is decomposable over orbit labels. Nothing proved so far prevents $P_{\Omega, \lambda}$ from having a component on a nonzero orbit. Pointwise translation triviality and exact zero-mode identification remain SP7.

D. Direct-integral disintegration over Lorentz orbits

We use the standard direct-integral decomposition of commutative von Neumann algebras and decomposable operators¹³; the orbit-support and seam statements below are proved for the present spectral system.

Theorem VI.9 (Orbit disintegration of the Poincaré spectral system). *Assume $\mathcal{H}_\lambda$ separable. There exist a Borel measure ν_λ on $\mathfrak{T}_{\text{orb}}$, a measurable field of Hilbert spaces $\{\mathcal{H}_{\lambda, \tau}\}_{\tau \in \mathfrak{T}_{\text{orb}}}$, and a unitary*

$$\mathcal{D}_\lambda : \mathcal{H}_\lambda \longrightarrow \int_{\mathfrak{T}_{\text{orb}}}^{\oplus} \mathcal{H}_{\lambda, \tau} d\nu_\lambda(\tau)$$

such that, for ν_λ -almost every τ :

(i)

$$\mathcal{D}_\lambda F_\lambda(C) \mathcal{D}_\lambda^* = \int_{\mathfrak{T}_{\text{orb}}}^{\oplus} \mathbf{1}_C(\tau) \mathbf{1}_{\mathcal{H}_{\lambda, \tau}} d\nu_\lambda(\tau);$$

(ii) *the full Poincaré representation is decomposable,*

$$\mathcal{D}_\lambda U_\lambda(g) \mathcal{D}_\lambda^* = \int_{\mathfrak{T}_{\text{orb}}}^{\oplus} U_{\lambda, \tau}(g) d\nu_\lambda(\tau);$$

(iii) the translation PVM is decomposable,

$$\mathcal{D}_\lambda E_\lambda(B) \mathcal{D}_\lambda^* = \int_{\mathfrak{T}_{\text{orb}}}^\oplus E_{\lambda,\tau}(B) d\nu_\lambda(\tau),$$

with

$$E_{\lambda,\tau}(\mathcal{O}_\tau) = \mathbf{1}_{\mathcal{H}_{\lambda,\tau}};$$

(iv) $(U_{\lambda,\tau}|_{\mathcal{L}}, E_{\lambda,\tau})$ is a transitive system of imprimitivity for the Lorentz action on $\mathcal{O}_\tau$;

(v)

$$\mathcal{D}_\lambda M_\lambda^2 \mathcal{D}_\lambda^* = \int_{\mathfrak{T}_{\text{orb}}}^\oplus q(\tau) \mathbf{1}_{\mathcal{H}_{\lambda,\tau}} d\nu_\lambda(\tau)$$

as a self-adjoint direct-integral operator.

The measure class of ν_λ and the field of transitive systems are unique up to the usual null-set and measurable-unitary equivalences.

Proof. The commutative von Neumann algebra generated by F_λ is represented on a separable Hilbert space and therefore admits a standard direct-integral diagonalization over $\mathfrak{T}_{\text{orb}}$. By [Theorem VI.6](#), every $U_\lambda(g)$ commutes with this algebra, hence is decomposable. Since every $E_\lambda(B)$ commutes with every $F_\lambda(C)$, the translation PVM is also decomposable.

For any Borel $C \subset \mathfrak{T}_{\text{orb}}$,

$$E_\lambda(\vartheta^{-1}(C)) = F_\lambda(C).$$

After disintegration this becomes

$$E_{\lambda,\tau}(\vartheta^{-1}(C)) = \mathbf{1}_C(\tau) \mathbf{1}_{\mathcal{H}_{\lambda,\tau}}$$

for almost every τ on a countable Borel determining class. Removing one common null set and applying the monotone-class theorem gives

$$(\vartheta_* E_{\lambda,\tau})(C) = \delta_\tau(C) \mathbf{1}_{\mathcal{H}_{\lambda,\tau}} \quad (C \in \mathfrak{B}(\mathfrak{T}_{\text{orb}})).$$

Hence $E_{\lambda,\tau}$ is concentrated on $\vartheta^{-1}(\{\tau\}) = \mathcal{O}_\tau$. Lorentz covariance disintegrates fiberwise, so each fiber is a transitive system of imprimitivity. Finally M^2 is multiplication by p^2 , which is constant and equal to $q(\tau)$ on $\mathcal{O}_\tau$. $\square$

Remark VI.10 (Choice of orbit measure). On each homogeneous orbit choose any standard quasi-invariant measure μ_τ . All such measures lie in the canonical quasi-invariant measure class relevant to the induced representation. No invariant-measure normalization enters any spectral threshold and no numerical gap constant is changed by this choice.

E. Wigner–Mackey induction on each orbit

The induced-representation and imprimitivity framework is the Wigner–Mackey classification mechanism^{14,15}. For $p \in \mathbb{R}^4$, define the translation character

$$\chi_p(a) := e^{i\langle p, a \rangle}.$$

The stabilizer subgroup in the Poincaré group is

$$\mathcal{P}_\tau := \mathbb{R}^{3,1} \rtimes \mathcal{L}_\tau.$$

Theorem VI.11 (Sector-free Wigner–Mackey fiber theorem). *For ν_λ -almost every $\tau \in \mathfrak{T}_{\text{orb}}$, there exists a Hilbert space $\mathcal{K}_{\lambda,\tau}$ and a strongly continuous, possibly reducible, unitary representation*

$$\sigma_{\lambda,\tau} : \mathcal{L}_\tau \rightarrow \mathcal{U}(\mathcal{K}_{\lambda,\tau})$$

such that the transitive fiber from [Theorem VI.9](#) is unitarily equivalent to

$$U_{\lambda,\tau} \simeq \text{Ind}_{\mathcal{P}_\tau}^{\mathcal{P}_+^\dagger} (\chi_{p_\tau} \otimes \sigma_{\lambda,\tau}).$$

In a Borel-section model this representation acts on

$$L^2(\mathcal{O}_\tau, \mu_\tau; \mathcal{K}_{\lambda,\tau})$$

with

$$(E_{\lambda,\tau}(B)\psi)(p) = \mathbf{1}_B(p)\psi(p),$$

and

$$(U_{\lambda,\tau}(a, \ell)\psi)(p) = e^{i\langle p, a \rangle} \Delta_\tau(\ell, p)^{1/2} \sigma_{\lambda,\tau}(c_\tau(\ell, p))\psi(\Lambda_\ell^{-1}p).$$

Here c_τ is the standard little-group cocycle determined by the chosen Borel section and Δ_τ is the Radon–Nikodym factor of the chosen quasi-invariant orbit measure.

Proof. By [Theorem VI.9](#), $(U_{\lambda,\tau}|_{\mathcal{L}}, E_{\lambda,\tau})$ is a transitive system of imprimitivity for the action of $\mathcal{L}$ on $\mathcal{L}/\mathcal{L}_\tau \simeq \mathcal{O}_\tau$. Mackey’s imprimitivity theorem gives a unitary representation $\sigma_{\lambda,\tau}$ of the stabilizer and identifies the system with the induced homogeneous-space system. Restoring the normal translation subgroup multiplies each momentum fiber by its character χ_p , which is exactly the induced representation from $\chi_{p_\tau} \otimes \sigma_{\lambda,\tau}$. The displayed formula is the standard Borel-section realization and verifies the translation PVM directly. $\square$

Corollary VI.12 (Irreducible refinement). *Whenever the little-group representation $\sigma_{\lambda,\tau}$ is disintegrated into irreducibles, the corresponding Poincaré fiber is the direct integral of the associated irreducible Wigner–Mackey representations. No irreducibility, finite spin, helicity, or exclusion of continuous-spin representations is required for the mass-support statements of this phase.*

Scope firewall VI.13 (No premature particle interpretation). The induced fibers classify spectral representation types. They need not correspond to asymptotic one-particle states unless additional scattering, isolation, and particle-weight hypotheses are proved. Paper V uses the orbit decomposition only to sharpen the invariant-mass and no-leakage statements.

F. Common cores under orbit disintegration

Proposition VI.14 (The Phase-V.II core in the induced model). *Let $\mathcal{C}_\lambda^{\text{spec}}$ be the common invariant core constructed in Phase V.II and let*

$$\widehat{\mathcal{C}^{\text{spec}}}_\lambda := \mathcal{D}_\lambda \mathcal{C}_\lambda^{\text{spec}}.$$

Then $\widehat{\mathcal{C}^{\text{spec}}}_\lambda$ is a global common core for the direct-integral momentum generators and for the direct-integral operator M^2 . Moreover one may choose a countable family $\{\Phi_n\} \subset \widehat{\mathcal{C}^{\text{spec}}}_\lambda$ such that the $L^\infty(\mathfrak{T}_{\text{orb}}, \nu_\lambda)$ -module generated by the Φ_n is graph dense for every P_μ . For almost every τ , the graph closure of the fiber values $\{\Phi_n(\tau)\}$ is the full graph of multiplication by p_μ on the induced orbit model. On each fixed orbit fiber $M^2 = q(\tau)\mathbf{1}$, so every dense fiber subspace is a core for M^2 .

Proof. Unitary transport preserves the global operator-core identities proved in Phase V.II. The graph Hilbert spaces of the decomposable operators P_μ are separable direct integrals. Choose a countable family from $\widehat{\mathcal{C}^{\text{spec}}}_\lambda$ that is dense in the finite sum of the four graph norms. Its bounded $L^\infty(\mathfrak{T}_{\text{orb}})$ -module span remains in the direct-integral graph domains because orbit-label multipliers commute with every momentum multiplication operator, and it is still graph dense because it contains the original countable family. The standard fiber-density criterion for decomposable Hilbert spaces then gives graph density of the fiber values almost everywhere. The last assertion follows from the scalar identity for M^2 on a fixed orbit. $\square$

Scope firewall VI.15 (No new Paper-IV form invariance). The $L^\infty(\mathfrak{T}_{\text{orb}})$ -module enlargement used to obtain fiberwise momentum cores is not asserted to lie in the Paper-IV prepared form

core. Phase V.II already supplies the required global common core. Fiberwise decomposability of K^{thin} would require a separate Poincaré/form-commutation theorem and is not used here.

Remark VI.16 (Paper-IV form remains non-diagonal). Nothing in the orbit theorem proves that K^{thin} is decomposable with respect to F , much less that it equals or is dominated by M^2 . Poincaré covariance of the closed thin form and the precise form-to-mass comparison remain separate Phase-V.IV obligations.

G. Exact seams and induced-fiber intertwiners

Theorem VI.17 (Decomposability of exact Poincaré intertwiners). *Let $W : \mathcal{H}_\lambda \rightarrow \mathcal{H}_\mu$ be a bounded exact Poincaré intertwiner that is either an isometry or a unitary. Then*

$$WF_\lambda(C) = F_\mu(C)W \quad (C \in \mathfrak{B}(\mathfrak{T}_{\text{orb}})).$$

For an isometry, the source orbit measure class is absolutely continuous with respect to the target class on $\text{Ran } W$. For a unitary, the two orbit measure classes agree. After choosing one dominating measure ν and compatible disintegrations,

$$\mathcal{D}_\mu W \mathcal{D}_\lambda^* = \int_{\mathfrak{T}_{\text{orb}}}^\oplus W_\tau d\nu(\tau),$$

where, for almost every τ , W_τ is respectively an isometric or unitary intertwiner between the transitive systems and hence between the induced representations

$$\text{Ind}_{\mathcal{P}_\tau}^{\mathcal{P}_+^\uparrow}(\chi_{p_\tau} \otimes \sigma_{\lambda,\tau}) \quad \text{and} \quad \text{Ind}_{\mathcal{P}_\tau}^{\mathcal{P}_+^\uparrow}(\chi_{p_\tau} \otimes \sigma_{\mu,\tau}).$$

Consequently an exact intertwiner never moves a vector from one orbit label to another. A unitary branch seam preserves orbit support exactly; an isometric inclusion may have additional target fibers orthogonal to its range.

Proof. The first identity follows from the all-Borel spectral intertwining theorem of Phase V.II and the definition of F . The scalar spectral measures of vectors in $\text{Ran } W$ are the pushforwards of the corresponding source measures. This gives absolute continuity for an isometry and equality of measure classes for a unitary. The decomposability theorem for intertwiners of multiplication representations yields the measurable field W_τ after passage to a dominating measure. Fiberwise Poincaré intertwining follows by disintegrating $WU_\lambda(g) = U_\mu(g)W$ on a countable dense subset of $\mathcal{P}_+^\uparrow$ and extending by strong continuity. Mackey equivalence then identifies each W_τ as an induced-system intertwiner. $\square$

Corollary VI.18 (Arrowwise orbit preservation). *Every exact regional or Cauchy isometry preserves the orbit label of the embedded subrepresentation. Every unitary chart, corridor, strictification, or coefficient-line branch seam preserves the full orbit support and every invariant-mass label exactly. This is an arrowwise theorem only; external direct sums or direct integrals, products, central quotients, and correspondence fusion still require the positive-essential-infimum analysis of SP9.*

H. Strict and sectorwise cases

1. Global ordinary observable system

Theorem VI.19 (Global observable orbit-induction theorem). *For every branch for which Phase V.I.A constructed the global ordinary observable vacuum representation, there is one global decomposition*

$$\mathcal{H}_{G,\xi}^{\text{obs}} \simeq \int_{\mathfrak{T}_{\text{orb}}}^{\oplus} \text{Ind}_{\mathcal{P}_{\tau}^+}^{\mathcal{P}_{\tau}^+}(\chi_{p_{\tau}} \otimes \sigma_{G,\xi,\tau}^{\text{obs}}) d\nu_{G,\xi}^{\text{obs}}(\tau).$$

The joint momentum PVM is multiplication by momentum on the orbit fibers and M^2 is multiplication by $q(\tau)$. No represented strictification or twist is needed to define this global observable decomposition.

Proof. Apply [Theorems VI.9](#) and [VI.11](#) to the global observable system. □

2. Strict Sector I

Theorem VI.20 (Sector-I global represented induction theorem). *Every fully certified Sector-I branch admits one global represented decomposition*

$$\mathcal{H}_{G,\xi}^{\text{I}} \simeq \int_{\mathfrak{T}_{\text{orb}}}^{\oplus} \text{Ind}_{\mathcal{P}_{\tau}^+}^{\mathcal{P}_{\tau}^+}(\chi_{p_{\tau}} \otimes \sigma_{G,\xi,\tau}^{\text{I}}) d\nu_{G,\xi}^{\text{I}}(\tau).$$

All strictification seams are decomposable induced-fiber intertwiners. The global strict represented orbit support is therefore well defined independently of chart and corridor presentations.

Proof. Phase V.I.B supplies the global represented Poincaré representation and Phase V.II supplies its joint PVM. Apply [Theorems VI.9](#), [VI.11](#) and [VI.17](#). □

3. Sector II

Theorem VI.21 (Sector-II corridorwise induced systems). *Let (G, ξ, II) be a physically verified Sector-II branch. Every corridor $\mathfrak{c}$ admits a decomposition*

$$\mathcal{H}_{\mathfrak{c}}^{\text{II}} \simeq \int_{\mathfrak{T}_{\text{orb}}}^{\oplus} \text{Ind}_{\mathcal{P}_{\tau}}^{\mathcal{P}_{\tau}^{\dagger}} (\chi_{p_{\tau}} \otimes \sigma_{\mathfrak{c}, \tau}^{\text{II}}) d\nu_{\mathfrak{c}}^{\text{II}}(\tau).$$

Every certified relative corridor seam identifies the orbit measure classes and decomposes into induced-fiber intertwiners. The nonzero relative chart-scale class $\widehat{\kappa}_{\xi}^{\text{cs}}$ remains in the composition law of those fiber intertwiners and is not removed by equality of orbit support.

Proof. Apply the orbit theorem inside each corridor and [Theorem VI.17](#) to every relative transition. Fiberwise uniqueness of the spectral system does not provide a simultaneous trivialization of the corridor tensorators, so the obstruction class persists. $\square$

Scope firewall VI.22 (Sector-II conditionality). The theorem is unconditional only after the physical represented corridor systems and their relative Poincaré seams have been verified. The abstract Sector-II transfer architecture alone does not supply a universal physical witness.

4. Scalar-twisted Sector III

For a coefficient-line seam write

$$V_{\alpha\beta}^{\mathfrak{c}} : \mathcal{H}_{\beta}^{\mathfrak{c}} \widehat{\otimes} L_{\alpha\beta}^{\mathfrak{c}} \longrightarrow \mathcal{H}_{\alpha}^{\mathfrak{c}}.$$

The certified scalar coefficient line carries trivial translation action; the identity factor is displayed below.

Theorem VI.23 (Sector-III coefficient-line induced system). *For every fully certified scalar Sector-III branch, each chart-corridor object $(\alpha, \mathfrak{c})$ admits*

$$\mathcal{H}_{\alpha}^{\mathfrak{c}} \simeq \int_{\mathfrak{T}_{\text{orb}}}^{\oplus} \text{Ind}_{\mathcal{P}_{\tau}}^{\mathcal{P}_{\tau}^{\dagger}} (\chi_{p_{\tau}} \otimes \sigma_{\alpha, \tau}^{\mathfrak{c}}) d\nu_{\alpha}^{\mathfrak{c}}(\tau).$$

For each overlap, the measure classes agree and

$$V_{\alpha\beta}^{\mathfrak{c}} = \int_{\mathfrak{T}_{\text{orb}}}^{\oplus} V_{\alpha\beta, \tau}^{\mathfrak{c}} d\nu^{\mathfrak{c}}(\tau),$$

with

$$V_{\alpha\beta, \tau}^{\mathfrak{c}} : \mathcal{H}_{\beta, \tau}^{\mathfrak{c}} \widehat{\otimes} L_{\alpha\beta}^{\mathfrak{c}} \longrightarrow \mathcal{H}_{\alpha, \tau}^{\mathfrak{c}}$$

intertwining the induced systems on the same orbit $\mathcal{O}_\tau$. On triple overlaps,

$$V_{\alpha\beta,\tau}^c(\mathbf{1} \otimes V_{\beta\gamma,\tau}^c) = \Omega_{\alpha\beta\gamma}^c V_{\alpha\gamma,\tau}^c \mu_{\alpha\beta\gamma}^c$$

for almost every τ . Thus the scalar associator class is retained on every occupied orbit fiber, while orbit support and invariant mass are chart independent. No global Sector-III physical Hilbert space is constructed.

Proof. Apply [Theorems VI.9](#) and [VI.11](#) objectwise. The coefficient-line version of [Theorem VI.17](#) gives decomposability after tensoring by the identity on the line. The source twisted associativity equation is an equality of bounded decomposable operators, hence holds fiberwise almost everywhere. Multiplication by the scalar $\Omega_{\alpha\beta\gamma}^c$ commutes with the momentum multiplication algebra and therefore changes neither orbit label nor support. $\square$

Remark VI.24 (Strict versus twisted induced data). Strict and scalar-twisted branches may have the same orbit support and even equivalent little-group multiplicity fields locally. They are not thereby globally unitarily equivalent: Sector III retains a nontrivial coefficient-line associator in the gluing of those local induced systems.

I. Exact reformulation of the next spectral obligations

Define the allowed forward orbit subset

$$\mathfrak{T}_{\text{orb}}^+ := \{\mathbf{0}, \mathbf{n}^+\} \sqcup \left(\{\mathbf{t}^+\} \times (0, \infty) \right).$$

Define the forbidden subset

$$\mathfrak{T}_{\text{orb}}^{\text{forb}} := \{\mathbf{n}^-\} \sqcup \left(\{\mathbf{t}^-\} \times (0, \infty) \right) \sqcup \left(\{\mathbf{s}\} \times (0, \infty) \right).$$

Theorem VI.25 (Orbit form of the forward-cone condition). *For every branch object λ ,*

$$\text{supp } E_\lambda \subseteq \overline{V}_+$$

if and only if

$$F_\lambda(\mathfrak{T}_{\text{orb}}^{\text{forb}}) = 0,$$

or equivalently

$$\nu_\lambda(\mathfrak{T}_{\text{orb}}^{\text{forb}}) = 0$$

after discarding zero multiplicity fibers.

Proof. The inverse image $\vartheta^{-1}(\mathfrak{T}_{\text{orb}}^+)$ is exactly $\overline{V}_+$. Apply [Theorem VI.5](#). $\square$

Corollary VI.26 (Orbit form of the future mass-gap theorem). *After SP5, SP6, and SP7 are proved, a mass-square threshold $m_*^2 > 0$ is equivalent to the conjunction*

$$F(\{\mathbf{0}\}) = P_\Omega, \quad F(\{\mathbf{n}^+\}) = 0, \quad F(\{(t^+, m) : 0 < m^2 < m_*^2\}) = 0.$$

This phase proves only the representation-theoretic meaning of those three conditions, not their validity.

Remark VI.27 (Why the orbit theorem is load bearing). The equality $E_{M^2}(\{0\}) = P_\Omega$ is not merely a statement about the point $p = 0$. It simultaneously states that the zero orbit is exactly the complete-ground sector and that every nonzero null induced sector is absent. The form comparison $M^2 \geq cK^{\text{thin}}$ is correctly directed precisely because it eliminates both kinds of unwanted zero-mass fiber whenever $\ker K^{\text{thin}} = \text{Ran } P_\Omega$.

J. Main theorem and phase verdict

Theorem VI.28 (Branchwise Lorentz-Orbit Disintegration and Sector-Correct Induction Theorem). *Assume the definitive Paper-IV handoff and Phases V.I.A–V.II. Then:*

- (1) *Every global ordinary observable Poincaré representation disintegrates over the explicit Lorentz-orbit space $\mathfrak{T}_{\text{orb}}$ into Wigner–Mackey systems induced from the momentum stabilizers.*
- (2) *Every fully certified Sector-I represented system has one global induced-orbit decomposition.*
- (3) *Every physically verified Sector-II represented system has one induced-orbit decomposition per corridor; relative seams are decomposable fiber intertwiners and retain $\widehat{\kappa}_\xi^{\text{CS}}$.*
- (4) *Every fully certified scalar Sector-III branch has chart–corridor induced-orbit decompositions; coefficient-line seams and their nontrivial scalar associator disintegrate fiberwise without changing orbit support.*
- (5) *On every orbit fiber, M^2 is the scalar $q(\tau)$.*

- (6) *Exact branch seams cannot create new orbit labels or move spectral support between different invariant-mass values.*
- (7) *The joint zero-momentum projection is the zero-orbit projection, while the zero-mass projection also includes nonzero null orbit fibers.*
- (8) *The forward-cone theorem is exactly the vanishing of the past-timelike, past-null, and spacelike orbit projections.*
- (9) *No positive-energy statement, form-to-mass comparison, null-fiber exclusion, vacuum identification, or mass gap is inferred.*

Proof. Items (1)–(5) are [Theorems VI.9, VI.19 to VI.21 and VI.23](#). Item (6) is [Theorems VI.17 and VI.18](#). Item (7) is [Theorem VI.7](#). Item (8) is [Theorem VI.25](#). Item (9) is enforced by the scope firewalls and [Theorem VI.26](#). $\square$

Proposition VI.29 (Obligation ledger after Phase V.II.A). *The phase has the following effect.*

- (1) *SP4 is strengthened by a complete Lorentz-orbit disintegration and reduction of the joint spectral system to little-group representations.*
- (2) *The Poincaré-covariant sharp-spectral portion of SP8 is strengthened to induced-fiber covariance.*
- (3) *The arrowwise part of SP9 is strengthened: exact seams preserve orbit measures and induced mass labels fiberwise.*
- (4) *SP5 remains missing but is reduced to the single exact projection identity*

$$F(\mathfrak{T}_{\text{orb}}^{\text{forb}}) = 0.$$

- (5) *SP6 remains missing; no comparison constant $c_{G,\xi}$ is introduced.*
- (6) *SP7 remains missing and now visibly contains two independent exclusions: no extra zero-orbit vectors and no nonzero null fibers.*
- (7) *κ_* is unchanged and unused. No physical mass threshold is defined.*

VII. THIN FORWARD-CONE SPECTRUM THEORY

Reader orientation. Positive energy at finite thickness is not yet a thin spectrum condition. Fourier smearing and compact-uniform coefficient convergence transfer closed spectral support to the thin joint PVM. The conclusion is support in the forward cone; it does not yet identify the zero-mass face or exclude nonzero future-null spectrum.

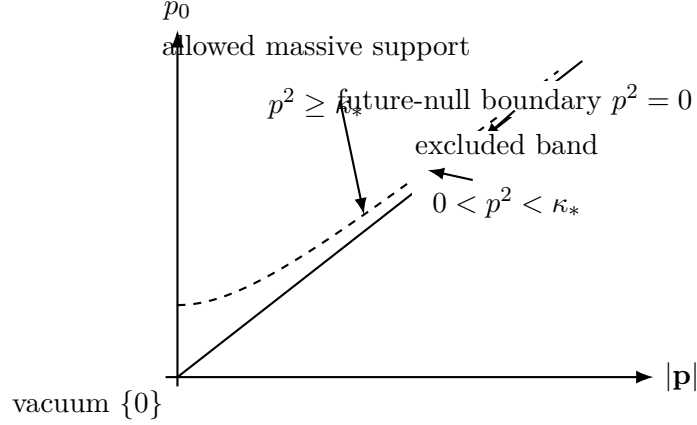


FIG. 2. The three spectral tasks are distinct: forward-cone support, exact identification of the zero-mass face, and quantitative separation of the positive spectrum.

A. Positive-thickness spectral-transfer packet

The branch-object label is

$$\lambda = \begin{cases} \text{obs}, & \text{global ordinary observable system,} \\ *, & \mathbf{S} = \text{I}, \\ \mathbf{c}, & \mathbf{S} = \text{II}, \\ (\alpha, \mathbf{c}), & \mathbf{S} = \text{III}. \end{cases}$$

For positive thickness $r > 0$, write

$$(\mathcal{H}_{r,\lambda}, U_{r,\lambda}, E_{r,\lambda}, \Omega_{r,\lambda})$$

for the corresponding thick Poincaré spectral system. At $r = 0$, use the Phase-V.I–V.II system.

Hypothesis VII.1 (Definitive positive-thickness spectral-transfer packet). On every fixed finite regional-chart-corridor-Cauchy diagram and one selected cofinal tail $r \downarrow 0$, the following hold.

- (S1) **Thick forward-cone spectrum.** Every positive-thickness constituent vacuum representation satisfies

$$\text{supp } E_{r,\lambda} \subseteq \overline{V}_+.$$

For a selected countable barycenter of invariant complete-ground states, each constituent satisfies the same support condition.

- (S2) **Recovered dense cyclic packets.** There is a norm-dense smooth recovery $*$ -algebra $\mathcal{A}_{\text{sm},0,\lambda}$ such that

$$\mathcal{D}_{0,\lambda}^{\text{cyc}} := \text{span}\{\pi_{0,\lambda}(A)\Omega_{0,\lambda} : A \in \mathcal{A}_{\text{sm},0,\lambda}\}$$

is dense in $\mathcal{H}_{0,\lambda}$. Every $A \in \mathcal{A}_{\text{sm},0,\lambda}$ has uniformly norm-bounded coherent lifts $A_{r,\lambda}$, and the corresponding vectors

$$\Psi_{r,A} := \pi_{r,\lambda}(A_{r,\lambda})\Omega_{r,\lambda}$$

are uniformly bounded in norm.

- (S3) **Compact-uniform translation coefficients.** For all $A, B \in \mathcal{A}_{\text{sm},0,\lambda}$ and every compact $K \subset \mathbb{R}^{3,1}$,

$$\sup_{a \in K} |\langle \Psi_{r,A}, U_{r,\lambda}(a)\Psi_{r,B} \rangle - \langle \Psi_{0,A}, U_{0,\lambda}(a)\Psi_{0,B} \rangle| \longrightarrow 0.$$

The convergence is uniform over the finitely many objects and arrows in the fixed diagram.

- (S4) **Exact thick and thin seams.** Every allowed unitary chart, corridor, coefficient-line, product, or quotient seam intertwines the corresponding Poincaré representations. Regional and Cauchy arrows are isometries intertwining translations on their ranges.
- (S5) **Branch typing.** Sector I strictifies globally. Sector II is strict inside each corridor and retains $\widehat{\kappa}_{\xi}^{\text{cs}}$ across corridor germs. Scalar Sector III is chart-corridor banded and retains its coefficient-line associator. No global Sector-III represented Hilbert-space descent is assumed.

Remark VII.2 (Source allocation). Clause (S1) is the retained upstream positive-thickness spectral certificate carried by the Paper-IV handoff shell; Paper III is its provenance source. Clauses (S2)–(S3) are the smooth-packet and compact-uniform coefficient theorems of Phases V.I.A–V.I.B. Clause (S4) is the exact Poincaré/seam theorem of Phases V.I.B–V.II. The present phase supplies the closed-support passage $r > 0 \rightarrow 0$.

B. Barycentric positive energy at positive thickness

The Phase-V.I.A vacuum state may be a countable positive barycenter rather than an extreme state. Its thick positive-energy property must therefore be checked.

Lemma VII.3 (Positive energy survives a countable invariant barycenter). *Let*

$$\omega_r = \sum_{j \geq 1} c_j \omega_r^j, \quad c_j > 0, \quad \sum_j c_j = 1,$$

where every ω_r^j is Poincaré invariant and its covariant GNS translation representation U_r^j satisfies

$$\text{supp } E_r^j \subseteq \bar{V}_+.$$

Then the covariant GNS translation representation of ω_r also has spectrum in $\bar{V}_+$.

Proof. Form the Hilbert direct sum

$$\mathcal{K}_r := \bigoplus_{j \geq 1} \mathcal{H}_r^j, \quad \tilde{U}_r := \bigoplus_{j \geq 1} U_r^j, \quad \tilde{\Omega}_r := \bigoplus_{j \geq 1} \sqrt{c_j} \Omega_r^j.$$

The vector state of $\tilde{\Omega}_r$ on the diagonal representation is ω_r . Its cyclic subspace $\mathcal{H}_r^{\text{cyc}}$ is reducing for $\tilde{U}_r$ because $\tilde{\Omega}_r$ is invariant and the algebra action is covariant. The GNS representation of ω_r is unitarily equivalent to this cyclic subrepresentation. The direct-sum PVM is

$$\tilde{E}_r(B) = \bigoplus_j E_r^j(B),$$

so $\tilde{E}_r(\mathbb{R}^4 \setminus \bar{V}_+) = 0$. Compression to a reducing subspace preserves this vanishing. $\square$

Remark VII.4 (No uniform gap is needed). A direct sum or direct integral preserves forward-cone support without any uniform positive mass lower bound. Uniform essential lower bounds become necessary only in the later gap-transfer phase.

C. Fourier-smearing calculus

Use the convention

$$\widehat{f}(p) := \int_{\mathbb{R}^4} e^{i\langle p, a \rangle} f(a) da, \quad f(a) = (2\pi)^{-4} \int_{\mathbb{R}^4} e^{-i\langle p, a \rangle} \widehat{f}(p) dp.$$

For $f \in L^1(\mathbb{R}^4)$, define the bounded Bochner integral

$$U_{r,\lambda}(f) := \int_{\mathbb{R}^4} f(a) U_{r,\lambda}(a) da.$$

Lemma VII.5 (Spectral functional calculus for translation smearing). *For $f \in L^1(\mathbb{R}^4)$,*

$$U_{r,\lambda}(f) = \int_{\mathbb{R}^4} \widehat{f}(p) dE_{r,\lambda}(p).$$

In particular, if $\widehat{f}$ vanishes on $\text{supp } E_{r,\lambda}$, then $U_{r,\lambda}(f) = 0$.

Proof. For vectors Ψ, Φ , Fubini's theorem applies because

$$|f(a) \langle \Psi, U_{r,\lambda}(a) \Phi \rangle| \leq |f(a)| \|\Psi\| \|\Phi\|.$$

Insert the joint spectral representation of $U_{r,\lambda}(a)$ and exchange the finite complex spectral measure with the L^1 integral. The scalar identity for all pairs of vectors gives the operator formula. $\square$

Lemma VII.6 (Compact-plus-tail coefficient passage). *Let $c_r, c_0 : \mathbb{R}^4 \rightarrow \mathbb{C}$ satisfy*

$$\sup_r \sup_a |c_r(a)| \leq C, \quad \sup_{a \in K} |c_r(a) - c_0(a)| \rightarrow 0$$

for every compact K . Then, for every $f \in L^1(\mathbb{R}^4)$,

$$\int f(a) c_r(a) da \longrightarrow \int f(a) c_0(a) da.$$

Proof. The pointwise limit obeys $|c_0(a)| \leq C$. Given $\varepsilon > 0$, choose compact K with

$$2C \int_{K^c} |f(a)| da < \varepsilon/2.$$

On K ,

$$\left| \int_K f(a) (c_r(a) - c_0(a)) da \right| \leq \|f\|_{L^1(K)} \sup_K |c_r - c_0|,$$

which is less than $\varepsilon/2$ on a cofinal tail. The complement contributes less than $\varepsilon/2$. $\square$

Proposition VII.7 (Smeared matrix-coefficient transfer). *For $A, B \in \mathcal{A}_{\text{sm},0,\lambda}$ and every $f \in L^1(\mathbb{R}^4)$,*

$$\langle \Psi_{r,A}, U_{r,\lambda}(f) \Psi_{r,B} \rangle \longrightarrow \langle \Psi_{0,A}, U_{0,\lambda}(f) \Psi_{0,B} \rangle.$$

Proof. Apply [Theorem VII.6](#) to the compact-uniform coefficients in (S3). Their uniform bound follows from

$$|\langle \Psi_{r,A}, U_{r,\lambda}(a) \Psi_{r,B} \rangle| \leq \|\Psi_{r,A}\| \|\Psi_{r,B}\|,$$

and the uniform packet norm bound in (S2). $\square$

D. Closed-support transfer theorem

Let

$$C := \overline{V}_+ = \{p \in \mathbb{R}^4 : p_0 \geq 0, p^2 \geq 0\}.$$

The proof is stated for an arbitrary closed set because this identifies the exact functional-analytic mechanism.

Theorem VII.8 (Closed spectral support cannot escape under recovered coefficient convergence). *Let $C \subset \mathbb{R}^4$ be closed. Assume, for every $r > 0$,*

$$\text{supp } E_{r,\lambda} \subseteq C,$$

and assume (S2)–(S3). Then

$$\boxed{\text{supp } E_{0,\lambda} \subseteq C.}$$

The conclusion is objectwise and uniform over every fixed finite branch diagram.

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^4 \setminus C)$ and let $f \in \mathcal{S}(\mathbb{R}^4)$ be its inverse Fourier transform, so $\hat{f} = \varphi$. By [Theorem VII.5](#) and the thick support condition,

$$U_{r,\lambda}(f) = \int \varphi(p) dE_{r,\lambda}(p) = 0.$$

Therefore, for all recovered cyclic packet vectors,

$$\langle \Psi_{r,A}, U_{r,\lambda}(f) \Psi_{r,B} \rangle = 0.$$

By [Theorem VII.7](#),

$$\langle \Psi_{0,A}, U_{0,\lambda}(f) \Psi_{0,B} \rangle = 0.$$

The recovered cyclic packet space is dense and $U_{0,\lambda}(f)$ is bounded, so

$$U_{0,\lambda}(f) = 0.$$

Using [Theorem VII.5](#) again,

$$\int \varphi(p) dE_{0,\lambda}(p) = 0 \quad \text{for every } \varphi \in C_c^\infty(\mathbb{R}^4 \setminus C).$$

Now let $K \subset \mathbb{R}^4 \setminus C$ be compact. Choose $\varphi \in C_c^\infty(\mathbb{R}^4 \setminus C)$ with $0 \leq \varphi \leq 1$ and $\varphi = 1$ on K . Positivity of the spectral integral gives

$$0 = \int \varphi dE_{0,\lambda} \geq E_{0,\lambda}(K),$$

so $E_{0,\lambda}(K) = 0$. Since the open set $\mathbb{R}^4 \setminus C$ is a countable increasing union of compact sets and PVMs are strongly countably additive,

$$E_{0,\lambda}(\mathbb{R}^4 \setminus C) = 0.$$

This is the desired support inclusion. Every estimate used above is uniform over a fixed finite diagram because (S2)–(S3) are. $\square$

Referee audit VII.9 (Why no norm-resolvent theorem is involved). The proof uses only scalar matrix-coefficient convergence for bounded smeared translation operators $U(f)$. It never claims norm convergence of $U_r(f)$, resolvent convergence of $P_{\mu,r}$, or convergence of arbitrary sharp spectral projections. The sharp support conclusion follows from annihilation of a separating family of smooth functions and PVM regularity.

Referee audit VII.10 (Why compact-uniform convergence is sufficient). The inverse Fourier transform of a compactly supported momentum test is Schwartz rather than compactly supported in spacetime. Compact-uniform coefficient convergence alone would not permit direct integration without tail control. The uniform unitary coefficient bound and the L^1 tail estimate in [Theorem VII.6](#) supply exactly the missing global control.

E. Thin forward-cone spectrum theorem

Theorem VII.11 (Global ordinary Thin Forward-Cone Spectrum Theorem). *For every fully certified Sector-I or scalar Sector-III branch, and for every physically verified Sector-II branch, the global ordinary observable vacuum representation satisfies*

$$\boxed{\text{supp } E_{G,\xi,0}^{\text{obs}} = \text{sp } U_{G,\xi}^{\text{obs}}|_{\mathbb{R}^{3,1}} \subseteq \overline{V}_+}.$$

Equivalently,

$$F_{G,\xi,0}^{\text{obs}}(\mathfrak{T}_{\text{orb}}^{\text{forb}}) = 0.$$

Proof. Apply [Theorem VII.3](#) at each positive thickness to the selected invariant complete-ground barycenter, and then apply [Theorem VII.8](#) with $C = \overline{V}_+$. The orbit equivalence is the Phase-V.II.A identity

$$\vartheta^{-1}(\mathfrak{T}_{\text{orb}}^+) = \overline{V}_+.$$

□

1. Sector I: global strict represented support

Theorem VII.12 (Sector-I global represented forward cone). *For every declared fully certified Sector-I exact-dilation branch,*

$$\boxed{\text{supp} E_{G,\xi,0}^{\text{I}} \subseteq \overline{V}_+}.$$

The global strict represented orbit measure has no past-timelike, past-null, or spacelike fibers.

Proof. The Sector-I strictification provides one global represented thick spectral packet and the compact-uniform represented coefficient transfer of (S2)–(S3). Apply [Theorem VII.8](#). Total strictification affects only the assembly of the branch objects and does not modify the support argument. □

2. Sector II: corridorwise strict support with relative twisting

Theorem VII.13 (Sector-II corridorwise forward cone). *Let a Sector-II branch possess a physically verified positive-thickness represented realization satisfying [Theorem VII.1](#). Then, for every corridor $\mathfrak{c}$,*

$$\boxed{\text{supp} E_{G,\xi,0}^{\mathfrak{c}} \subseteq \overline{V}_+}.$$

Every unitary corridor seam preserves this support and the induced orbit measure class. The nonzero relative chart-scale class $\widehat{\kappa}_{\xi}^{\text{cs}}$ is retained and is not converted into a global strictification.

Proof. Apply [Theorem VII.8](#) inside each corridor, where the represented system is strict. If $V_{\mathfrak{c}\mathfrak{d}}$ is a unitary corridor seam, Phase V.II gives

$$V_{\mathfrak{c}\mathfrak{d}} E_{\mathfrak{d}}(B) = E_{\mathfrak{c}}(B) V_{\mathfrak{c}\mathfrak{d}}$$

for every Borel B . Hence forbidden support vanishes on one side exactly when it vanishes on the other. The composition law of these seams still carries the relative obstruction. $\square$

Scope firewall VII.14 (Sector-II conditionality). The theorem is unconditional only for a Sector-II branch whose positive-thickness physical represented system and invariant spectral packet are independently certified. The abstract corridor architecture alone does not establish physical non-vacuity.

3. *Sector III: scalar coefficient-line twisted support*

Theorem VII.15 (Sector-III chart–corridor forward cone). *For every declared fully certified scalar Sector-III branch and each chart–corridor object $(\alpha, \mathfrak{c})$,*

$$\boxed{\text{supp} E_{\alpha, G, \xi, 0}^{\mathfrak{c}} \subseteq \overline{V}_+}.$$

For every overlap coefficient-line unitary,

$$V_{\alpha\beta}^{\mathfrak{c}} \left(\mathbf{1}_{L_{\alpha\beta}^{\mathfrak{c}}} \otimes E_{\beta}^{\mathfrak{c}}(B) \right) = E_{\alpha}^{\mathfrak{c}}(B) V_{\alpha\beta}^{\mathfrak{c}}, \quad B \in \mathfrak{B}(\mathbb{R}^4).$$

Thus the scalar associator class is retained while no chart or corridor carries forbidden momentum support.

Proof. Apply [Theorem VII.8](#) objectwise to every chart–corridor packet. The displayed all-Borel intertwining identity is the Phase-V.II sharp spectral seam theorem. Multiplication by a scalar associator commutes with the momentum multiplication algebra; it cannot alter a spectral projection or move support between orbit strata. The result remains a banded family and does not descend to one global represented Hilbert space. $\square$

Corollary VII.16 (Strict and twisted support agree at the observable level). *The global ordinary observable representations of certified strict Sector I and scalar-twisted Sector III satisfy the same forward-cone support condition. Equality of this kinematic support constraint does not imply unitary equivalence of their represented systems or triviality of the Sector-III associator.*

F. Regional, Cauchy, product, and quotient stability

Proposition VII.17 (Isometric arrows preserve the allowed support on their ranges). *Let $W : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be an isometry satisfying*

$$WU_1(a) = U_2(a)W.$$

Then

$$WE_1(B) = E_2(B)W$$

for all Borel B . If $E_2(\mathbb{R}^4 \setminus \overline{V}_+) = 0$, then the source also has forward-cone support. Conversely, source support controls the target only on $\text{Ran } W$; the orthogonal complement may carry additional target spectrum.

Proof. The spectral intertwining follows from uniqueness of Fourier transforms of matrix-coefficient measures. The remaining assertions follow by taking $B = \mathbb{R}^4 \setminus \overline{V}_+$ and using $W^*W = \mathbf{1}$. $\square$

Proposition VII.18 (Finite products and central quotients at the support level). *Suppose a declared compact semisimple branch is assembled from finitely many factor spectral systems, each supported in $\overline{V}_+$, by the source-certified tensor/product and finite central-quotient observable construction. Then the assembled translation spectrum is supported in $\overline{V}_+$.*

Proof. For tensor-product translations,

$$U(a) = \bigotimes_i U_i(a),$$

and the joint spectral measure is the pushforward of the product measure under momentum addition. The closed forward cone is convex and closed under finite addition, so the pushforward support lies in $\overline{V}_+$. A finite central quotient selects a reducing observable subspace or finite invariant sum of such subspaces in the certified representation. Reduction cannot enlarge spectral support. This proves only cone support; it does not identify a later mass threshold or prove that the quotient is spectrally identical to the covering system. $\square$

Proposition VII.19 (Direct sums and direct integrals). *Any Hilbert direct sum or direct integral of measurable translation representations supported in $\overline{V}_+$ is again supported in $\overline{V}_+$. No essential-infimum hypothesis is needed for this statement.*

Proof. The forbidden spectral projection is respectively the direct sum or decomposable integral of the fiber forbidden projections, all of which vanish almost everywhere. $\square$

G. Consequences for invariant mass and orbit induction

Theorem VII.20 (Nonnegativity of the invariant-mass square). *On every global or branch-local spectral system covered above,*

$$M^2 = P_0^2 - \sum_{j=1}^3 P_j^2 = \int_{\bar{V}_+} p^2 dE(p)$$

is a nonnegative self-adjoint operator. Therefore

$$M := (M^2)^{1/2}$$

is well defined and self-adjoint, with

$$\text{Dom}(M) = \left\{ \Psi : \int_{\bar{V}_+} p^2 d\langle \Psi, E(p)\Psi \rangle < \infty \right\}.$$

Proof. The Borel function $p \mapsto p^2$ is nonnegative on $\bar{V}_+$. The joint functional calculus therefore makes M^2 nonnegative. The square root and its domain follow from the spectral theorem. $\square$

Corollary VII.21 (Allowed induced fibers after Phase V.III). *The Wigner–Mackey disintegration of Phase V.II.A reduces to*

$$\{\mathbf{0}\} \sqcup \{\mathbf{n}^+\} \sqcup \left(\{\mathbf{t}^+\} \times (0, \infty) \right).$$

All past-timelike, past-null, and spacelike little-group fibers vanish.

Corollary VII.22 (Zero mass remains larger than zero momentum a priori). *After the forward-cone theorem,*

$$E_{M^2}(\{0\}) = F(\{\mathbf{0}, \mathbf{n}^+\}).$$

Thus SP7 must still eliminate the nonzero future-null orbit projection and identify the zero-orbit projection with P_Ω .

H. Tube analyticity as a consequence

The spectrum-condition/tube-analyticity background is standard in algebraic quantum field theory¹⁶; here analyticity is derived only after the support theorem. Let V_+° be the open future cone and

$$\mathcal{T}_+ := \mathbb{R}^4 + iV_+^\circ.$$

Theorem VII.23 (Forward-tube analyticity of translation coefficients). *For $\Psi, \Phi \in \mathcal{H}$, define the finite complex spectral measure*

$$\mu_{\Psi, \Phi}(B) := \langle \Psi, E(B)\Phi \rangle.$$

Then

$$F_{\Psi, \Phi}(z) := \int_{\bar{V}_+} e^{i\langle p, z \rangle} d\mu_{\Psi, \Phi}(p), \quad z \in \mathcal{T}_+,$$

is holomorphic and satisfies

$$|F_{\Psi, \Phi}(x + iy)| \leq \|\Psi\| \|\Phi\|.$$

Its boundary value as $y \rightarrow 0$ inside V_+° is

$$\langle \Psi, U(x)\Phi \rangle$$

pointwise and in the sense of tempered distributions.

Proof. For $p \in \bar{V}_+$ and $y \in V_+^\circ$, self-duality of the cone gives $p \cdot y \geq 0$, so

$$|e^{ip \cdot (x + iy)}| = e^{-p \cdot y} \leq 1.$$

On compact subsets of the tube, $p \cdot y$ has a uniform positive lower bound proportional to $|p|$ away from $p = 0$, giving dominated differentiation under the finite measure. Hence the integral is holomorphic. The norm bound follows from the total variation estimate for the coefficient measure. Dominated convergence gives the boundary value. $\square$

Remark VII.24 (Logical direction). The paper does not use tube analyticity to prove the spectrum condition because the uniform holomorphic thick-to-thin convergence package was not independently available. Instead, the support theorem proves analyticity afterward. This preserves the anti-circularity firewall.

I. Arveson-spectrum audit

The transfer-spectrum terminology in this audit follows Arveson's automorphism-spectrum framework¹⁷.

Referee audit VII.25 (The naive two-sided Arveson route is rejected). For the observable automorphism group α_a , the ordinary Arveson spectrum records energy-momentum transfers

of observables. If the represented translation spectrum lies in $\overline{V}_+$, those transfers lie in differences

$$\overline{V}_+ - \overline{V}_+,$$

which are not contained in $\overline{V}_+$ and may fill all of $\mathbb{R}^4$. Therefore the implication

$$\mathrm{Sp}_{\mathrm{Arv}}(\alpha) \subseteq \overline{V}_+$$

is neither a standard consequence of positive energy nor an appropriate primitive target for a local observable net.

Proposition VII.26 (One-sided vacuum replacement). *The valid one-sided object is the vacuum translation representation from Phase V.I.A. Equivalently, use the coefficient measures*

$$a \longmapsto \omega_{\Omega,0}(A^* \alpha_a(B)).$$

The support-transfer theorem proves that every such recovered coefficient measure is supported in $\overline{V}_+$ and then extends the result by cyclic density to the full GNS representation.

Proof. This is precisely the test-function argument in [Theorem VII.8](#), expressed in the cyclic GNS coordinates

$$\Psi_A = \pi(A)\Omega, \quad \Psi_B = \pi(B)\Omega.$$

□

J. Gauss–Ward, surgery, and common-core compatibility at this phase

Proposition VII.27 (Spectral support respects the prepared Gauss–Ward category). *The forward-cone projection identity is compatible with every exact unitary branch seam and with the common spectral/Gauss–Ward core of Phase V.II. In particular, applying a smooth spectral multiplier $\varphi(P)$ to a common-core vector preserves the declared graph domains whenever φ is a Schwartz multiplier covered by the Phase-V.II convolution theorem.*

Proof. All-Borel seam covariance was proved in Phase V.II. For a Schwartz multiplier $\varphi = \widehat{f}$, one has

$$\varphi(P) = U(f),$$

and the translation-convolution construction of $\mathcal{C}^{\mathrm{spec}}$ proves graph-domain preservation. The forbidden multipliers vanish identically by [Theorem VII.8](#), so they cannot create a Gauss–Ward component. □

Scope firewall VII.28 (Nonlinear surgery is not yet a spectral operator). Poincaré covariance of state surgery and the forward-cone support theorem imply that surgery maps the selected invariant prepared state face within a positive-energy representation category. They do not yet imply a linear operator identity

$$E(B)\mathcal{S}^{\text{st,as}} = \mathcal{S}^{\text{st,as}}E(B).$$

Sharp surgery/spectral restriction remains in SP8 and will be formulated on state faces rather than by pretending that surgery is a bounded Hilbert-space operator.

K. Principal theorem and obligation ledger

Theorem VII.29 (Branchwise Thin Forward-Cone Spectrum Theorem). *Assume the certified Paper-IV-to-Paper-V intake of Theorem II.2, including its retained upstream positive-thickness spectral certificate, and Phases V.I.A–V.II.A. Then:*

- (1) *For every fully certified Sector-I or scalar Sector-III branch, the global ordinary observable vacuum representation satisfies*

$$\text{sp } U_{G,\xi}^{\text{obs}}|_{\mathbb{R}^{3,1}} \subseteq \bar{V}_+.$$

The same holds for every physically verified Sector-II ordinary branch.

- (2) *Every declared Sector-I represented system has one global strict positive-energy representation.*
- (3) *Every physically verified Sector-II represented system has positive-energy translation spectrum on each corridor, with its relative chart–scale obstruction retained.*
- (4) *Every declared scalar Sector-III represented system has positive-energy spectrum on every chart–corridor object; coefficient-line intertwiners preserve every Borel spectral projection and retain the scalar associator.*
- (5) *Equivalently, on every valid object,*

$$F_{G,\xi,0}(\mathfrak{T}_{\text{orb}}^{\text{forb}}) = 0.$$

- (6) *The invariant-mass square M^2 is nonnegative and $M = (M^2)^{1/2}$ is well defined.*

- (7) *Regional and Cauchy isometries preserve the result on their ranges; finite products, certified central-quotient reductions, direct sums, and direct integrals do not create forbidden support.*
- (8) *The proof is independent of $\bar{q}_0$, K^{thin} , and κ_* . It uses no form-to-mass comparison.*
- (9) *No conclusion is drawn about $E_{M^2}(\{0\})$, nonzero future-null fibers, a positive mass threshold, vacuum uniqueness, or a mass gap.*

Proof. Items (1)–(4) are Theorems VII.11 to VII.13 and VII.15. Item (5) follows from Phase V.II.A and Theorems VII.11 to VII.13 and VII.15. Item (6) is Theorem VII.20. Item (7) is Theorems VII.17 to VII.19. Items (8)–(9) are the scope firewalls enforced throughout the proof. $\square$

Proposition VII.30 (SP ledger after Phase V.III). *The phase changes the master obligation ledger as follows.*

- (1) *SP5 is closed for fully certified Sector-I and scalar Sector-III branches and conditionally closed for physically verified Sector-II branches.*
- (2) *SP4 is strengthened: M^2 is now nonnegative and the physical mass operator $M = (M^2)^{1/2}$ exists.*
- (3) *The sharp Poincaré/seam part of SP8 remains closed and is now restricted to the allowed orbit subset.*
- (4) *The arrowwise portion of SP9 is strengthened: exact seams cannot introduce forbidden orbit support.*
- (5) *SP6 remains missing. No $c_{G,\xi}$ has been defined.*
- (6) *SP7 remains missing. The zero-mass projection may still contain the nonzero future-null orbit.*
- (7) *SP10–SP12 remain missing.*
- (8) *κ_* is unchanged and unused; the final mass-gap constant is still undefined.*

VIII. HILBERT-GERBE GLUE–MOSCO–RIESZ STABILITY

Reader orientation. The spectral argument must preserve the complete-ground projection while forms, Hilbert spaces, and seams vary with thickness and branch. The order is finite diagram first, simultaneous Mosco recovery second, resolvent and Riesz control third, and global assembly only afterward. This prevents isolated chartwise limits from manufacturing new zero-form directions.

A. Hilbert-gerbe form datum

Definition VIII.1 (Branchwise form gerbe). For each thickness parameter $r \geq 0$ and each valid branch object λ , let

$$(\mathcal{H}_{r,\lambda}, q_{r,\lambda}, K_{r,\lambda}, P_{r,\lambda})$$

consist of a Hilbert space, a densely defined closed nonnegative quadratic form, its associated nonnegative self-adjoint operator, and its complete-ground projection. For a unitary branch seam $a : \mu \rightarrow \lambda$, write

$$V_{r,a} : L_{r,a} \otimes \mathcal{H}_{r,\mu} \longrightarrow \mathcal{H}_{r,\lambda},$$

where $L_{r,a} = \mathbb{C}$ in the strict case and is a coefficient line in the scalar-twisted case. Regional or Cauchy arrows may instead be isometries onto reducing ranges.

The association of a densely defined closed semibounded form with its self-adjoint operator is the standard Friedrichs–Kato representation theory^{18,19}.

Hypothesis VIII.2 (Definitive Glue–Mosco packet). On every fixed finite regional–chart–corridor diagram and one selected cofinal tail $r \downarrow 0$:

(GM1) There are uniformly bounded comparison maps $J_{r,\lambda}$ and adjoint comparison maps $J_{r,\lambda}^\dagger$ implementing the Paper-IV comparison-strong and comparison-weak topologies.

(GM2) If $\Psi_r \rightarrow_J \Psi_0$ and $\sup_r (q_{r,\lambda}[\Psi_r] + \|\Psi_r\|^2) < \infty$, then

$$q_{0,\lambda}[\Psi_0] \leq \liminf_{r \downarrow 0} q_{r,\lambda}[\Psi_r].$$

(GM3) Every $\Psi_0 \in \text{Dom } q_{0,\lambda}$ has a recovery family $\Psi_r \rightarrow_J \Psi_0$ with

$$q_{r,\lambda}[\Psi_r] \rightarrow q_{0,\lambda}[\Psi_0].$$

The recovery is simultaneous on every fixed finite diagram.

(GM4) Complete-ground transport satisfies full TT1:

$$J_{r,\lambda} P_{r,\lambda} \Psi_r - P_{0,\lambda} J_{r,\lambda} \Psi_r \longrightarrow 0$$

for every graph-bounded comparison family, and likewise for the complements.

(GM5) Uniform separation holds:

$$q_{r,\lambda}[\Psi] \geq \kappa_* \|(\mathbf{1} - P_{r,\lambda})\Psi\|^2$$

with one $\kappa_* > 0$ on the declared branch family.

(GM6) For each exact seam,

$$q_{r,\lambda}[V_{r,a}(\ell \otimes \Psi)] = q_{r,\mu}[\Psi] \|\ell\|^2.$$

For a controlled seam, the relative form defect tends to zero uniformly on the form unit ball over the finite diagram.

(GM7) On the exact-dilation and other norm-resolvent-certified branches, the relative form defect is uniform on the full form unit ball. On the general certified branch, only comparison strong-resolvent convergence is asserted.

Scope firewall VIII.3 (Finite diagrams before global assembly). Glue–Mosco convergence is proved first on every fixed finite diagram. Global statements are obtained by the established exhaustion and common-refinement system. This prevents a diagonal family from changing charts or corridors faster than the compactness argument can track it.

B. Glue–Mosco convergence

The variational convergence notion is Mosco convergence, including its comparison-space extension used in spectral geometry^{20,21}.

Definition VIII.4 (Compatible section space). For a finite diagram D , let $\mathcal{H}_{r,D}^{\text{gl}}$ denote the closed subspace of the Hilbert direct sum of its objects consisting of compatible sections. For exact seams this means

$$\Psi_{r,\lambda} = V_{r,a}(\ell_a \otimes \Psi_{r,\mu})$$

for every arrow $a : \mu \rightarrow \lambda$, with the prescribed normalized coefficient-line element or balanced tensor interpretation. In the asymptotic comparison category, use the graph closure of families whose seam defects tend to zero. Define

$$q_{r,D}^{\text{gl}}[\Psi_r] := \sum_{\lambda \in D} w_\lambda q_{r,\lambda}[\Psi_{r,\lambda}], \quad w_\lambda > 0.$$

Theorem VIII.5 (Finite-diagram Glue–Mosco theorem). *Under [Theorem VIII.2](#), the glued forms satisfy*

$$q_{r,D}^{\text{gl}} \xrightarrow{\text{Mosco}} q_{0,D}^{\text{gl}}$$

in the comparison topology. Their kernels are exactly the glued complete-ground sections, and

$$q_{0,D}^{\text{gl}}[\Psi] \geq \kappa_D \|(\mathbf{1} - P_{0,D})\Psi\|^2, \quad \kappa_D := \kappa_* \min_{\lambda \in D} w_\lambda.$$

If the weights are normalized objectwise rather than summed, the inherited constant is exactly κ_* .

Proof. Let $\Psi_r \rightarrow_J \Psi_0$ be graph bounded in the glued spaces. Objectwise comparison compactness and the finite nature of D give one subnet converging on every object. The seam defects converge to zero, and the seam maps converge on the prepared comparison packet, so the limiting family is compatible. Summing the objectwise liminf inequalities gives

$$q_{0,D}^{\text{gl}}[\Psi_0] \leq \liminf_r q_{r,D}^{\text{gl}}[\Psi_r].$$

For the recovery direction, the joint finite-diagram recovery clause gives one family recovering all object components simultaneously; its seam defects vanish by construction. The weighted energies converge termwise and hence in the finite sum.

TT1 identifies the glued ground projection. The objectwise lower bounds give the displayed coercive estimate. Conversely, a glued ground section has zero energy on every object and is recovered by the zero-energy complete-ground family, so the kernel is exactly the glued complete-ground space. $\square$

Corollary VIII.6 (No zero-form leakage through finite gluing). *A normalized family $\Psi_r \in \mathcal{H}_{r,D}^{\text{gl}}$ satisfying*

$$P_{r,D}\Psi_r = 0, \quad \sup_r q_{r,D}^{\text{gl}}[\Psi_r] < \infty,$$

cannot converge comparison-strongly to a nonzero vector in $\text{Ran } P_{0,D}$.

Proof. TT1 gives $P_{0,D}\Psi_0 = 0$ for every comparison-strong limit. Equivalently, the limiting ground component cannot be created by moving among objects of the diagram. $\square$

C. Riesz stability and sharp ground projections

We use the standard resolvent and Riesz functional calculus for isolated spectral sets^{11,19}. Fix

$$0 < \rho < \frac{\kappa_*}{2}, \quad \Gamma_\rho := \{z \in \mathbb{C} : |z| = \rho\}$$

with positive orientation.

Definition VIII.7 (Ground Riesz projection). For every object and thickness define

$$\Pi_{r,\lambda} := \frac{1}{2\pi i} \oint_{\Gamma_\rho} (z - K_{r,\lambda})^{-1} dz.$$

Lemma VIII.8 (Riesz projection equals the complete-ground projection). *For every $r \geq 0$ and valid λ ,*

$$\Pi_{r,\lambda} = P_{r,\lambda}.$$

Proof. The uniform lower bound and exact kernel imply

$$\sigma(K_{r,\lambda}) \subseteq \{0\} \cup [\kappa_*, \infty).$$

The contour encloses only the isolated spectral set $\{0\}$. The Riesz functional calculus therefore gives the spectral projection onto $\ker K_{r,\lambda} = \text{Ran } P_{r,\lambda}$. $\square$

Theorem VIII.9 (Strong and norm Riesz stability). *Under [Theorem VIII.2](#):*

(1) *On every fully certified branch,*

$$J_{r,\lambda} P_{r,\lambda} J_{r,\lambda}^\dagger \longrightarrow P_{0,\lambda}$$

in the comparison-strong operator topology.

(2) *On every branch with transported norm-resolvent convergence uniform on Γ_ρ ,*

$$\|J_{r,\lambda}P_{r,\lambda}J_{r,\lambda}^\dagger - P_{0,\lambda}\| \longrightarrow 0.$$

(3) *Both convergences are uniform on each fixed finite diagram whenever the underlying comparison estimates are diagram-uniform.*

Proof. Choose $\chi \in C_c(\mathbb{R})$ with $\chi = 1$ on $[-\rho, \rho]$ and $\chi = 0$ on $[\kappa_*/2, \infty)$. By the spectral separation,

$$\chi(K_{r,\lambda}) = P_{r,\lambda}, \quad \chi(K_{0,\lambda}) = P_{0,\lambda}.$$

Comparison strong-resolvent convergence implies comparison-strong convergence of bounded continuous functional calculus, proving (1). Under norm-resolvent convergence, the resolvents converge uniformly on the compact contour, and contour integration gives

$$\|J_r P_r J_r^\dagger - P_0\| \leq \frac{\text{length}(\Gamma_\rho)}{2\pi} \sup_{z \in \Gamma_\rho} \|J_r(z - K_r)^{-1} J_r^\dagger - (z - K_0)^{-1}\|,$$

which tends to zero. Finite-diagram uniformity follows by taking the maximum over finitely many objects and arrows. $\square$

Referee audit VIII.10 (Strong versus norm sharp-projection convergence). Mosco convergence alone yields comparison strong-resolvent convergence and hence strong convergence of the isolated ground projection. It does not yield operator-norm convergence. Norm Riesz convergence is asserted only on branches carrying the stronger uniform relative-form or norm-resolvent certificate.

D. Resolvent and Riesz naturality under gerbe seams

Lemma VIII.11 (Exact resolvent intertwining). *Suppose $V : L \otimes \mathcal{H}_\mu \rightarrow \mathcal{H}_\lambda$ is unitary and*

$$K_\lambda V = V(\mathbf{1}_L \otimes K_\mu)$$

on an operator core. Then for every $z \in \mathbb{C} \setminus [0, \infty)$,

$$(z - K_\lambda)^{-1} V = V(\mathbf{1}_L \otimes (z - K_\mu)^{-1}),$$

and consequently

$$P_\lambda V = V(\mathbf{1}_L \otimes P_\mu).$$

Proof. The core intertwining closes because both operators are self-adjoint and V is unitary. Multiplying the closed intertwining relation by the resolvents gives the first identity. Integrating around Γ_ρ gives the Riesz identity. $\square$

Lemma VIII.12 (Asymptotic resolvent seam estimate). *Let $R_{r,\lambda}(z) = (z - K_{r,\lambda})^{-1}$. If the seam defect extends in form-resolvent norm as*

$$\delta_{r,a}(z) := \|R_{r,\lambda}(z)V_{r,a} - V_{r,a}(\mathbf{1} \otimes R_{r,\mu}(z))\|,$$

with

$$\sup_{z \in \Gamma_\rho} \delta_{r,a}(z) \rightarrow 0,$$

then

$$\|P_{r,\lambda}V_{r,a} - V_{r,a}(\mathbf{1} \otimes P_{r,\mu})\| \leq \rho \sup_{z \in \Gamma_\rho} \delta_{r,a}(z) \rightarrow 0.$$

Proof. Integrate the resolvent defect around the circle and use $\text{length}(\Gamma_\rho) = 2\pi\rho$. $\square$

Theorem VIII.13 (Hilbert-Gerbe Glue–Mosco–Riesz Stability Theorem). *For every fully certified Sector-I or scalar Sector-III branch, and corridorwise for every physically verified Sector-II branch:*

- (1) *the Paper-IV forms satisfy finite-diagram glue–Mosco convergence;*
- (2) *the complete-ground projection is the isolated Riesz projection of the thin form operator;*
- (3) *the transported ground projections converge strongly, and in norm on norm-resolvent-certified branches;*
- (4) *every exact chart, corridor, coefficient-line, product, and central-quotient seam intertwines the thin resolvent and ground Riesz projection;*
- (5) *controlled seams satisfy the corresponding asymptotic identity on the selected cofinal tail;*
- (6) *scalar Sector-III associators remain nontrivial but cancel in every conjugation identity for resolvents and Riesz projections;*
- (7) *no new zero-form direction is created by chart change, corridor change, coefficient-line reassociation, regional exhaustion, finite product, certified central quotient, or a direct integral whose gap has positive essential infimum.*

Proof. Items (1)–(3) are [Theorems VIII.5](#) and [VIII.9](#). Items (4)–(5) are [Theorems VIII.11](#) and [VIII.12](#). In Sector III, a triple composite differs by a scalar $\Omega_{\alpha\beta\gamma}\mathbf{1}$; scalar multiplication commutes with every resolvent and projection, so both parenthesizations induce the same conjugation law while retaining the coefficient-line associator. Exhaustion follows by compatibility on the directed local system. Product, quotient, and direct-integral assertions are proved definitively in [Theorems XI.16](#), [XI.18](#) and [XI.22](#). $\square$

What has now been established. The complete-ground projection is now a stable Riesz object of the entire finite descent diagram, not a chartwise projection glued after the limit. Exact and controlled seams transport its resolvent calculus, while scalar associators survive without changing conjugation identities.

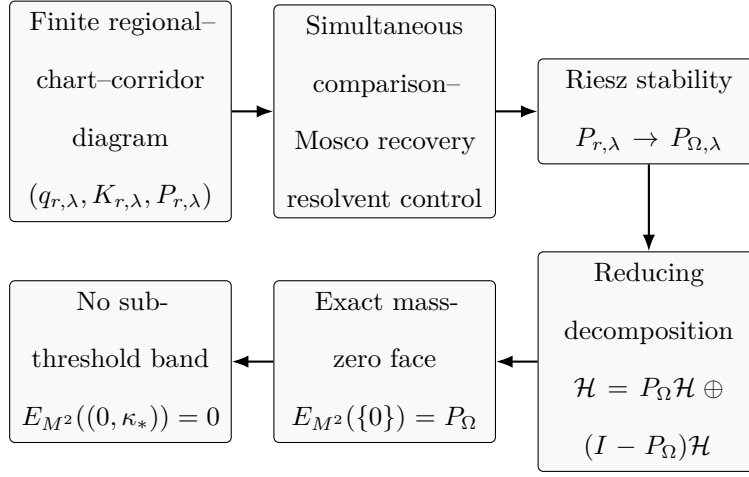


FIG. 3. The zero projection is transported through finite diagrams before the sharp spectral decomposition is asserted. This is the mechanism that prevents chart, corridor, or coefficient-line gluing from creating new zero modes.

E. Dimensional normalization

Referee audit VIII.14 (Dimensions of the comparison). In the stiffness-completed normalization of the internal physical hyperbolic realization,

$$[K] = [P_0^2] = [M^2] = \text{mass}^2, \quad [c_E] = 1, \quad [\kappa_*] = \text{mass}^2.$$

Therefore

$$[m_{G,\xi}^2] = [c_E \kappa_*] = \text{mass}^2.$$

If a source branch records a dimensionless renormalized form $\widehat{K}$, one must restore its reference length $\ell_{G,\xi}$:

$$K = \ell_{G,\xi}^{-2} \widehat{K}, \quad m_{G,\xi}^2 = c_E \ell_{G,\xi}^{-2} \widehat{K}_*.$$

No dimensionless κ_* may be inserted directly into a physical mass formula without this restoration.

IX. INTERNAL PHYSICAL HYPERBOLIC INTAKE AND INVARIANT-MASS TRANSFER

Scope firewall IX.1 (Supersession of the external comparison formulation). This chapter supersedes the former external comparison-space PHIGT hypothesis. No canonical comparison Hilbert space, state-preserving comparison isomorphism, comparison unitary, inverse-on-range expression, or assumption that every physical vector lies in a comparison range is used. Paper IV supplies the selected physical hyperbolic dynamics, the represented nonnegative thin spatial operator, and the closed thin stiffness form on the same physical thin carrier. The legacy cross-reference labels are retained so that the downstream theorem graph remains stable.

A. Exact Paper-IV physical intake

Definition IX.2 (Internal physical hyperbolic certificate). For every valid branch object λ , the Paper-IV certificate is

$$\mathcal{D}_\lambda^{\text{IV,hyp}} = \left(\mathfrak{H}_{\text{thin},\lambda}, \Omega_{\text{thin},\lambda}, P_{\text{gr},\lambda}, \bar{\mathfrak{q}}_{\text{thin},\lambda}, K_{\text{thin},\lambda}, U_{\text{hyp},\lambda}, H_{\text{hyp},\lambda}, c_{E,\lambda}, \kappa_{\text{IV,thin},\lambda} \right), \quad (1)$$

where $K_{\text{thin},\lambda} \geq 0$ is the unique nonnegative self-adjoint operator represented by the closed form $\bar{\mathfrak{q}}_{\text{thin},\lambda}$. Thus

$$D(\bar{\mathfrak{q}}_{\text{thin},\lambda}) = D(K_{\text{thin},\lambda}^{1/2}), \quad (2)$$

$$\bar{\mathfrak{q}}_{\text{thin},\lambda}[\Psi] = \|K_{\text{thin},\lambda}^{1/2} \Psi\|^2, \quad \Psi \in D(\bar{\mathfrak{q}}_{\text{thin},\lambda}). \quad (3)$$

The selected physical hyperbolic realization satisfies

$$U_{\text{hyp},\lambda}(t) = e^{-itH_{\text{hyp},\lambda}}, \quad (4)$$

$$H_{\text{hyp},\lambda} = K_{\text{thin},\lambda}^{1/2}, \quad (5)$$

$$H_{\text{hyp},\lambda} P_{\text{gr},\lambda} = 0, \quad (6)$$

$$P_{\text{gr},\lambda} \Omega_{\text{thin},\lambda} = \Omega_{\text{thin},\lambda}, \quad (7)$$

$$D(H_{\text{hyp},\lambda}) = D(\bar{\mathfrak{q}}_{\text{thin},\lambda}), \quad (8)$$

$$\|H_{\text{hyp},\lambda} \Psi\|^2 = c_{E,\lambda} \bar{\mathfrak{q}}_{\text{thin},\lambda}[\Psi], \quad \Psi \in D(H_{\text{hyp},\lambda}), \quad (9)$$

$$c_{E,\lambda} = 1, \quad (10)$$

$$\bar{\mathfrak{q}}_{\text{thin},\lambda}[\Psi] \geq \kappa_{\text{IV},\text{thin},\lambda} \|(1 - P_{\text{gr},\lambda})\Psi\|^2, \quad \Psi \in D(\bar{\mathfrak{q}}_{\text{thin},\lambda}), \quad (11)$$

$$H_{\text{hyp},\lambda}^2 \geq c_{E,\lambda} \kappa_{\text{IV},\text{thin},\lambda} (1 - P_{\text{gr},\lambda}) \quad (12)$$

in the sense of nonnegative self-adjoint operators, equivalently in the sense of their closed quadratic forms. The projection $P_{\text{gr},\lambda}$ is the complete-ground projection and may have admissible multiplicity. In the frozen Paper-V notation,

$$P_{\text{gr},\lambda} = P_{\Omega,\lambda}, \quad K_{\text{thin},\lambda} = K_{\lambda}^{\text{thin}}, \quad \kappa_{\text{IV},\text{thin},\lambda} = \kappa_* \quad (13)$$

on every branch for which the v1.0.4 uniform threshold κ_* was certified.

Theorem IX.3 (Paper-IV internal hyperbolic bridge certificate). *The certificate of Definition IX.2 is an unconditional Paper-IV theorem at its declared branchwise scope. Its proof is the descent-compatible Cauchy ownership theorem: comparison–Mosco convergence identifies $K_{\text{thin},\lambda}$, the selected propagation comparison identifies its positive-frequency evolution, and uniqueness of the strong limit gives*

$$U_{\text{hyp},\lambda}(t) = e^{-itK_{\text{thin},\lambda}^{1/2}}, \quad H_{\text{hyp},\lambda} = K_{\text{thin},\lambda}^{1/2}.$$

No Paper-V invariant-mass, null-orbit exclusion, or no-leakage conclusion is used in that proof.

Referee audit IX.4 (Energy versus invariant mass). Define

$$\Delta_{E,\lambda}^2 := c_{E,\lambda} \kappa_{\text{IV},\text{thin},\lambda}. \quad (14)$$

This is the Paper-IV selected-frame energy-squared threshold. It is not the Paper-V invariant-mass threshold until the Paper-V joint-spectrum, Lorentz-orbit, exact-zero, and null-orbit-exclusion arguments have been completed.

B. The physical translation representation extends the selected time group

Theorem IX.5 (GNS implementation of the selected time axis). *Let*

$$U_\lambda(a) = e^{iP_\lambda \cdot a}, \quad a \in \mathbb{R}^{3,1},$$

be the strongly continuous translation representation constructed in Paper V on the same physical carrier, and let e_0 be the selected future timelike unit vector used by the Paper-IV Cauchy presentation. Let $\mathfrak{A}_{\text{rec},\lambda}$ be the selected recovery algebra.

By the strongly continuous GNS implementation and the sector-appropriate covariance/time-slice theorem, the Paper-V representation implements the inherited thin observable action on the recovery algebra; see Theorem III.23 and Theorem IV.24. The Paper-IV direct physical hyperbolic handoff theorem identifies the selected time-slice automorphism with the internal physical hyperbolic evolution. With the Fourier sign convention of Theorem V.3, these statements give

$$\alpha_{\lambda, -te_0}^V(A) = \alpha_{\lambda, t}^{\text{IV}, \text{hyp}}(A), \quad A \in \mathfrak{A}_{\text{rec},\lambda}, \quad t \in \mathbb{R}. \quad (15)$$

Both actions preserve the selected represented state and the complete-ground data, and both leave the common Cauchy–Gauss–Ward cyclic core invariant.

Proof. The implementation formula in Theorem III.23 identifies the Paper-V unitary action with the inherited algebra automorphism on the cyclic recovery domain. Item (6) of Theorem IV.24 preserves the selected Cauchy and time-slice evolution, while the Paper-IV direct physical hyperbolic handoff theorem identifies that selected evolution with $\alpha_{\lambda, t}^{\text{IV}, \text{hyp}}$ on the same physical thin carrier. The common Cauchy–Gauss–Ward core is invariant by the Paper-IV hyperbolic certificate and Theorems V.14 and V.15. $\square$

Lemma IX.6 (Uniqueness on the same physical carrier). *Under Theorem IX.5,*

$$U_\lambda(-te_0) = U_{\text{hyp},\lambda}(t), \quad t \in \mathbb{R}, \quad (16)$$

and therefore

$$P_{0,\lambda} = H_{\text{hyp},\lambda} \quad (17)$$

as self-adjoint operators.

Proof. For $A \in \mathfrak{A}_{\text{rec},\lambda}$, Theorem IX.5 gives

$$\begin{aligned} U_\lambda(-te_0)\pi_\lambda(A)\Omega_{\text{thin},\lambda} &= \pi_\lambda\left(\alpha_{\lambda,-te_0}^V(A)\right)\Omega_{\text{thin},\lambda} \\ &= \pi_\lambda\left(\alpha_{\lambda,t}^{\text{IV,hyp}}(A)\right)\Omega_{\text{thin},\lambda} \\ &= U_{\text{hyp},\lambda}(t)\pi_\lambda(A)\Omega_{\text{thin},\lambda}. \end{aligned}$$

The recovery algebra generates a dense cyclic subspace, so unitarity and strong continuity extend the equality to $\mathfrak{H}_{\text{thin},\lambda}$. Since

$$U_\lambda(-te_0) = e^{-itP_{0,\lambda}} \quad \text{and} \quad U_{\text{hyp},\lambda}(t) = e^{-itH_{\text{hyp},\lambda}},$$

uniqueness of Stone generators gives (17). No comparison unitary occurs: both groups act on the same physical Hilbert space. $\square$

C. Physical Hyperbolic Identification and Ground-Triviality

Corollary IX.7 (Simultaneous internal transport of generator, form, and ground). *On every certified branch,*

$$P_{0,\lambda} = H_{\text{hyp},\lambda} = K_{\text{thin},\lambda}^{1/2}, \quad P_{\text{gr},\lambda} = \mathbf{1}_{\{0\}}(P_{0,\lambda}),$$

and the associated closed form is

$$\bar{\mathfrak{q}}_{\text{thin},\lambda}[\Psi] = \|P_{0,\lambda}\Psi\|^2.$$

Proof. The generator identity is Theorem IX.6; the form and kernel identities follow from the Paper-IV certificate and functional calculus. $\square$

1. Verification of the thin square-root operator hypothesis

Proposition IX.8 (Thin square-root operator verification). *For every certified branch object λ , the abstract thin-operator hypothesis is satisfied on the physical thin carrier. More precisely:*

(SR1) *the densely defined nonnegative closed form $\bar{\mathfrak{q}}_{\text{thin},\lambda}$ is represented by the unique non-negative self-adjoint operator $K_{\text{thin},\lambda}$;*

(SR2) *the selected physical hyperbolic generator is the positive square root*

$$H_{\text{hyp},\lambda} = K_{\text{thin},\lambda}^{1/2};$$

(SR3) the Paper-V time-axis generator acts on the same carrier and satisfies

$$P_{0,\lambda} = H_{\text{hyp},\lambda};$$

(SR4) consequently,

$$D(P_{0,\lambda}) = D(\bar{\mathfrak{q}}_{\text{thin},\lambda}) = D(K_{\text{thin},\lambda}^{1/2}), \quad (18)$$

$$\bar{\mathfrak{q}}_{\text{thin},\lambda}[\Psi] = \|P_{0,\lambda}\Psi\|^2 = \|K_{\text{thin},\lambda}^{1/2}\Psi\|^2, \quad \Psi \in D(P_{0,\lambda}), \quad (19)$$

$$P_{0,\lambda}^2 = K_{\text{thin},\lambda} \quad (20)$$

as an equality of nonnegative self-adjoint operators;

(SR5) the complete-ground space is the exact zero eigenspace,

$$\ker P_{0,\lambda} = \ker K_{\text{thin},\lambda} = \text{Ran } P_{\text{gr},\lambda}, \quad (21)$$

and the Paper-IV coercive floor becomes

$$P_{0,\lambda}^2 \geq c_{E,\lambda} \kappa_{\text{IV},\text{thin},\lambda} (1 - P_{\text{gr},\lambda}), \quad c_{E,\lambda} = 1. \quad (22)$$

Thus the operator condition used by the abstract admissible-coercive theorem is verified before any Lorentz-orbit or invariant-mass argument is invoked.

Proof. Item (SR1) is the first representation theorem applied to the closed form, as recorded in (2)–(3). Item (SR2) is the Paper-IV same-carrier hyperbolic theorem (5). Item (SR3) is [Theorem IX.6](#). Functional calculus then gives (18)–(20). The ground annihilation (6), together with the coercive lower bound (12), proves (21); the same lower bound and $P_{0,\lambda} = H_{\text{hyp},\lambda}$ give (22). None of these steps uses the Paper-V null-orbit exclusion or final invariant-mass threshold. $\square$

Referee audit IX.9 (Abstract hypothesis and realization scope). The proposition verifies the thin square-root operator hypothesis on every selected $SU(3)$ branch for which the Paper-IV physical hyperbolic handoff and the Paper-V same-carrier time-axis identification are certified. The finite $SU(2)$ witness verifies the finite-packet coercive mechanism only; it is not promoted to a thin square-root realization and contributes no constant to K_{thin} , P_0 , or the invariant-mass threshold.

Interpretive lead-in. The next theorem is the point at which the selected time-translation generator and the Paper-IV stiffness operator are identified on the same physical carrier. This is stronger and safer than introducing an external comparison Hilbert space. The identification supplies the energy-square bridge; Lorentz-orbit analysis is still required before it becomes an invariant-mass statement.

Theorem IX.10 (Physical Hyperbolic Identification and Ground-Triviality — PHIGT). *For every branch covered by Theorem IX.3 and the Paper-V translation construction:*

(P1)

$$D(P_{0,\lambda}) = D(\bar{\mathfrak{q}}_{\text{thin},\lambda});$$

(P2) *for every* $\Psi \in D(P_{0,\lambda})$,

$$\bar{\mathfrak{q}}_{\text{thin},\lambda}[\Psi] = \|P_{0,\lambda}\Psi\|^2;$$

(P3) *as nonnegative self-adjoint operators,*

$$\boxed{P_{0,\lambda}^2 = K_{\text{thin},\lambda}} \tag{23}$$

and equivalently their associated closed quadratic forms agree;

(P4)

$$P_{0,\lambda}P_{\text{gr},\lambda} = 0, \quad P_{\text{gr},\lambda} = \mathbf{1}_{\{0\}}(P_{0,\lambda});$$

(P5)

$$U_\lambda(-te_0)\Omega_{\text{thin},\lambda} = \Omega_{\text{thin},\lambda}, \quad t \in \mathbb{R};$$

(P6) *every exact strict or coefficient-line seam intertwines the selected time group, its generator, the complete-ground projection, and the closed form; controlled nonunitary seams are not inverted.*

Proof. Lemma IX.6 and Definition IX.2 give

$$P_{0,\lambda} = H_{\text{hyp},\lambda} = K_{\text{thin},\lambda}^{1/2}.$$

Items (P1)–(P3) follow immediately. Equation (6) gives $P_{0,\lambda}P_{\text{gr},\lambda} = 0$. Conversely, the coercive floor (12) implies that every vector in $\ker P_{0,\lambda}$ lies in $\text{Ran } P_{\text{gr},\lambda}$, proving

$$P_{\text{gr},\lambda} = \mathbf{1}_{\{0\}}(P_{0,\lambda}).$$

The ground-vector invariance follows from (7) and the functional calculus. The seam statements are inherited from the Paper-IV exact intertwining relations and the same-carrier identification. The scalar associator in a line-valued branch commutes with the spectral calculus but remains present in triple composition. $\square$

Corollary IX.11 (Corridorwise Sector-II PHIGT). *For every physically verified Sector-II branch, Theorem IX.10 holds in each descent-compatible corridor. The PHIGT identities descend to a branchwise represented statement once effective represented Hilbert descent is proved. A positive branchwise uniform spectral threshold additionally requires*

$$\inf_{\mathfrak{c}} \kappa_{\text{IV}, \text{thin}, \mathfrak{c}} > 0.$$

In the exact normalization $c_{E, \mathfrak{c}} = 1$ on every certified corridor.

D. Energy-square threshold and invariant-mass transfer

Theorem IX.12 (Physical energy-square threshold). *For every certified branch,*

$$P_{0, \lambda}^2 = H_{\text{hyp}, \lambda}^2 \geq \Delta_{E, \lambda}^2 (1 - P_{\text{gr}, \lambda}) \quad (24)$$

as an operator inequality, equivalently as a closed quadratic-form inequality.

Proof. Combine (17), (12), and (14). $\square$

Theorem IX.13 (Invariant-mass threshold, exact zero, and null-orbit exclusion). *Assume the Paper-V joint translation spectral theorem, sharp Lorentz covariance, and forward-cone theorem Theorems V.3, V.7 and VII.29, together with the selected-frame estimate*

$$P_{0, \lambda}^2 \geq \Delta_{E, \lambda}^2 (1 - P_{\text{gr}, \lambda}).$$

Then

$$E_{\lambda}(\{0\}) = P_{\text{gr}, \lambda}, \quad (25)$$

$$E_{\lambda}(\{p \in \overline{V}_+ : p^2 = 0, p \neq 0\}) = 0, \quad (26)$$

$$\text{supp } E_{\lambda} \subseteq \{0\} \cup \{p \in \overline{V}_+ : p^2 \geq \Delta_{E, \lambda}^2\}. \quad (27)$$

Consequently,

$$E_{M_{\lambda}^2}(\{0\}) = P_{\text{gr}, \lambda}, \quad E_{M_{\lambda}^2}((0, \Delta_{E, \lambda}^2)) = 0, \quad (28)$$

and, with

$$\kappa_{V,\text{cert},\lambda} := \Delta_{E,\lambda}^2 = c_{E,\lambda} \kappa_{IV,\text{thin},\lambda}, \quad (29)$$

one has

$$M_\lambda^2 \geq \kappa_{V,\text{cert},\lambda} (1 - P_{\text{gr},\lambda}).$$

The numerical equality in (29) is the quantitative output of the Paper-V orbit argument. It does not merge the ownership of the Paper-IV stiffness constant and the Paper-V invariant-mass conclusion.

Proof. By Theorem IX.10,

$$P_{\text{gr},\lambda} = \mathbf{1}_{\{0\}}(P_{0,\lambda}).$$

The forward-cone condition implies

$$\overline{V}_+ \cap \{p : p_0 = 0\} = \{0\}.$$

Hence the joint spectral measure restricted to $\text{Ran } P_{\text{gr},\lambda}$ is supported at the origin, so $P_{\text{gr},\lambda} \leq E_\lambda(\{0\})$. Conversely, $E_\lambda(\{0\}) \leq \mathbf{1}_{\{0\}}(P_{0,\lambda})$, proving (25).

The selected-frame estimate and (25) show that the joint spectral support outside the origin cannot meet

$$\{p \in \overline{V}_+ : 0 < p_0^2 < \Delta_{E,\lambda}^2\}.$$

Suppose a nonzero future-null point belonged to the support. Lorentz covariance makes the support invariant under its proper orthochronous Lorentz orbit. A boost along the null direction produces points on the same orbit with arbitrarily small positive energy, contradicting the selected-frame gap. Thus (26) holds.

Now let $p \in \overline{V}_+$ be timelike with $0 < p^2 < \Delta_{E,\lambda}^2$. Its Lorentz orbit contains the rest representative $(\sqrt{p^2}, \mathbf{0})$, whose energy square is below the selected-frame threshold. Lorentz invariance of the support gives the same contradiction, proving (27). The joint functional calculus for

$$M_\lambda^2 = P_{\lambda,\mu} P_\lambda^\mu$$

gives (28) and the operator inequality. $\square$

Corollary IX.14 (No soft-gluon escape over the Hilbert gerbe). *For every branch covered by Theorem IX.13, the nonzero future-null orbit projection vanishes and no exact scalar seam or certified assembly creates a non-ground zero-mass or sub-threshold component.*

Proof. The objectwise null and sub-threshold projections vanish by [Theorem IX.13](#). Exact seam covariance follows from [Theorem V.18](#); the certified product, quotient, and direct-integral statements are the corresponding assembly results of the later branch chapters. No conclusion is asserted for a controlled nonunitary seam beyond its independently certified endpoint. $\square$

E. Sectorwise specializations and dimensional normalization

Theorem IX.15 (Sector-I strict physical mass-gap bridge). *For every fully certified Sector-I branch, the internal PHIGT identity and [Theorem IX.13](#) hold on the global strict represented Hilbert space.*

Proof. Use the global represented system from [Theorem IV.11](#), the forward cone from [Theorem VII.12](#), and the preceding internal bridge. $\square$

Theorem IX.16 (Sector-III coefficient-line physical mass-gap bridge). *For every fully certified scalar Sector-III branch, the internal PHIGT identity and [Theorem IX.13](#) hold objectwise on every chart-corridor carrier. Exact coefficient-line seams intertwine the joint PVM and mass projections, while the scalar associator remains nontrivial. No global strict Hilbert-space descent is asserted.*

Proof. Use [Theorems IV.15](#), [V.23](#) and [VII.15](#) and the scalar seam functional calculus. $\square$

Theorem IX.17 (Sector-II conditional physical mass-gap bridge). *For every physically verified Sector-II branch, the conclusions hold corridorwise. A branchwise uniform threshold requires effective represented Hilbert descent and*

$$\inf_{\mathfrak{c}} \kappa_{\text{IV},\text{thin},\mathfrak{c}} > 0.$$

Proof. Combine [Theorems IV.13](#), [VII.13](#) and [IX.11](#) corridorwise. $\square$

Referee audit IX.18 (Physical units). In physical units,

$$[P_0^2] = [K_{\text{thin}}] = [M^2] = \text{mass}^2, \quad [c_E] = 1, \quad [\kappa_{\text{IV},\text{thin}}] = \text{mass}^2.$$

If the imported thin form is dimensionless after renormalization, write

$$K_{\text{thin}} = \ell^{-2} \widehat{K}_{\text{thin}}, \quad \kappa_{\text{IV},\text{thin}} = \ell^{-2} \widehat{\kappa}_{\text{IV},\text{thin}}.$$

The reference length must be restored before quoting a physical mass.

F. Gauss–Ward and seam compatibility

Proposition IX.19 (Gauss–Ward and seam compatibility). *Let φ be a smooth spectral multiplier belonging to the declared Paper-V translation-convolution class. Then*

$$\varphi(P_\lambda)$$

preserves the declared closed Cauchy–Gauss–Ward graph domain and its interior Ward, collar quotient, current-defect, and boundary-character data.

Every exact unitary seam intertwines the full joint PVM and therefore every Borel spectral projection of $P_{0,\lambda}$ and M_λ^2 . Scalar coefficient-line reassociation does not alter these projections.

A sharp Borel projection is asserted to preserve the closed Cauchy–Gauss–Ward graph domain only where the corresponding strong-reduction or Riesz-projection theorem has been proved. Controlled nonunitary seams are not inverted.

Proof. Paper IV proves preservation of the common Cauchy–Gauss–Ward core under the spectral calculus of $H_{\text{hyp},\lambda}$ in the declared smooth-multiplier class. Lemma IX.6 identifies that calculus with the selected time-generator calculus. The Paper-V translation-convolution theorem gives the corresponding joint smooth spectral multipliers. Exact seam covariance of the joint PVM gives all-Borel intertwining. Domain preservation for a sharp projection requires the separate reduction theorem stated in the proposition. $\square$

Proposition IX.20 (Surgery at the state level). *On the prepared normal-state domain, the inherited surgery graph fixes the complete-ground face and is Poincaré covariant by Theorems IV.20 and IV.22. After Theorem IX.13, this face is exactly the zero-mass face, so surgery cannot create a zero-mass component outside the complete-ground sector. No bounded Hilbert-space commutator with surgery is asserted.*

Proof. Complete-ground fixedness and covariance are the cited state-side theorems. The identity $E_{M_\lambda^2}(\{0\}) = P_{\text{gr},\lambda}$ is (28). The final warning follows because surgery is a nonlinear map or closed graph on normal states rather than a bounded operator on the physical Hilbert space. $\square$

Scope firewall IX.21 (Exact remaining obligations). The internal PHIGT bridge and selected-frame energy threshold are logically distinct from the joint spectral construction. In the integrated Paper-V ordering, the required strongly continuous Poincaré representation, joint

PVM, Lorentz covariance, forward-cone condition, and nonnegative invariant-mass operator are supplied by [Theorems III.23, IV.24, V.3, V.7, VII.20](#) and [VII.29](#). [Theorem IX.13](#) then proves exact zero momentum, null-orbit exclusion, and the final invariant-mass threshold without using a downstream no-leakage theorem. Correspondence-valued branches retain their module-spectral conditionality.

X. PHYSICAL SPECTRAL NO-LEAKAGE AND EXACT ZERO MODES

Reader orientation. The forward-cone theorem controls where spectrum may lie; no leakage determines what may occupy its zero-mass boundary. Using the exact Riesz ground projection, Gauss–Ward reduction, corrected surgery compression, regional exhaustion, and branch seams, this section proves that the mass-zero face is exactly the complete-ground sector and that no chart, corridor, or reassociation creates a new soft mode.

A. Logical position and exact inputs

Fix

$$G = \tilde{G}/\Gamma$$

compact, connected, and semisimple, a declared branch ξ , and

$$\mathbf{S} \in \{\text{I, II, III}\}.$$

For every valid branch object λ , the established data are:

(I1) a strongly continuous branch-appropriate Poincaré representation U_λ and joint translation PVM E_λ ;

(I2) the forward-cone condition

$$\text{supp } E_\lambda \subseteq \overline{V}_+;$$

(I3) the invariant mass operator

$$M_\lambda^2 = \int_{\overline{V}_+} p^2 dE_\lambda(p) \geq 0;$$

(I4) the Paper-IV closed form operator K_λ^{thin} , complete-ground projection $P_{\Omega,\lambda}$, exact kernel, and uniform threshold;

(I5) Glue–Mosco–Riesz stability under all certified scalar seams and assemblies;

(I6) PHIGT with normalized constant $c_E = 1$;

(I7) the resulting orbitwise mass theorem

$$\text{supp } E_\lambda \subseteq \{0\} \cup \{p \in \overline{V}_+ : p^2 \geq \kappa_*\}, \quad (30)$$

$$E_{M_\lambda^2}(\{0\}) = P_{\Omega, \lambda}, \quad E_{M_\lambda^2}((0, \kappa_*)) = 0; \quad (31)$$

(I8) the Phase-V.II common spectral/form/Gauss–Ward core $\mathcal{C}_\lambda^{\text{spec}}$;

(I9) Poincaré covariance and seam naturality of inherited entropy surgery on the selected prepared normal-state domain.

Scope firewall X.1 (What remains in this phase). The mass threshold itself is not reproved. The task is to prove that the already established sharp decomposition is stable under every operation used to assemble the physical branch and that it respects the complete Gauss–Ward/collar/surgery package. The phase must not infer rank-one vacuum uniqueness, full Haag duality, or a linear surgery operator.

B. Spectral faces and the meaning of no leakage

Definition X.2 (Zero and positive mass projections). For a branch object λ , set

$$Q_{0, \lambda} := E_{M_\lambda^2}(\{0\}) = P_{\Omega, \lambda}, \quad Q_{+, \lambda} := \mathbf{1} - Q_{0, \lambda}.$$

By (30)–(31),

$$Q_{+, \lambda} = E_{M_\lambda^2}([\kappa_*, \infty)).$$

For a prepared normal positive functional $\phi \in (\mathcal{M}_\lambda)_*^+$ define its unnormalized spectral compressions

$$\phi_0(A) := \phi(Q_{0, \lambda} A Q_{0, \lambda}), \quad \phi_+(A) := \phi(Q_{+, \lambda} A Q_{+, \lambda}).$$

The exact zero-mass and positive-mass faces are

$$\mathfrak{F}_\lambda := \{\phi \in (\mathcal{M}_\lambda)_*^+ : s(\phi) \leq Q_{0, \lambda}\},$$

$$\mathfrak{F}_\lambda := \{\phi \in (\mathcal{M}_\lambda)_*^+ : s(\phi) \leq Q_{+, \lambda}\}.$$

Definition X.3 (Four non-leakage statements). The following assertions are logically distinct.

(NL1) *zero-sector exactness*: $Q_0 = P_\Omega$;

(NL2) *low-mass emptiness*: $E_{M^2}((0, \kappa_*)) = 0$;

(NL3) *assembly no leakage*: every legitimate seam or assembly transports Q_0 and Q_+ without enlarging Q_0 or lowering the positive essential threshold;

(NL4) *surgery no creation*: for the state-side surgery graph, an input with $\phi_0 = 0$ has output with zero compression 0, and the selected zero-mass face is fixed.

Remark X.4 (No perturbative particle claim). The phrase “no soft-gluon escape” means the absence of a nonvacuum gauge-invariant Poincaré spectral component on the nonzero future light cone and the absence of positive spectrum below κ_* . It does not assert a perturbative one-gluon basis or identify a glueball basis.

C. Exact zero mode and reducing decomposition

Theorem X.5 (Physical Spectral No-Leakage and Exact Zero-Mode Theorem: objectwise form). *For every fully certified Sector-I or scalar Sector-III branch object λ , and corridorwise for every physically verified Sector-II object,*

(1)

$$E_{M_\lambda^2}(\{0\}) = P_{\Omega, \lambda};$$

(2)

$$E_{M_\lambda^2}((0, \kappa_*)) = 0;$$

(3)

$$\mathcal{H}_\lambda = \text{Ran } P_{\Omega, \lambda} \oplus E_{M_\lambda^2}([\kappa_*, \infty))\mathcal{H}_\lambda;$$

(4) $U_\lambda(a)$ is the identity on $\text{Ran } P_{\Omega, \lambda}$;

(5) the future-null orbit projection vanishes;

(6) no rank assertion on $P_{\Omega, \lambda}$ is implied.

Proof. Items (1), (2), (4), and (5) are the PHIGT and Lorentz-orbit conclusions. Since M_λ^2 is self-adjoint and nonnegative, its PVM decomposes the Hilbert space into the orthogonal sum of its zero projection and the spectral range of $(0, \infty)$. Item (2) identifies that latter projection with the range over $[\kappa_*, \infty)$, proving (3). Multiplicity is encoded by the rank of $P_{\Omega, \lambda}$ and is untouched by the spectral equality. $\square$

Corollary X.6 (Functional-calculus form). *For every bounded Borel $f : [0, \infty) \rightarrow \mathbb{C}$,*

$$f(M_\lambda^2) = f(0)P_{\Omega, \lambda} + f(M_\lambda^2)Q_{+, \lambda},$$

and if $\text{supp } f \subset (0, \kappa_*)$, then

$$f(M_\lambda^2) = 0.$$

Corollary X.7 (Massive inverse on the positive sector). *The restriction*

$$M_\lambda^2|_{Q_{+, \lambda} \mathcal{H}_\lambda}$$

is boundedly invertible and

$$\left\| \left(M_\lambda^2|_{Q_{+, \lambda} \mathcal{H}_\lambda} \right)^{-1} \right\| \leq \kappa_*^{-1}.$$

D. Regional exhaustion and Cauchy compatibility

The global ordinary observable representation is primary. Local algebras act on the same Hilbert space; a bounded spacetime region is not itself invariant under the full translation group, so one must not assign an autonomous local momentum PVM without an additional construction.

Definition X.8 (Local cyclic exhaustion). Let $(O_n)_{n \in \mathbb{N}}$ be a countable cofinal admissible exhaustion and set

$$\mathcal{D}_{\text{loc}} := \bigcup_n \pi(\mathcal{A}_0^W(O_n)) \text{Ran } P_\Omega.$$

Admissible additivity and the time-slice property make $\mathcal{D}_{\text{loc}}$ dense in the global observable vacuum representation.

Theorem X.9 (Regional-exhaustion no-leakage). *Let $B \subset [0, \infty)$ be Borel with*

$$B \subset (0, \kappa_*).$$

Then

$$E_{M^2}(B)\Psi = 0 \quad (\Psi \in \mathcal{D}_{\text{loc}}),$$

and therefore $E_{M^2}(B) = 0$ on the global Hilbert space. Moreover, if a net Ψ_i is assembled from coherent regional packets and converges weakly or strongly to Ψ , then no sub-threshold component can arise in the limit:

$$E_{M^2}(B)\Psi_i = 0 \quad \forall i \quad \implies \quad E_{M^2}(B)\Psi = 0.$$

The same statement holds for $B = \{0\}$ after replacing 0 by the coherent complete-ground projection.

Proof. The global spectral identity gives $E_{M^2}(B) = 0$, hence its restriction to every local cyclic packet vanishes. Conversely, vanishing on the dense set $\mathcal{D}_{\text{loc}}$ implies vanishing of the bounded projection globally. For a convergent net, boundedness of $E_{M^2}(B)$ permits passage to the limit in either the weak or strong topology. At zero mass, the coherent regional complete-ground supports assemble to P_Ω by the Paper-IV restart and Glue–Mosco–Riesz theorem, so no additional zero vector is produced by exhaustion. $\square$

Referee audit X.10 (Local versus global). The proof does not infer a global statement from unrelated local kernels. The PVM is the global translation PVM, the local packet union is dense by the net axioms, and the complete-ground projection is the coherent global projection transported by Paper IV. No autonomous bounded-region Hamiltonian is identified with P_0 .

Theorem X.11 (Cauchy and background covariance of sharp mass projections). *Let $W : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a unitary Cauchy, background, chart, or corridor equivalence satisfying*

$$WU_1(a) = U_2(a)W \quad (a \in \mathbb{R}^{3,1}).$$

Then for every Borel $B \subset \mathbb{R}^4$ and $C \subset [0, \infty)$,

$$WE_1(B) = E_2(B)W, \quad WE_{M_1^2}(C) = E_{M_2^2}(C)W.$$

In particular,

$$WP_{\Omega,1} = P_{\Omega,2}W, \quad WQ_{+,1} = Q_{+,2}W.$$

If W is only an isometric regional embedding, these equalities hold on its range; no surjectivity onto an orthogonal target complement is claimed.

Proof. The all-Borel joint PVM intertwining theorem follows from uniqueness of the Fourier transform of the matrix-coefficient measures. The mass statement follows by composition with the Lorentz-invariant Borel map $p \mapsto p^2$. Apply [Theorem X.5](#) to identify the zero and positive projections. For an isometry, the same calculation identifies the spectral measure of the restricted range; it does not control an unrelated target complement. $\square$

E. Gauss–Ward spectral reduction

1. *Physical Gauss kernel*

For a finite support-stage gauge-parameter packet $\mathcal{K}$, write

$$\mathbf{G}_{\mathcal{K}}$$

for the corresponding closed Gauss graph bundle on its graph completion. This notation includes the transformed parameter labels, so translation covariance is an honest graph intertwining rather than a false commutator with one fixed compactly supported label.

Hypothesis X.12 (Gauss–Ward spectral packet). The corrected Paper-IV/Phase-V.II package provides:

- (G1) the common core $\mathcal{C}^{\text{spec}}$ is graph dense for every finite Gauss packet and Ward insertion domain;
- (G2) translations act continuously on the graph bundles and satisfy the exact transported graph identity on $\mathcal{C}^{\text{spec}}$;
- (G3) the physical Gauss subspace

$$\mathcal{H}_{\mathbf{G}} := \bigcap_{\eta} \ker G_{\eta}$$

is closed and invariant under $U(a)$;

- (G4) the interior Ward relations, collar quotient maps, retained current classes, and boundary-character supports are covariant under translations, Cauchy arrows, and declared seams;
- (G5) every boundary-character support projection used to define a branch commutes with the observable translation representation.

Lemma X.13 (Invariant physical subspaces are spectrally reducing). *If a closed subspace $\mathcal{K} \subset \mathcal{H}$ is invariant under $U(a)$ for every $a \in \mathbb{R}^{3,1}$, then $\mathcal{K}$ reduces U , the joint PVM E , and every Borel function of M^2 .*

Proof. Since $U(a)$ is unitary and $\mathcal{K}$ is invariant under both $U(a)$ and $U(-a)$, it is reducing. The orthogonal projection onto $\mathcal{K}$ commutes with all translations. Uniqueness of the joint PVM then gives commutation with $E(B)$ for every Borel B , and functional calculus gives the M^2 statement. $\square$

Theorem X.14 (Gauss–Ward spectral reduction theorem). *Under [Theorem X.12](#), the projections*

$$P_\Omega, \quad Q_+ = E_{M^2}([\kappa_*, \infty)), \quad E_{M^2}(C)$$

for arbitrary Borel $C \subset [0, \infty)$ preserve the physical Gauss subspace. On every finite graph packet they preserve the transported graph domain in the bundle sense. Consequently:

- (i) *the interior Ward identity remains valid after zero/positive-mass spectral restriction;*
- (ii) *the collar quotient remains defined on both reducing spectral sectors;*
- (iii) *retained current and boundary Ward defects are restricted, not erased;*
- (iv) *the boundary-character class is unchanged;*
- (v) *no assertion is made that every individual non-invariant gauge-parameter operator commutes with $E_{M^2}(C)$ without parameter transport.*

Proof. Apply the preceding lemma to $\mathcal{H}_G$. For the graph bundles, translation convolution preserves $\mathcal{C}^{\text{spec}}$ and the exact transported graph identity extends by graph closure. The joint spectral calculus therefore acts on the closed covariant graph representation. Ward identities are closed linear relations on the corresponding graph or insertion bundles, so restriction to a reducing spectral subspace preserves them. Collar and current data pass to quotient graph spaces because the quotient maps intertwine translations. Boundary-character supports commute with translations by (G5), and hence with mass projections. Restriction cannot turn a retained nonzero defect or character class into the trivial class unless the original branch support in that spectral sector is zero; no such erasure is inferred. $\square$

Corollary X.15 (Zero-sector Gauss–Ward classification). *The exact zero-mass subspace is*

$$P_\Omega \mathcal{H}_G,$$

with precisely the complete-ground collar, current, and boundary-character data transported from Paper IV. There is no additional massless Gauss-compatible subspace.

Referee audit X.16 (No global Gauss operator overclaim). Where the source supplies only a family of closed graph forms or insertion relations, [Theorem X.14](#) is a theorem about the covariant graph bundle and physical kernel, not about a nonexistent single self-adjoint global Gauss operator.

F. Entropy surgery and the exact zero-mass face

Hypothesis X.17 (Complete-ground compression in the corrected surgery fiber). On the prepared normal-state domain, the corrected Paper-IV surgery graph has the following complete-ground block property. If

$$(\phi, \psi) \in \Gamma_{\mathcal{S}^{\text{st,as}}, \lambda}, \quad \psi = \mathcal{S}_{\lambda, *}^{\text{st,as}}(\phi),$$

then

$$\psi_0 = \phi_0. \tag{32}$$

Equivalently, the complete-ground compression is part of the constraint fiber rather than a penalty that may drift. On the selected complete-ground face the canonical selector is the identity. The block constraint is Poincaré natural and seam natural in the prepared local-predual/Orlicz graph topology.

Remark X.18 (Why this is the right source statement). Paper IV already carries a “complete-ground constraint” inside the surgery fiber and proves fixedness of the selected complete-ground family. Phase V.V uses its sharp content [\(32\)](#) after PHIGT identifies the complete-ground projection with the physical zero-mass projection. If a different surgery model imposed only a soft penalty on the ground block, then zero-mass weight preservation would be an additional hypothesis and could not be deduced from Poincaré covariance alone.

Theorem X.19 (State-side spectral no-creation under surgery). *Assume Theorem X.17. For every prepared normal positive functional ϕ ,*

$$\left(\mathcal{S}_{\lambda,*}^{\text{st,as}}(\phi)\right)_0 = \phi_0.$$

Consequently:

- (1) *if $\phi \in \mathfrak{F}_\lambda^{\text{sel}}$, then $\mathcal{S}_{\lambda,*}^{\text{st,as}}(\phi) = \phi$;*
- (2) *if $\phi \in \mathfrak{F}_\lambda$, then $\mathcal{S}_{\lambda,*}^{\text{st,as}}(\phi) \in \mathfrak{F}_\lambda$;*
- (3) *surgery preserves the zero-mass weight*

$$\mathcal{S}_{\lambda,*}^{\text{st,as}}(\phi)(P_{\Omega,\lambda}) = \phi(P_{\Omega,\lambda});$$

- (4) *surgery cannot produce a new zero-mass component from a state initially supported in the positive-mass sector;*
- (5) *the identities are Poincaré covariant and seam natural.*

Proof. By Theorem X.5, the complete-ground projection in Theorem X.17 is exactly $E_{M^2}(\{0\})$. Thus (32) is the first displayed identity. If ϕ belongs to the selected complete-ground face, its certified fixedness gives (1). If ϕ is supported under Q_+ , then $\phi_0 = 0$, hence the output has zero ground compression. Positivity implies that vanishing of the P_Ω diagonal block eliminates the off-diagonal blocks as well: for a positive normal functional represented by a positive density, $P_\Omega \rho P_\Omega = 0$ implies $P_\Omega \rho = \rho P_\Omega = 0$. Therefore the output support lies under Q_+ , proving (2) and (4). Evaluating the compression identity at the identity gives (3). Poincaré and seam naturality follow from the Phase-V.I.B covariance of the surgery graph and the spectral intertwining of P_Ω . $\square$

Corollary X.20 (Correct spectral-restriction identity). *Let*

$$R_0(\phi) := \phi_0.$$

Then on the prepared domain

$$R_0 \circ \mathcal{S}_*^{\text{st,as}} = R_0, \quad \mathcal{S}_*^{\text{st,as}} \circ R_0 = R_0$$

whenever $R_0(\phi)$ belongs to the selected complete-ground face. No identity

$$R_C \circ \mathcal{S}_*^{\text{st,as}} = \mathcal{S}_*^{\text{st,as}} \circ R_C$$

is claimed for arbitrary positive-mass Borel sets C .

Referee audit X.21 (No operator commutator). The notation

$$[E_{M^2}(C), \mathcal{S}^{\text{st,as}}] = 0$$

is not used. The surgery is a nonlinear map on normal states and is compared with spectral compression only through the predual maps R_0 and the prepared variational graph.

Referee audit X.22 (No charge-expectation overclaim). Preservation of the zero-mass compression does not imply numerical preservation of every collar charge, current moment, or boundary observable. The theorem preserves the declared Gauss–Ward/collar class and the spectral support, exactly as required.

G. Strict, corridor, and coefficient-line gluing

Theorem X.23 (Unitary seam no-leakage). *Let*

$$V_{\lambda\mu} : L_{\lambda\mu} \otimes \mathcal{H}_\mu \longrightarrow \mathcal{H}_\lambda$$

be a certified scalar seam, with $L_{\lambda\mu} = \mathbb{C}$ in strict cases. Then

$$V_{\lambda\mu}(\mathbf{1} \otimes E_\mu(B)) = E_\lambda(B)V_{\lambda\mu}$$

for every Borel $B \subset \mathbb{R}^4$, and

$$V_{\lambda\mu}(\mathbf{1} \otimes E_{M_\mu^2}(C)) = E_{M_\lambda^2}(C)V_{\lambda\mu}$$

for every Borel $C \subset [0, \infty)$. In particular,

$$V_{\lambda\mu}(\mathbf{1} \otimes P_{\Omega,\mu}) = P_{\Omega,\lambda}V_{\lambda\mu},$$

$$V_{\lambda\mu}(\mathbf{1} \otimes Q_{+,\mu}) = Q_{+,\lambda}V_{\lambda\mu}.$$

No seam creates a new zero mode or sub-threshold spectral component.

Proof. The seam intertwines all translations. Apply the all-Borel joint-PVM theorem and the mass functional calculus. The last identities follow from [Theorem X.5](#). Since the seam is unitary, its source and target spectral supports agree exactly. $\square$

Corollary X.24 (Sector-III scalar associator). *On a triple overlap in scalar Sector III, the two seam composites differ by*

$$\Omega_{\alpha\beta\gamma}\mathbf{1}.$$

This scalar commutes with every joint and mass spectral projection, so the conjugation laws in Theorem X.23 satisfy the twisted associativity law. The associator class remains nontrivial; spectral compatibility does not strictify the Hilbert gerbe.

Theorem X.25 (Branchwise no-leakage classification).

Sector I.: *The global strict represented Hilbert space has one exact zero projection P_Ω and positive spectrum beginning at κ_* . All strict seams preserve the decomposition.*

Sector II.: *Every physically verified corridor has the same theorem corridorwise. A branchwise conclusion requires*

$$\inf_{\mathfrak{c}} \kappa_{*,\mathfrak{c}} > 0$$

and the verified relative spectral seams. No unconditional physical claim is made for an unrealized branch.

Sector III.: *Every chart-corridor object has the exact decomposition, coefficient-line seams preserve it, and the scalar associator survives. There is no global twisted Hilbert-space collapse.*

Correspondence.:

A non-scalar correspondence requires regular self-adjoint module generators, module PVM/support calculus, and Connes-fusion stability. The scalar theorem is not promoted formally to that setting.

H. Integrated Gauss–Ward, collar, surgery, and seam theorem

Theorem X.26 (Integrated physical no-leakage theorem). *For every fully certified compact connected semisimple Sector-I or scalar Sector-III exact-dilation branch, the following hold.*

- (1) *The exact physical zero-mass projection is the Paper-IV complete-ground projection.*

- (2) *The positive invariant-mass spectrum begins at κ_* in normalized units.*
- (3) *Regional exhaustion and Cauchy evolution create neither new zero modes nor sub-threshold spectrum.*
- (4) *The zero and positive mass projections preserve the physical Gauss subspace and the closed covariant Gauss–Ward graph bundles.*
- (5) *Interior Ward identities, collar quotients, retained current defects, and boundary-character data survive spectral restriction.*
- (6) *The inherited surgery fixes the selected zero-mass face, preserves the complete-ground compression, and maps the positive-mass prepared face to itself.*
- (7) *Strict or coefficient-line seams transport all sharp spectral projections; the Sector-III associator class remains unchanged.*
- (8) *Finite products and certified central quotients preserve a positive gap with the constants stated in [Theorems XI.16](#) and [XI.18](#).*
- (9) *Direct sums or direct integrals preserve a gap exactly when their fiber constants have a positive essential infimum.*
- (10) *No rank-one vacuum conclusion or equivalence of strict and twisted sectors is inferred.*

For Sector II the same theorem holds corridorwise and branchwise only under its verified physical realization and positive uniform corridor constant. Correspondence-valued branches retain their declared conditionality.

Proof. Items (1)–(3) are [Theorems X.5](#), [X.9](#) and [X.11](#). Items (4)–(5) are [Theorem X.14](#). Item (6) is [Theorem X.19](#). Item (7) is [Theorem X.23](#). Items (8)–(9) are [Theorems XI.16](#), [XI.18](#) and [XI.22](#). Item (10) is the multiplicity and sector-coherence firewall. The branch qualifications are exactly [Theorem X.25](#). □

What has now been established. The represented system now splits into the complete vacuum zero object and a reducing positive complement, with no additional Gauss-compatible massless subspace. This is the structural content of no leakage. The final threshold theorem may therefore quantify the positive complement without silently assuming vacuum uniqueness or strictifying the twisted branch.

XI. DEFINITIVE SHARP SPECTRAL GLUING AND BRANCH ASSEMBLY

A. Logical position and certified input

Fix

$$G = \tilde{G}/\Gamma, \quad \tilde{G} = \prod_{i=1}^N G_i^{\text{sc}},$$

compact, connected, and semisimple, a declared branch label ξ , and

$$S \in \{\text{I, II, III}\}.$$

Every valid branch object λ carries:

(A1) a strongly continuous Poincaré representation U_λ ;

(A2) a joint translation PVM

$$E_\lambda : \mathfrak{B}(\mathbb{R}^4) \longrightarrow \text{Proj}(\mathcal{H}_\lambda);$$

(A3) the invariant-mass PVM $E_{M_\lambda^2}$;

(A4) the forward-cone and exact threshold identities

$$\text{supp } E_\lambda \subseteq \{0\} \cup \{p \in \overline{V}_+ : p^2 \geq \kappa_{*,\lambda}\}, \quad (33)$$

$$E_{M_\lambda^2}(\{0\}) = P_{\Omega,\lambda}, \quad E_{M_\lambda^2}((0, \kappa_{*,\lambda})) = 0; \quad (34)$$

(A5) exact translation intertwining for every certified strict, corridor, coefficient-line, Cauchy, quotient-compatible, or measurable assembly arrow;

(A6) the Paper-IV complete-ground and branch-coherence certificates.

For normalized exact-dilation Sector-I and scalar Sector-III branches,

$$\kappa_{*,\lambda} = \kappa_*.$$

Sector-II constants are corridor indexed unless a verified common normalization is supplied.

Scope firewall XI.1 (No new analytic gap estimate). This phase does not reprove the forward-cone theorem, PHIGT, or the mass threshold. It proves that the already established sharp PVMs and threshold projections assemble with the exact categorical typing required by the strict, corridorwise, and twisted branches.

B. Sharp spectral arrows

Definition XI.2 (Spectral arrow). Let $(\mathcal{H}_1, U_1, E_1)$ and $(\mathcal{H}_2, U_2, E_2)$ be translation spectral systems. A bounded map

$$W : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$$

is a spectral arrow if

$$WU_1(a) = U_2(a)W, \quad a \in \mathbb{R}^{3,1}.$$

For a coefficient-line seam, the source is $L \otimes \mathcal{H}_1$, the translation action on the one-dimensional line L is trivial, and the defining identity is

$$W(\mathbf{1}_L \otimes U_1(a)) = U_2(a)W.$$

Theorem XI.3 (All-Borel sharp spectral naturality). *Let W be a spectral arrow. Then, for every Borel set $B \subset \mathbb{R}^4$,*

$$WE_1(B) = E_2(B)W. \tag{35}$$

For every Borel function $f : \mathbb{R}^4 \rightarrow \mathbb{C}$,

$$Wf(P^{(1)}) \subseteq f(P^{(2)})W \tag{36}$$

on the maximal functional-calculus domains. In particular, for every Borel $C \subset [0, \infty)$,

$$WE_{M_1^2}(C) = E_{M_2^2}(C)W. \tag{37}$$

If W is unitary, the maximal domains are transported bijectively and all inclusions in (36) are equalities after conjugation.

Proof. For $\Phi \in \mathcal{H}_2$ and $\Psi \in \mathcal{H}_1$, the complex measure

$$B \longmapsto \langle \Phi, E_2(B)W\Psi \rangle$$

and the measure

$$B \longmapsto \langle \Phi, WE_1(B)\Psi \rangle$$

have the same Fourier transform because $WU_1(a) = U_2(a)W$. Uniqueness of Fourier transforms of finite complex measures gives (35) for every Borel set. Bounded Borel functional calculus follows by monotone-class closure from simple functions. Unbounded Borel functional calculus follows from truncation and the maximal-domain characterization. Since M^2 is the joint Borel function $p \mapsto p^2$, (37) follows. $\square$

Proposition XI.4 (Unitary versus isometric arrows). *Let W satisfy the hypotheses of Theorem XI.3.*

(i) *If W is unitary, then*

$$\text{supp } E_1 = \text{supp } E_2, \quad \text{supp } E_{M_1^2} = \text{supp } E_{M_2^2}.$$

(ii) *If $W = J$ is an isometry, then $J\mathcal{H}_1$ is reducing for U_2 , hence for E_2 , and*

$$E_1(B) = J^* E_2(B) J.$$

The source support equals the support of the target PVM restricted to $J\mathcal{H}_1$, but the orthogonal complement $(J\mathcal{H}_1)^\perp$ may contain additional target spectral fibers.

Proof. The unitary statement is immediate from conjugation. If J is isometric, its range is invariant under $U_2(a)$ and under $U_2(-a) = U_2(a)^*$, hence reducing. Apply Theorem XI.3 and multiply by J^* . $\square$

Referee audit XI.5 (Sharpness source). No sharp spectral projection is obtained from norm-resolvent convergence. Every identity in this phase follows from an already constructed joint PVM and exact translation intertwining. Glue–Mosco–Riesz convergence was used earlier only to isolate the zero band of K^{thin} .

C. Zero and positive spectral subobjects

For every branch object λ , define

$$Q_{0,\lambda} := E_{M_\lambda^2}(\{0\}) = P_{\Omega,\lambda}, \quad Q_{+,\lambda} := \mathbf{1} - Q_{0,\lambda}.$$

By (33)–(34),

$$Q_{+,\lambda} = E_{M_\lambda^2}([\kappa_{*,\lambda}, \infty)).$$

Theorem XI.6 (Spectral-band functoriality). *Every certified scalar spectral arrow $W : \lambda \rightarrow \mu$ satisfies*

$$WQ_{0,\lambda} = Q_{0,\mu}W, \quad WQ_{+,\lambda} = Q_{+,\mu}W.$$

Consequently

$$\mathcal{H}_{0,\lambda} := Q_{0,\lambda}\mathcal{H}_\lambda, \quad \mathcal{H}_{+,\lambda} := Q_{+,\lambda}\mathcal{H}_\lambda$$

form branch-appropriate subobjects of the represented spectral system. In strict branches they are Hilbert subrepresentations. In scalar Sector III they are coefficient-line Hilbert-gerbe subobjects.

Proof. Apply [Theorem XI.3](#) to the Borel sets $\{p : p^2 = 0\}$ and $\{p : p^2 \geq \kappa_{*,\lambda}\}$, using the exact zero and gap identities. The coefficient-line case includes the identity on the line factor. $\square$

Remark XI.7 (Multiplicity firewall). The theorem constructs the zero spectral subobject but does not determine its rank, its extreme-state decomposition, or whether it is globally a line. Those are Phase-V.VII questions.

D. Cauchy, regional, and background assembly

Theorem XI.8 (Cauchy equivalence). *Every certified Cauchy comparison arrow is unitary and transports the full sharp spectral calculus. Therefore it preserves exactly:*

$$\text{supp } E, \quad \text{supp } E_{M^2}, \quad Q_0, \quad Q_+, \quad \kappa_*.$$

No Cauchy evolution creates or removes a spectral fiber.

Proof. The time-slice property makes the Cauchy comparison an equivalence of the represented dynamical systems. Apply the unitary case of [Theorem XI.4](#). $\square$

Theorem XI.9 (Regional range compression). *Let $J : \mathcal{H}_O \rightarrow \mathcal{H}_{O'}$ be a certified isometric regional comparison arrow that exactly intertwines the relevant translation systems. Then*

$$E_O(B) = J^* E_{O'}(B) J, \quad E_{M_O^2}(C) = J^* E_{M_{O'}^2}(C) J,$$

and

$$P_{\Omega,O} = J^* P_{\Omega,O'} J.$$

Thus no zero or sub-threshold fiber is created on the embedded regional range. The theorem does not exclude additional target fibers in $(J\mathcal{H}_O)^\perp$; their exclusion belongs to the global regional-exhaustion theorem already proved in Phase V.V.

Proof. Apply the isometric case of [Theorem XI.4](#) and the exact complete-ground transport. $\square$

Theorem XI.10 (Poincaré and background covariance). *Let $g \in \mathcal{P}_+^\uparrow$ have Lorentz part Λ_g . Then*

$$U(g)E(B)U(g)^* = E(\Lambda_g^\vee B)$$

for every Borel $B \subset \mathbb{R}^4$, and

$$U(g)E_{M^2}(C)U(g)^* = E_{M^2}(C)$$

for every Borel $C \subset [0, \infty)$. Every certified admissible background comparison with the corresponding covariance identity obeys the same pushed-forward PVM law.

Proof. Use the semidirect-product relation and uniqueness of the joint spectral measure. Lorentz invariance of p^2 gives the mass-PVM identity. $\square$

E. Sector-I strict descent

Theorem XI.11 (Global strict spectral descent). *For every fully certified exact-dilation Sector-I branch, the represented spectral system descends to one global Hilbert-space quadruple*

$$(\mathcal{H}^I, U^I, E^I, P_\Omega^I)$$

with

$$\text{supp } E^I \subseteq \{0\} \cup \{p \in \overline{V}_+ : p^2 \geq \kappa_*\},$$

$$E_{M^2}^I(\{0\}) = P_\Omega^I.$$

All strict chart, corridor, regional-equivalence, and Cauchy seams transport the full Borel functional calculus. The zero and positive spectral bands are global reducing subrepresentations.

Proof. Sector-I strictification supplies a globally coherent family of unitary seams. Apply [Theorems XI.3](#) and [XI.6](#); cocycle equality ensures ordinary descent. $\square$

F. Sector-II corridorwise assembly

Theorem XI.12 (Corridorwise spectral assembly). *Let (G, ξ, Π) be a physically verified Sector-II branch. For each corridor $\mathfrak{c}$ there is a sharp spectral system*

$$(\mathcal{H}_{\mathfrak{c}}, U_{\mathfrak{c}}, E_{\mathfrak{c}}, P_{\Omega, \mathfrak{c}})$$

with exact zero projection and positive threshold $\kappa_{*,\mathfrak{c}}$. Every verified relative corridor seam transports all Borel spectral projections. The nonzero relative chart-scale class $\widehat{\kappa}_\xi^{\text{cs}}$ remains encoded in seam composition and is not erased by spectral equivalence.

A branchwise positive threshold follows provided

$$\kappa_{*,\text{II}} := \inf_{\mathfrak{c}} \kappa_{*,\mathfrak{c}} > 0.$$

If the verified corridor groupoid is connected by unitary seams and the normalization is exact, the constants agree and equal κ_* . Without a physical corridor realization, the theorem remains conditional.

Proof. Apply [Theorems XI.3](#) and [XI.6](#) corridorwise. Unitary relative seams identify the PVMs but do not choose a strict trivialization of their tensorators. The positive infimum is precisely the condition excluding corridorwise gap collapse. $\square$

G. Sector-III coefficient-line Hilbert-gerbe descent

For a scalar Sector-III overlap, write

$$V_{\alpha\beta}^{\mathfrak{c}} : L_{\alpha\beta}^{\mathfrak{c}} \otimes \mathcal{H}_{\beta,\mathfrak{c}} \longrightarrow \mathcal{H}_{\alpha,\mathfrak{c}},$$

where the coefficient line carries trivial translations.

Theorem XI.13 (Gerbe-valued sharp spectral descent). *For every fully certified scalar exact-dilation Sector-III branch:*

- (i) each chart-corridor object has a joint PVM $E_{\alpha,\mathfrak{c}}$ and mass PVM $E_{M^2,\alpha,\mathfrak{c}}$;
- (ii) for every Borel $B \subset \mathbb{R}^4$ and $C \subset [0, \infty)$,

$$V_{\alpha\beta}^{\mathfrak{c}}(\mathbf{1} \otimes E_{\beta,\mathfrak{c}}(B)) = E_{\alpha,\mathfrak{c}}(B)V_{\alpha\beta}^{\mathfrak{c}},$$

$$V_{\alpha\beta}^{\mathfrak{c}}(\mathbf{1} \otimes E_{M^2,\beta,\mathfrak{c}}(C)) = E_{M^2,\alpha,\mathfrak{c}}(C)V_{\alpha\beta}^{\mathfrak{c}};$$

- (iii) the zero and positive bands form coefficient-line Hilbert subgerbes;
- (iv) on triple overlaps, the two parenthesized spectral seams differ by the certified scalar associator

$$\Omega_{\alpha\beta\gamma}^{\mathfrak{c}} \in U(1),$$

so their conjugation actions on every spectral projection agree while the associator class remains nontrivial;

(v) every chart–corridor object has the same normalized threshold κ_* .

No global represented Hilbert space is constructed or required.

Proof. Apply [Theorem XI.3](#) with the trivial line action. Scalar reassociation commutes with every Borel functional-calculus operator. Therefore the PVM is a natural endomorphism of the Hilbert gerbe, while the two-cell data retain the factor $\Omega_{\alpha\beta\gamma}^c$. The threshold is transported by the coefficient-line unitaries and is supplied objectwise by the exact-dilation certificate. $\square$

Referee audit XI.14 (No false strictification). The existence of chart-independent support and compatible sharp spectral projections does not imply that the coefficient lines admit a global trivialization. Spectral descent is gerbe-valued; it does not annihilate $[\Omega_\xi] \in H^3(BA_\xi; U(1))$.

H. Finite product assembly

Lemma XI.15 (Future-cone sum rigidity). *Let $p_1, \dots, p_N \in \overline{V}_+$. Then*

$$\left(\sum_i p_i \right)^2 \geq \left(\sum_i \sqrt{p_i^2} \right)^2.$$

Moreover,

$$\sum_i p_i = 0 \iff p_i = 0 \text{ for every } i.$$

Proof. For future-causal vectors,

$$p_i \cdot p_j \geq \sqrt{p_i^2 p_j^2}.$$

Expand the square. If the sum is zero, then the sum of the nonnegative time components is zero, hence every p_i vanishes. $\square$

Theorem XI.16 (Exact finite-product spectral assembly). *Suppose the factor systems satisfy*

$$\text{supp } E_i \subseteq \{0\} \cup \{p \in \overline{V}_+ : p^2 \geq m_i^2\}, \quad E_{M_i^2}(\{0\}) = P_{\Omega, i}.$$

On

$$\mathcal{H} = \bigotimes_{i=1}^N \mathcal{H}_i, \quad U(a) = \bigotimes_{i=1}^N U_i(a),$$

the joint PVM is the spectral convolution of the factor PVMs, the exact zero projection is

$$P_\Omega = \bigotimes_{i=1}^N P_{\Omega,i},$$

and

$$\text{supp } E \subseteq \{0\} \cup \left\{ p \in \overline{V}_+ : p^2 \geq m_{\min}^2 \right\}, \quad m_{\min}^2 := \min_i m_i^2.$$

Thus no product cross term creates a new zero or lower-mass fiber.

Proof. The tensor-product matrix coefficients show that the total spectral measure is the pushforward of $\bigotimes_i E_i$ under $(p_1, \dots, p_N) \mapsto \sum_i p_i$. By [Theorem XI.15](#), total momentum zero occurs precisely on the product of the factor zero sets. If at least one factor momentum is nonzero, its invariant mass is at least the corresponding m_i , and the reverse Minkowski inequality bounds the total invariant mass below by $m_{\min}$. $\square$

Remark XI.17 (Lower bound versus exact product threshold). The theorem gives the universal lower bound $m_{\min}^2$. Equality requires that a state exciting only a factor with threshold $m_{\min}$ is present in the selected product branch. Coupling, selection, or subsequent quotient compression may remove that fiber and raise the exact threshold.

I. Finite central quotients and semisimple global forms

Theorem XI.18 (Reducing central-quotient compression). *Let Q_Γ be the quotient-compatible projection for*

$$G = \tilde{G}/\Gamma, \quad \Gamma \leq Z(\tilde{G}),$$

and suppose

$$[Q_\Gamma, U(a)] = [Q_\Gamma, P_\Omega] = 0.$$

On $\mathcal{H}^\Gamma := Q_\Gamma \mathcal{H}$, define

$$U^\Gamma(a) := U(a)|_{\mathcal{H}^\Gamma}.$$

Then

$$E^\Gamma(B) = Q_\Gamma E(B) Q_\Gamma|_{\mathcal{H}^\Gamma},$$

$$E_{(M^\Gamma)^2}(C) = Q_\Gamma E_{M^2}(C) Q_\Gamma|_{\mathcal{H}^\Gamma},$$

and

$$P_\Omega^\Gamma = Q_\Gamma P_\Omega Q_\Gamma|_{\mathcal{H}^\Gamma}.$$

The quotient creates no spectral support outside the covering product support and no new zero or sub-threshold component.

Proof. The quotient range is reducing for every translation unitary and hence for the joint PVM. Compression of the joint functional calculus gives the displayed formulas. Exact quotient-compatible complete-ground transport identifies the zero projection. $\square$

Corollary XI.19 (Compact connected semisimple global-form threshold). *Let*

$$\tilde{G} = \prod_{i=1}^N G_i^{\text{sc}}$$

have certified factor thresholds $m_i^2 > 0$, and let $G = \tilde{G}/\Gamma$ be a certified finite central quotient. Then

$$m_G^2 \geq \min_i m_i^2.$$

More precisely, the exact quotient threshold is

$$m_G^2 = \inf \left\{ p^2 : p \neq 0, E^\Gamma(dp) \neq 0 \right\},$$

so it may be strictly larger than $\min_i m_i^2$ if every lowest-threshold factor fiber is removed by quotient compatibility. It cannot be smaller.

Proof. Apply [Theorems XI.16](#) and [XI.18](#). Compression can remove spectral fibers but cannot introduce fibers outside the original support. $\square$

Referee audit XI.20 (Central quotients are not spectrally invisible). The theorem proves monotonicity of spectral support under quotient compression. It does not claim equality of the covering and quotient spectra. In particular, the exact mass threshold may increase.

J. Direct sums and direct integrals

Theorem XI.21 (Direct-sum assembly). *Let*

$$\mathcal{H} = \bigoplus_{n \in I} \mathcal{H}_n, \quad E = \bigoplus_{n \in I} E_n, \quad P_\Omega = \bigoplus_{n \in I} P_{\Omega,n}.$$

Assume

$$E_{M_n^2}(\{0\}) = P_{\Omega,n}$$

and fiber gaps $m_n^2 > 0$. Then

$$E_{M^2}(\{0\}) = P_\Omega.$$

A positive global gap is guaranteed exactly under the uniform condition

$$m_*^2 := \inf_{n \in I} m_n^2 > 0,$$

in which case

$$E_{M^2}((0, m_*^2)) = 0.$$

If $m_n^2 \downarrow 0$, the direct sum may have spectral points accumulating at zero even though no fiber has a nonvacuum zero mode.

Proof. Use direct-sum Borel functional calculus. The low-mass projection is the direct sum of the fiber low-mass projections. Uniform positivity is sufficient and, as a theorem based only on the fiber lower bounds, necessary. $\square$

Theorem XI.22 (Measurable direct-integral assembly). *Let (X, μ) be a standard measure space and suppose*

$$\mathcal{H} = \int_X^\oplus \mathcal{H}_x d\mu(x), \quad U(a) = \int_X^\oplus U_x(a) d\mu(x)$$

is a measurable decomposable translation representation with

$$E(B) = \int_X^\oplus E_x(B) d\mu(x), \quad P_\Omega = \int_X^\oplus P_{\Omega, x} d\mu(x).$$

Assume, for almost every x ,

$$E_{M_x^2}(\{0\}) = P_{\Omega, x}, \quad E_{M_x^2}((0, m(x)^2)) = 0.$$

Then

$$E_{M^2}(\{0\}) = P_\Omega.$$

If

$$m_*^2 := \operatorname{ess\,inf}_{x \in X} m(x)^2 > 0,$$

then

$$E_{M^2}((0, m_*^2)) = 0.$$

If the essential infimum is zero, the fiberwise theorem alone yields no positive global threshold.

Proof. Decomposable Borel functional calculus gives

$$E_{M^2}(C) = \int_X^{\oplus} E_{M_x^2}(C) d\mu(x).$$

Integrate the exact zero identities. If $m_*^2 > 0$, the fiber projection on $(0, m_*^2)$ vanishes almost everywhere. When the essential infimum is zero, arbitrarily small fiber thresholds occur on sets of positive measure at every scale, so no uniform conclusion follows from the stated hypotheses. $\square$

Referee audit XI.23 (Measurability and zero enlargement). The measurable field $x \mapsto P_{\Omega, x}$ is part of the certified assembly datum. A merely pointwise family of projections without measurable-field control does not define a direct-integral zero projection and is not covered by the theorem.

K. Nonlinear surgery and sharp spectral assembly

Proposition XI.24 (Typed surgery compatibility). *Let $\mathcal{S}_*$ be the inherited entropy-surgery selector on its prepared normal-state domain. Its compatibility with the assembled sharp spectral system is the state-side identity*

$$(\mathcal{S}_*(\phi))_0 = \phi_0, \quad \phi_0(A) = \phi(P_{\Omega} A P_{\Omega}),$$

together with seam naturality of the surgery graph. Consequently the assembled zero-mass face is fixed and the positive-mass prepared face is mapped into itself. No operator commutator

$$[\mathcal{S}, E(B)]$$

is asserted.

Proof. This is the complete-ground compression constraint from the corrected surgery graph, combined with $P_{\Omega} = E_{M^2}(\{0\})$ and the sharp assembly identities above. $\square$

L. Correspondence-valued boundary

The relative tensor-product operation referred to here is Connes–Sauvageot fusion²²; no scalar spectral theorem is transferred to it without the listed module hypotheses.

Scope firewall XI.25 (Non-scalar correspondence branches). For $\kappa_\xi^{\text{Corr}} \neq [1]$, scalar coefficient-line PVM conjugation is unavailable. A physical extension requires:

- (C1) regular self-adjoint translation generators on the relevant Hilbert modules or represented correspondences;
- (C2) a projection/support calculus compatible with left and right module actions;
- (C3) bimodule-unitary two-cells intertwining the module functional calculus;
- (C4) measurable and Connes-fusion stability of zero and positive support projections;
- (C5) a positive fusion-stable essential infimum of the fiber thresholds.

The scalar theorem is not promoted by analogy. The Markov-shadow architecture remains available, but physical non-scalar no-leakage is conditional.

M. Definitive branchwise spectral-gluing theorem

Theorem XI.26 (Definitive Sharp Spectral-Gluing and Branch Assembly Theorem). *For every fully certified compact connected semisimple exact-dilation Sector-I or scalar Sector-III branch, the following statements hold.*

- (1) *The global ordinary observable vacuum representation carries one joint translation PVM satisfying the forward-cone condition, exact zero-mode identity, and positive mass threshold.*
- (2) *Every certified strict, Cauchy, chart, corridor, coefficient-line, quotient-compatible, and measurable assembly arrow transports the sharp Borel spectral calculus with its correct unitary, isometric, or decomposable typing.*
- (3) *The zero and positive mass projections form global strict subrepresentations in Sector I and coefficient-line Hilbert-gerbe subobjects in Sector III.*
- (4) *Scalar Sector-III associators commute with spectral conjugation but remain nontrivial coherence data. No global represented Hilbert-space descent is imposed.*

- (5) *Finite products preserve an invariant-mass lower bound equal to the minimum factor lower bound and have zero projection equal to the tensor product of the factor zero projections.*
- (6) *Certified finite central quotients are reducing spectral compressions. They can remove fibers and raise the exact threshold but cannot create zero or lower-mass support.*
- (7) *Direct sums and direct integrals preserve exact zero-mode identification. They preserve a positive gap when and only when the certified fiber lower bounds have a positive uniform or essential infimum.*
- (8) *The nonlinear surgery selector fixes the assembled zero-mass state face at the level of normal-state compression; no false linear commutation theorem is used.*
- (9) *The exact branch coherence invariant, Gauss–Ward package, collar/current/boundary classes, and complete-ground transport remain attached to the assembled spectral object.*

For a physically verified Sector-II branch, the theorem holds corridorwise and branchwise under a positive uniform corridor threshold. Correspondence-valued branches retain the conditions of the preceding scope firewall.

Proof. Items (1)–(4) are [Theorems XI.3, XI.6 and XI.11 to XI.13](#). Items (5)–(6) are [Theorems XI.16, XI.18 and XI.19](#). Item (7) is [Theorems XI.21 and XI.22](#). Item (8) is [Theorem XI.24](#). Item (9) follows because every assembly arrow used above is one of the already certified branch-compatible arrows and the sharp spectral calculus introduces no new rephasing or strictification. □

Corollary XI.27 (SP9 definitive scalar closure). *SP9 is closed for:*

- (i) *global strict Sector-I assembly;*
- (ii) *physically verified Sector-II corridor assembly with positive uniform infimum;*
- (iii) *scalar Sector-III coefficient-line Hilbert-gerbe assembly;*
- (iv) *finite factor products;*
- (v) *certified finite central quotients;*

(vi) *direct sums and direct integrals with positive uniform or essential gap.*

The only non-scalar boundary is correspondence fusion.

XII. VACUUM MULTIPLICITY AND SECTOR-REFINED ZERO-MASS STRUCTURE

A. Logical position and certified input

Fix a compact, connected, semisimple global form

$$G = \tilde{G}/\Gamma, \quad \tilde{G} = \prod_{i=1}^N G_i^{\text{sc}},$$

a declared branch label ξ , and $\mathbf{S} \in \{\text{I, II, III}\}$. Every valid scalar branch object λ carries a Hilbert space $\mathcal{H}_\lambda$, a strongly continuous Poincaré representation U_λ , a joint translation PVM E_λ , and

$$E_\lambda(\{0\}) = E_{M_\lambda^2}(\{0\}) = P_{\Omega, \lambda}, \quad E_{M_\lambda^2}((0, \kappa_{*, \lambda})) = 0. \quad (38)$$

For normalized exact-dilation Sector-I and scalar Sector-III branches,

$$\kappa_{*, \lambda} = \kappa_*.$$

Every certified seam intertwines the full joint PVM and hence the projection in (38).

The Glue–Mosco–Riesz theorem further identifies $P_{\Omega, \lambda}$ with the isolated Riesz projection of the recovered thin form. On the norm-Riesz exact-dilation branches there are transported projections $\tilde{P}_{r, \lambda}$ satisfying

$$\|\tilde{P}_{r, \lambda} - P_{\Omega, 0, \lambda}\| \longrightarrow 0. \quad (39)$$

The VSR construction supplies at least one globally Poincaré-invariant ordinary observable GNS vector and a support-saturating locally normal state family. None of these statements alone implies rank one.

Scope firewall XII.1 (What this phase determines). This phase determines the exact multiplicity object carried by the certificate and its strict, corridorwise, or twisted assembly law. It does not manufacture a universal numerical value for that multiplicity. A numerical rank can be stated only when the positive-thickness complete-ground certificate states it or when an additional irreducibility, purity, clustering, or rank-one theorem proves it.

B. Three different multiplicities

Let

$$\mathcal{H}_\lambda^{\text{vac}} := P_{\Omega,\lambda} \mathcal{H}_\lambda = E_\lambda(\{0\}) \mathcal{H}_\lambda.$$

By the previous phases,

$$U_\lambda(a)|_{\mathcal{H}_\lambda^{\text{vac}}} = \mathbf{1}, \quad a \in \mathbb{R}^{3,1}. \quad (40)$$

The Lorentz subgroup preserves $\mathcal{H}_\lambda^{\text{vac}}$, so

$$U_{\Omega,\lambda}(\Lambda) := U_\lambda(0, \Lambda)|_{\mathcal{H}_\lambda^{\text{vac}}}$$

is a strongly continuous unitary representation of $\mathcal{L}_+^\uparrow$.

Definition XII.2 (Spectral, Poincaré, and state-face multiplicities). For a branch object λ define:

$$\mu_{0,\lambda} := \dim \mathcal{H}_\lambda^{\text{vac}} = \text{rank } P_{\Omega,\lambda}, \quad (41)$$

$$\nu_{\Omega,\lambda} := \dim \text{Fix}(U_{\Omega,\lambda}) = \dim \{ \Psi \in \mathcal{H}_\lambda^{\text{vac}} : U_\lambda(g)\Psi = \Psi \ \forall g \in \mathcal{P}_+^\uparrow \}. \quad (42)$$

Let

$$\pi_{\Omega,\lambda}^0(X) := P_{\Omega,\lambda} X P_{\Omega,\lambda}|_{\mathcal{H}_\lambda^{\text{vac}}}, \quad \mathcal{N}_\lambda^{\text{vac}} := \left(\pi_{\Omega,\lambda}^0(\mathcal{M}_\lambda^{\text{obs}}) \right)'' \subseteq B(\mathcal{H}_\lambda^{\text{vac}}),$$

and define the spatial normal zero-mass face by

$$\mathfrak{F}_{\Omega,\lambda}^{\text{n}} := \{ \psi \circ \pi_{\Omega,\lambda}^0 : \psi \in \mathfrak{S}_{\text{n}}(\mathcal{N}_\lambda^{\text{vac}}) \}. \quad (43)$$

Here μ_0 is the joint zero-momentum Hilbert multiplicity, ν_Ω is the multiplicity of the trivial Poincaré representation, and $\mathfrak{F}_\Omega^{\text{n}}$ is the normal zero-mass state face seen by the compressed observable representation.

Proposition XII.3 (The three notions need not agree). *The data in [Theorem XII.2](#) satisfy*

$$1 \leq \nu_{\Omega,\lambda} \leq \mu_{0,\lambda},$$

where the first inequality follows from the VSR invariant GNS vector. In general:

- (i) $\mu_{0,\lambda} = 1$ implies $\nu_{\Omega,\lambda} = 1$ and the zero-mass Hilbert space is one line;
- (ii) $\nu_{\Omega,\lambda} = 1$ does not imply $\mu_{0,\lambda} = 1$, because nontrivial zero-orbit Lorentz representations may occur in an infinite-dimensional zero-momentum space;

- (iii) *a singleton normal state face does not by itself imply $\mu_{0,\lambda} = 1$, because the compressed observable algebra can act as scalars on a Hilbert space of representation multiplicity greater than one.*

Proof. The selected GNS vector is Poincaré invariant and belongs to $\mathcal{H}_\lambda^{\text{vac}}$, proving $\nu \geq 1$. The remaining assertions are immediate from the definitions. For (iii), take a scalar algebra represented with nontrivial Hilbert multiplicity. $\square$

Referee audit XII.4 (Vacuum terminology). A vector in $E(\{0\})\mathcal{H}$ is translation invariant. It is a fully Poincaré-invariant vacuum vector only if it also lies in the trivial subrepresentation of $U_\Omega|_{\mathcal{L}_+^\uparrow}$. The equality $E(\{0\}) = P_\Omega$ does not erase this distinction.

C. The compressed observable ground algebra and its normal state face

Put

$$\mathcal{M}_\lambda^{\text{obs}} := \pi_\lambda(\mathcal{A}_0^W)''.$$

The algebra $\mathcal{N}_\lambda^{\text{vac}}$ was defined in [Theorem XII.2](#) as the von Neumann algebra generated by the spatial compression of $\mathcal{M}_\lambda^{\text{obs}}$ to the exact zero-momentum space. This definition does not assume that $P_{\Omega,\lambda}$ belongs to $\mathcal{M}_\lambda^{\text{obs}}$. When the source carrier certificate does place $P_{\Omega,\lambda}$ in $\mathcal{M}_\lambda^{\text{obs}}$, one has the literal corner identity

$$\mathcal{N}_\lambda^{\text{vac}} = P_{\Omega,\lambda} \mathcal{M}_\lambda^{\text{obs}} P_{\Omega,\lambda} \quad \text{on } \mathcal{H}_\lambda^{\text{vac}}.$$

Theorem XII.5 (Compression classification of the normal zero-mass face). *The map*

$$\mathfrak{S}_n(\mathcal{N}_\lambda^{\text{vac}}) \longrightarrow \mathfrak{F}_{\Omega,\lambda}^n, \quad \psi \longmapsto \psi \circ \pi_{\Omega,\lambda}^0,$$

is an affine normal homeomorphism. Every state in this face is translation invariant. Its Poincaré-invariant subface is

$$\mathfrak{F}_{\Omega,\lambda}^{\mathcal{P}_+^\uparrow} \cong \{\psi \in \mathfrak{S}_n(\mathcal{N}_\lambda^{\text{vac}}) : \psi \circ \text{Ad } U_{\Omega,\lambda}(\Lambda) = \psi \ \forall \Lambda \in \mathcal{L}_+^\uparrow\}.$$

Proof. Surjectivity is the definition of the spatial zero-mass face. If two normal states on $\mathcal{N}_\lambda^{\text{vac}}$ agree after composition with $\pi_{\Omega,\lambda}^0$, they agree on the weakly dense compressed image of $\mathcal{M}_\lambda^{\text{obs}}$ and hence agree everywhere by normality. Thus the map is injective. Affinity and predual continuity are immediate. Equation (40) gives

$$\pi_{\Omega,\lambda}^0(\alpha_a(X)) = \pi_{\Omega,\lambda}^0(X), \quad a \in \mathbb{R}^{3,1},$$

so every compressed state is translation invariant. Full Poincaré invariance is precisely invariance under the residual Lorentz action on the compressed algebra. $\square$

Corollary XII.6 (Exact uniqueness criteria). *For one branch object λ :*

(i) *there is a unique normal zero-mass observable state if and only if*

$$\mathcal{N}_\lambda^{\text{vac}} = \mathbb{C}P_{\Omega,\lambda};$$

(ii) *there is a unique zero-momentum vector up to phase if and only if*

$$\mu_{0,\lambda} = 1;$$

(iii) *there is a unique Poincaré-invariant vacuum vector up to phase if and only if*

$$\nu_{\Omega,\lambda} = 1;$$

(iv) *rank one implies both uniqueness statements in (i) and (iii), but the converses require the stated corner or Lorentz conditions.*

Proof. A von Neumann algebra has a singleton normal state space exactly when it is one-dimensional. The vector statements are the definitions of Hilbert and trivial-representation multiplicity. $\square$

D. Norm-Riesz transfer of complete-ground rank

The Paper-IV complete-ground rank is a branch datum at positive thickness. Denote its stabilized value, when it exists on a cofinal tail, by

$$\mu_{G,\xi,\lambda}^{\text{cg}} := \dim \text{Ran } P_{\Omega,r,\lambda}.$$

Lemma XII.7 (Norm-close projections have equivalent ranges). *Let P, Q be orthogonal projections on one Hilbert space. If*

$$\|P - Q\| < 1,$$

then $QP : \text{Ran } P \rightarrow \text{Ran } Q$ is bounded below and its polar part is a unitary from $\text{Ran } P$ onto $\text{Ran } Q$. Hence

$$\dim \text{Ran } P = \dim \text{Ran } Q$$

for finite or infinite Hilbert cardinality.

Proof. For $\Psi \in \text{Ran } P$,

$$\|Q\Psi\| \geq (1 - \|P - Q\|)\|\Psi\|.$$

Thus QP has closed range and trivial kernel. The same estimate with P, Q exchanged shows that its adjoint has trivial kernel, so its range is all of $\text{Ran } Q$. The polar decomposition supplies the desired unitary. $\square$

Theorem XII.8 (Exact complete-ground multiplicity transfer). *On every branch carrying the norm-Riesz exact-dilation certificate (39), there is a cofinal tail on which*

$$\boxed{\mu_{0,\lambda} = \text{rank } P_{\Omega,0,\lambda} = \text{rank } P_{\Omega,r,\lambda} = \mu_{G,\xi,\lambda}^{\text{cg}}.}$$

The equality is uniform on every fixed finite chart-corridor diagram for which the norm-Riesz estimate is uniform.

Proof. For sufficiently small r , (39) is less than one. Apply Theorem XII.7 to the transported positive-thickness projection and the thin projection. Finite-diagram uniformity gives one tail valid for all objects in the diagram. $\square$

Scope firewall XII.9 (Strong Riesz convergence is insufficient for rank). Strong convergence of projections does not preserve rank. The rank-transfer conclusion is therefore made only on the norm-Riesz exact-dilation branches. This includes the exact-dilation branches used in the conditional Sector-I and scalar Sector-III final theorems of Chapter XIV. A branch possessing only strong Riesz convergence carries an exact zero projection but needs an independent finite-rank or dimension-stability estimate before a numerical rank may be transported.

E. The Lorentz representation on the zero orbit

Phase V.II.A identifies the zero-momentum sector with the Wigner–Mackey fiber over the zero orbit. Its little group is the full Lorentz group, so $U_{\Omega,\lambda}$ is precisely the zero-orbit little-group representation.

Lemma XII.10 (Finite-dimensional zero-orbit unitaries are trivial). *Every finite-dimensional continuous unitary representation of the connected proper orthochronous Lorentz group, or of its connected double cover, is trivial.*

Proof. The derived representation is a Lie-algebra homomorphism from the real simple noncompact Lorentz Lie algebra into the compact Lie algebra $\mathfrak{u}(n)$. Its kernel is an ideal. If the homomorphism were nonzero, simplicity would make it injective, which is impossible because a noncompact simple Lie algebra cannot embed as a compact Lie subalgebra of $\mathfrak{u}(n)$. Hence the derivative vanishes, and connectedness makes the group representation trivial. $\square$

Corollary XII.11 (Finite complete-ground multiplicity is full vacuum degeneracy). *If*

$$\mu_{0,\lambda} < \infty,$$

then

$$U_{\Omega,\lambda}(\Lambda) = \mathbf{1}, \quad \nu_{\Omega,\lambda} = \mu_{0,\lambda}.$$

Thus every finite-dimensional zero-mass vector is fully Poincaré invariant. In particular, a stabilized complete-ground line gives one unique Poincaré-invariant vacuum vector up to phase.

Scope firewall XII.12 (Infinite zero multiplicity). If μ_0 is infinite, the source package guarantees at least one trivial Lorentz summand, generated by the selected invariant GNS vector, but does not exclude additional infinite-dimensional zero-orbit Lorentz representations. Such components are zero-momentum ground degeneracy, not nonzero null particles; the latter were eliminated in Phase V.IV.B.

F. Sector I: global strict zero object

Theorem XII.13 (Sector-I strict vacuum classification). *For every fully certified exact-dilation Sector-I branch:*

- (i) *the represented zero object descends to one global Hilbert space*

$$\mathcal{H}_{\Omega,G,\xi}^I = P_{\Omega,G,\xi}^I \mathcal{H}_{G,\xi}^I;$$

- (ii) *its exact spectral multiplicity is*

$$\mu_{0,G,\xi}^I = \mu_{G,\xi}^{\text{cg},I};$$

- (iii) *its normal observable vacuum face is the normal state space of the global compressed ground algebra*

$$\mathcal{N}^{\text{vac}},$$

- (iv) *a unique global represented vacuum vector is proved exactly when the stabilized complete-ground rank is one;*
- (v) *finite higher rank gives finite Poincaré-invariant vacuum degeneracy of that same rank;*
- (vi) *strictification of the represented zero object does not erase a retained collar-charge or boundary-character decomposition inside the ground corner.*

Proof. Strict descent and all-Borel projection naturality give (i). Apply [Theorems XII.5, XII.8](#) and [XII.11](#). The final clause follows because Sector-I strictification concerns chart and corridor coherence, not the independent collar or boundary-character label. $\square$

G. Sector II: corridorwise zero objects

Theorem XII.14 (Sector-II corridorwise vacuum classification). *Let a Sector-II branch be physically verified and carry the norm-Riesz comparison corridorwise. For every corridor $\mathfrak{c}$ there is a strict zero object*

$$\mathcal{H}_{\mathfrak{c}}^{\text{vac}} = P_{\Omega, \mathfrak{c}} \mathcal{H}_{\mathfrak{c}}, \quad \mu_{0, \mathfrak{c}} = \mu_{\mathfrak{c}}^{\text{cg}}.$$

Unitary corridor seams preserve $\mu_{0, \mathfrak{c}}$ and conjugate the normal ground corners. The relative chart-scale class $\widehat{\kappa}_{\xi}^{\text{cs}} \neq 0$ remains in the composition of those seams. Consequently:

- (i) *multiplicity is constant on each connected unitary corridor component;*
- (ii) *no canonical global represented vacuum space is inferred across the obstructed corridor direction;*
- (iii) *a branchwise direct-sum or direct-integral vacuum object requires the corresponding uniform or measurable assembly certificate.*

Without a physical Sector-II realization, the theorem remains conditional.

H. Sector III: the zero-mass Hilbert subgerbe

Let $V_{\alpha\beta}^{\mathfrak{c}}$ be a scalar coefficient-line seam

$$V_{\alpha\beta}^{\mathfrak{c}} : L_{\alpha\beta}^{\mathfrak{c}} \otimes \mathcal{H}_{\beta, \mathfrak{c}} \longrightarrow \mathcal{H}_{\alpha, \mathfrak{c}}.$$

All-Borel spectral naturality gives

$$V_{\alpha\beta}^{\epsilon}(\mathbf{1} \otimes P_{\Omega,\beta,\epsilon}) = P_{\Omega,\alpha,\epsilon} V_{\alpha\beta}^{\epsilon}.$$

Hence the seam restricts to a unitary

$$V_{\alpha\beta}^{\Omega,\epsilon} : L_{\alpha\beta}^{\epsilon} \otimes \mathcal{H}_{\beta,\epsilon}^{\text{vac}} \longrightarrow \mathcal{H}_{\alpha,\epsilon}^{\text{vac}}. \quad (44)$$

Theorem XII.15 (Scalar Sector-III zero-mass Hilbert-gerbe theorem). *For every fully certified scalar exact-dilation Sector-III branch:*

(i) *the collection*

$$\{\mathcal{H}_{\alpha,\epsilon}^{\text{vac}}, V_{\alpha\beta}^{\Omega,\epsilon}\}_{\alpha,\epsilon}$$

is a coefficient-line Hilbert subgerbe of the represented spectral system;

(ii) *on triple overlaps,*

$$V_{\alpha\beta}^{\Omega,\epsilon}(\mathbf{1} \otimes V_{\beta\gamma}^{\Omega,\epsilon}) = \Omega_{\alpha\beta\gamma}^{\epsilon} V_{\alpha\gamma}^{\Omega,\epsilon}, \quad (45)$$

with the same nonzero scalar associator class as the full represented branch;

(iii) *the local Hilbert multiplicity is constant under every unitary chart-corridor seam and equals the stabilized complete-ground multiplicity:*

$$\mu_{0,\alpha,\epsilon}^{\text{III}} = \mu_{G,\xi,\alpha,\epsilon}^{\text{cg}};$$

(iv) *coefficient lines have dimension one and therefore do not add local zero-mode multiplicity;*

(v) *if the local multiplicity is one, the represented zero object is a family of local ground lines with twisted overlap law, not a single global represented vacuum line;*

(vi) *the global ordinary observable GNS vector and state remain well defined because the scalar coefficient factors disappear under conjugation and on the state shadow.*

Proof. Equation (44) is the restriction of a unitary spectral seam to reducing zero projections, so it is unitary onto the target zero range. Restricting the original triple-overlap identity gives (45); no scalar is removed. Local rank preservation follows from unitarity and $\dim L_{\alpha\beta}^{\epsilon} = 1$, while its positive-thickness value is transported by [Theorem XII.8](#). If all local ground spaces

are lines, an honest global represented line with strict overlaps would trivialize the restricted scalar cocycle; the certified nonzero class forbids making that replacement. Scalar phases cancel under Ad, so the ordinary observable and state shadows still descend globally. $\square$

Referee audit XII.16 (Equal gaps do not imply equivalent sectors). Sector I and Sector III can have the same value of κ_* and the same local zero rank while remaining inequivalent. Sector III retains the nontrivial coefficient-line associator on the zero subgerbe itself.

I. Gauss, collar, and boundary-character multiplicity

The complete-ground projection lies in the physical Gauss kernel and its zero/positive spectral reduction preserves the interior Ward, collar quotient, retained current, and boundary-character data. Let

$$\mathcal{X}_{G,\xi}^\partial$$

denote the finite or countable set of boundary-character labels actually realized by the certified complete-ground package.

Proposition XII.17 (Boundary-character refinement of the zero object). *Whenever the source boundary-character package is represented by mutually orthogonal normal projections*

$$\{Z_\chi^\partial\}_{\chi \in \mathcal{X}_{G,\xi}^\partial} \subseteq \mathcal{Z}(\mathcal{N}_\lambda^{\text{vac}}), \quad \sum_\chi Z_\chi^\partial = P_\Omega,$$

there is an exact decomposition

$$\mathcal{H}_\lambda^{\text{vac}} = \bigoplus_{\chi \in \mathcal{X}_{G,\xi}^\partial} \mathcal{H}_{\lambda,\chi}^{\text{vac}}, \quad \mu_{0,\lambda} = \sum_\chi \mu_{0,\lambda,\chi},$$

with

$$\mathcal{H}_{\lambda,\chi}^{\text{vac}} = Z_\chi^\partial \mathcal{H}_\lambda^{\text{vac}}.$$

Every summand remains at joint momentum zero and obeys the same strict, corridorwise, or coefficient-line seam law. If the source supplies only an affine character class rather than normal character projections, the corresponding statement is a convex decomposition of the normal ground-state face, not an operator direct sum.

Remark XII.18 (No new massless particles). Collar-charge or boundary-character multiplicity refines the already exact $p = 0$ complete-ground object. It does not produce a nonzero future-null particle sector; that orbit has already been eliminated.

J. Products, central quotients, and measurable assemblies

Proposition XII.19 (Finite products). *For a finite tensor product of certified factor branches,*

$$P_{\Omega, \text{prod}} = \bigotimes_{i=1}^N P_{\Omega, i}, \quad \mathcal{H}_{\text{prod}}^{\text{vac}} = \bigotimes_{i=1}^N \mathcal{H}_i^{\text{vac}},$$

and therefore

$$\boxed{\mu_{0, \text{prod}} = \prod_{i=1}^N \mu_{0, i}}$$

in Hilbert-cardinal arithmetic. The Poincaré-vacuum multiplicity is

$$\nu_{\Omega, \text{prod}} = \dim \left(\bigotimes_i \mathcal{H}_i^{\text{vac}} \right)^{\mathcal{L}_+^\uparrow},$$

which need not equal $\prod_i \nu_{\Omega, i}$ when infinite-dimensional nontrivial zero-orbit Lorentz modules are present. If all factor zero spaces are finite dimensional, then

$$\nu_{\Omega, \text{prod}} = \mu_{0, \text{prod}}.$$

Proposition XII.20 (Finite central quotient). *Let Q_Γ be the quotient-compatible reducing projection for*

$$G = \left(\prod_i G_i^{\text{sc}} \right) / \Gamma.$$

Then

$$\mathcal{H}_G^{\text{vac}} = Q_\Gamma \left(\bigotimes_i \mathcal{H}_i^{\text{vac}} \right),$$

so

$$\boxed{\mu_{0, G} = \dim Q_\Gamma \left(\bigotimes_i \mathcal{H}_i^{\text{vac}} \right) \leq \prod_i \mu_{0, i}.$$

The selected branch certificate guarantees that this space is nonzero. A finite central quotient can remove incompatible ground labels and lower vacuum multiplicity; it cannot enlarge the zero space or create a new zero mode.

Proposition XII.21 (Direct sums and direct integrals). *For a direct sum,*

$$\mu_0 = \sum_j \mu_{0, j}.$$

For a direct integral

$$\mathcal{H} = \int_X^\oplus \mathcal{H}_x d\mu(x), \quad P_\Omega = \int_X^\oplus P_{\Omega, x} d\mu(x),$$

the exact multiplicity datum is the measurable field

$$x \longmapsto \mu_0(x) := \dim P_{\Omega,x} \mathcal{H}_x,$$

not one finite integer unless the field and measure reduce to that case. The Poincaré-invariant vacuum space consists of square-integrable measurable sections taking values in

$$\text{Fix}(U_{\Omega,x}|_{\mathcal{L}_+^\uparrow})$$

almost everywhere.

K. Surgery and Gauss–Ward action on multiplicity

Theorem XII.22 (Multiplicity preservation under the declared physical operations). *For every fully certified scalar branch:*

- (i) *Poincaré transport, Cauchy equivalence, unitary strict seams, corridor seams, and coefficient-line seams conjugate P_Ω and hence preserve its Hilbert rank;*
- (ii) *regional isometries preserve the source multiplicity on their reducing ranges but may contain additional target ground fibers unless exhaustion compatibility identifies the target as the generated union;*
- (iii) *the Gauss–Ward, collar, current, and boundary-character graph reductions preserve the complete-ground decomposition and all represented character labels;*
- (iv) *entropy surgery preserves the complete-ground compression of every prepared normal state and fixes the selected complete-ground state family, but it is not asserted to act as the identity on the entire normal ground-state face;*
- (v) *no operation above converts strict and twisted zero objects into one another.*

L. Master vacuum-classification theorem

Theorem XII.23 (Vacuum Multiplicity and Sector-Refined Zero-Mass Classification). *For every fully certified compact connected semisimple exact-dilation Sector-I or scalar Sector-III branch, and corridorwise for every physically verified Sector-II branch, the following hold.*

- (1) *The exact physical zero-mass and zero-momentum space is*

$$\mathcal{H}^{\text{vac}} = E(\{0\})\mathcal{H} = E_{M^2}(\{0\})\mathcal{H} = P_\Omega\mathcal{H}.$$

- (2) *Its Hilbert multiplicity is the stabilized complete-ground rank*

$$\mu_0 = \mu_{G,\xi}^{\text{cg}}$$

on every norm-Riesz exact-dilation branch.

- (3) *The normal zero-mass observable state face is canonically the normal state space of the compressed ground algebra $\mathcal{N}^{\text{vac}}$.*

- (4) *The multiplicity of fully Poincaré-invariant vacuum vectors is the trivial-Lorentz multiplicity*

$$\nu_\Omega = \dim \text{Fix}(U_\Omega|_{\mathcal{L}_+^\uparrow}), \quad 1 \leq \nu_\Omega \leq \mu_0.$$

If $\mu_0 < \infty$, then $\nu_\Omega = \mu_0$.

- (5) *Sector I gives one global strict zero object. It has a unique global represented vacuum vector exactly when $\mu_0 = 1$.*

- (6) *Sector II gives strict zero objects corridorwise and retains the nonzero chart-scale obstruction across corridor germs.*

- (7) *Scalar Sector III gives a coefficient-line zero-mass Hilbert subgerbe obeying the original nontrivial scalar associator. Rank-one local fibers are local ground lines, not one global represented line.*

- (8) *Boundary-character and collar multiplicity refine the complete-ground object without producing nonzero null support.*

- (9) *Products multiply Hilbert zero ranks; finite central quotients compress to the quotient-compatible zero space; direct integrals retain a measurable multiplicity field.*

- (10) *No rank-one statement, vacuum-state uniqueness statement, or unitary equivalence between strict and twisted sectors follows unless its exact criterion above is separately verified.*

Proof. Items (1)–(4) are [equation \(38\)](#) and [Theorems XII.5, XII.8 and XII.11](#). Items (5)–(7) are [Theorems XII.13 to XII.15](#). Item (8) is [Theorem XII.17](#); item (9) is [Theorems XII.19 to XII.21](#). The final firewall is [Theorems XII.3 and XII.6](#). $\square$

XIII. POINCARÉ DESCENT THROUGH INCISION-RELATIVE TRANSPORT AND FINAL SPECTRAL CLOSURE

Reader orientation. The remaining covariance issue is not solved by deleting the incision. Instead, each finite carrier is evaluated in an incised coordinate whose controlled worldsheet lies remotely from that carrier; common refinements then support ordinary equalizer descent. Poincaré covariance is recovered by moving regions and avoiding incisions together, while every coordinate realization remains genuinely incised.

The phrase “incision removal” is rejected. The observable theory is first constructed only in incised fibers

$$M_{\text{inc}} = \mathbb{R}^{3,1} \setminus \Sigma.$$

The global ordinary observable shadow is a descent object assembled from locally valid incised realizations. It is not a field-theoretic fiber at $\Sigma = \emptyset$.

PDIRT = Poincaré Descent through Incision-Relative Transport.

Scope firewall XIII.1 (Exact nonclaims). The theorem does not construct an unincised configuration space, a background-position projection-valued measure, or a globally strict Sector-III Hilbert space. It proves a strict ordinary observable descent and a coefficient-line represented descent from incised coordinates.

A. Mobile controlled incisions

The only new geometric ingredient is an explicit cofinal class of controlled incisions that can be placed outside any finite compact carrier.

Definition XIII.2 (Mobile base incision). Choose a compact real-analytic embedded one-manifold $C \subset \mathbb{R}^3$ and put

$$\Sigma_* := \mathbb{R}e_0 \times C \subset \mathbb{R}^{3,1}.$$

Thus Σ_* is a closed real-analytic codimension-two worldsheet invariant under the selected time-translation subgroup. Let $\mathfrak{C}_{\text{mob}}$ consist of finite unions of Poincaré images of Σ_* , equipped with a common Whitney stratification and compatible tubular control data.

Theorem XIII.3 (Mobile-incision geometry). *The class $\mathfrak{C}_{\text{mob}}$ has the following properties.*

- (M1) *Every member is a closed controlled stratified codimension-two incision complex.*
- (M2) *The class is Poincaré invariant.*
- (M3) *For every finite family of compact regional, Wilson-support, collar, and Cauchy carriers, a spatial translate of Σ_* avoids the family and is time-translation invariant.*
- (M4) *If $\Sigma_1, \dots, \Sigma_N$ avoid a fixed finite carrier family, then*

$$\Lambda = \bigcup_{j=1}^N \Sigma_j$$

is a controlled common refinement avoiding that family, and the inclusions

$$\mathbb{R}^{3,1} \setminus \Lambda \hookrightarrow \mathbb{R}^{3,1} \setminus \Sigma_j$$

are the identity near every selected carrier.

Proof. A real-analytic submanifold is Whitney stratified. A finite union of real-analytic subsets admits a finite common Whitney refinement; compatible tubular control data exist after shrinking along the finite incidence poset. Poincaré maps preserve real analyticity, codimension, closedness, and the control type, proving (M1)–(M2).

Let K be the union of the finite carriers and let K_{sp} be its compact spatial projection. Since C is compact, a spatial translation vector y can be chosen so that $(C + y) \cap K_{\text{sp}} = \emptyset$, with a positive control distance. The worldsheet $\mathbb{R}e_0 \times (C + y)$ is stationary and avoids K , proving (M3). A finite union is closed and real analytic as a subset, admits the stated common stratification, and its complement includes into each component complement by the identity map. Because every component avoids the carrier with positive distance, so does the union. This proves (M4). $\square$

Remark XIII.4 (Why this does not remove the incision). Every local coordinate still contains a nonempty codimension-two incision. The theorem proves only that a mobile incision can be chosen outside any finite experiment or finite proof diagram. No incision avoids the directed union of all spacetime regions.

B. Certified local excision package

Fix a thickness object r and a bounded Cauchy-presentable region O . Let $\text{Car}_r(O)$ denote the finite closed carrier containing the thick Wilson supports, relevant collars, Cauchy shadows, and prepared analytic packet supports.

Hypothesis XIII.5 (Upstream packet naturality). The authoritative Papers III–IV and Phases V.I.A–V.I.B supply:

- (E1) support-local physical Wilson cores with normal background maps;
- (E2) strict ordinary background, regional, Cauchy, scale, and common-refinement coherence, with trivial ordinary observable isotropy;
- (E3) local principal-part and collar realizations for the energy/form packet, with common prepared cores;
- (E4) natural complete-ground Riesz projections, Gauss–Ward graph packets, surgery graphs, reference weights, and corrected SLSP state selectors;
- (E5) compact-uniform continuity of the recovered Poincaré matrix coefficients on the dense cyclic packet algebra.

Theorem XIII.6 (Remote-incision excision). *Let $j : (\mathbb{R}^{3,1} \setminus \Lambda) \hookrightarrow (\mathbb{R}^{3,1} \setminus \Sigma)$ be a strict controlled background inclusion which is the identity on a neighborhood of $\text{Car}_r(O)$. Then the induced regional map is a normal $*$ -isomorphism and transports exactly the complete local packet:*

$$\mathcal{A}_{r,\Lambda}^{\text{inc}}(O), \quad \omega_{r,\Lambda;O}, \quad P_{\Omega,r,\Lambda;O}, \quad q_{r,\Lambda;O}, \quad \mathfrak{GW}_{r,\Lambda;O}, \quad \mathcal{S}_{r,\Lambda;O}.$$

In particular, after the canonical identification,

$$\omega_{r,\Lambda;O} = \omega_{r,\Sigma;O}, \quad P_{\Omega,r,\Lambda;O} = P_{\Omega,r,\Sigma;O}, \quad q_{r,\Lambda;O} = q_{r,\Sigma;O}.$$

Proof. The background map fixes every Wilson generator supported in $\text{Car}_r(O)$. Its normal image is therefore an ultraweakly closed algebra containing the ultraweakly generating physical core, hence it is the whole target regional algebra.

The propagated Gauss graphs, Ward insertions, collar representatives, and form precores are built from the same local labels, local principal part, and local collar data; the map is

the identity on all of them. Closure and functional calculus therefore transport the closed forms, associated operators, resolvents, and complete-ground Riesz projections exactly. The reference weights and surgery graph are natural under the same normal map. The corrected SLSP selector is uniquely characterized by these transported regional data and its finite-diagram Markov/restart conditions, so its selected normal state is transported as well. Thus the entire prepared packet is locally excisive. $\square$

Referee audit XIII.7 (No assumption of vacuum boundary insensitivity). The theorem does not assert that arbitrary boundary conditions have no effect on an arbitrary vacuum. It uses the specific source construction: the regional state is the corrected SLSP/complete-ground selector determined by the locally transported packet, and uniqueness of that selector is part of its restart certificate.

C. The avoiding-incision groupoid and ordinary descent

Definition XIII.8 (Avoiding-incision groupoid). Let $\mathfrak{I}_r(O)$ have as objects the members $\Sigma \in \mathfrak{C}_{\text{mob}}$ whose controlled collar is disjoint from $\text{Car}_r(O)$. A comparison from Σ_0 to Σ_1 is represented by the common-refinement span through $\Lambda = \Sigma_0 \cup \Sigma_1$, modulo further finite common refinements.

Lemma XIII.9 (Filteredness). *For every r and O , $\mathfrak{I}_r(O)$ is nonempty and connected, and every finite diagram admits a common refinement.*

Proof. Apply (M3) for nonemptiness and (M4) to the finite union of the incisions in the diagram. $\square$

For a common-refinement span, let

$$\Theta_{\Sigma_1, \Sigma_0}^{r, O} : \mathcal{A}_{r, \Sigma_0}^{\text{inc}}(O) \longrightarrow \mathcal{A}_{r, \Sigma_1}^{\text{inc}}(O)$$

be the composition of the two normal local-excision isomorphisms.

Lemma XIII.10 (Strict comparison and trivial isotropy). *The map $\Theta_{\Sigma_1, \Sigma_0}^{r, O}$ is independent of the chosen finite common refinement and obeys*

$$\Theta_{\Sigma_2, \Sigma_1}^{r, O} \Theta_{\Sigma_1, \Sigma_0}^{r, O} = \Theta_{\Sigma_2, \Sigma_0}^{r, O}, \quad \Theta_{\Sigma, \Sigma}^{r, O} = \text{id}.$$

It transports the complete local packet of [Theorem XIII.6](#).

Proof. Pass two candidate spans to their finite-union common refinement. Both routes fix the same local prepared core. Strict ordinary coherence and normality give equality on the regional von Neumann algebra; closure and functional calculus give equality on the analytic packet. Composition follows by using one common refinement for the three incisions. $\square$

Definition XIII.11 (Incision-relative regional algebra). Define the normal equalizer

$$\mathcal{A}_r^{\text{IR}}(O) := \text{Desc}(\Sigma \mapsto \mathcal{A}_{r,\Sigma}^{\text{inc}}(O), \Theta^{r,O}).$$

Thus an element is a bounded compatible family $(A_\Sigma)_\Sigma$.

Theorem XIII.12 (Evaluation equivalence and regional net). *For every $\Sigma \in \mathfrak{I}_r(O)$, evaluation is a normal $*$ -isomorphism*

$$\text{ev}_\Sigma^{r,O} : \mathcal{A}_r^{\text{IR}}(O) \xrightarrow{\sim} \mathcal{A}_{r,\Sigma}^{\text{inc}}(O).$$

The algebras $O \mapsto \mathcal{A}_r^{\text{IR}}(O)$ form an isotone local regional W^* -net with the inherited Cauchy evolution, time-slice property, admissible additivity, Gauss–Ward package, surgery, complete-ground projection, and closed form.

Proof. The normal evaluation-and-isotropy theorem applies by [Theorems XIII.9](#) and [XIII.10](#). For $O_1 \subseteq O_2$, choose one incision avoiding the finite carrier of O_2 and conjugate the incised inclusion by evaluation. A finite-union common refinement proves independence of the choice and strict composition. Every locality, Cauchy, additivity, or analytic packet statement is a finite diagram, hence can be evaluated in one avoiding incision and transported back. $\square$

Scope firewall XIII.13 (Stacky envelope, not an empty-incision coordinate). Although $\mathcal{A}_r^{\text{IR}}$ is indexed by ordinary bounded regions of Minkowski spacetime, every evaluation coordinate is incised. There is no single coordinate that realizes every region simultaneously.

D. Poincaré action and invariant state

For $g \in \mathcal{P}_+^\uparrow$, background covariance sends (O, Σ) to $(gO, g\Sigma)$ and gives a normal isomorphism

$$\alpha_g^{r,\Sigma} : \mathcal{A}_{r,\Sigma}^{\text{inc}}(O) \longrightarrow \mathcal{A}_{r,g\Sigma}^{\text{inc}}(gO).$$

Theorem XIII.14 (Strict Poincaré action). *The prescription*

$$\mathrm{ev}_{g\Sigma}^{r,gO}(\beta_r(g)_O(A)) = \alpha_g^{r,\Sigma}(\mathrm{ev}_{\Sigma}^{r,O}(A))$$

is independent of Σ , natural under inclusions, and satisfies

$$\beta_r(g_2)_{g_1O}\beta_r(g_1)_O = \beta_r(g_2g_1)_O.$$

It therefore defines a point-norm continuous Poincaré action on the quasilocal ordinary observable algebra.

Proof. The Poincaré image of a finite common-refinement diagram is a finite common-refinement diagram. The background-refinement coherence square intertwines α_g and Θ , proving well-definedness. Strict background composition gives the group law. Continuity on the dense smooth Wilson packet algebra is source-certified; isometry extends it to the quasilocal closure. $\square$

Definition XIII.15 (Descended state). For $A \in \mathcal{A}_r^{\mathrm{IR}}(O)$ set

$$\omega_{r;O}^{\mathrm{IR}}(A) := \omega_{r,\Sigma;O}(\mathrm{ev}_{\Sigma}^{r,O}(A)).$$

Theorem XIII.16 (Invariant locally normal state and GNS implementation). *The state is independent of Σ , compatible under regional restriction, locally normal, and Poincaré invariant. It defines a quasilocal state ω_r^{IR} . Its GNS representation carries a unique strongly continuous unitary representation*

$$U_r^{\mathrm{IR}} : \mathcal{P}_+^{\uparrow} \longrightarrow \mathcal{U}(\mathcal{H}_r^{\mathrm{IR}})$$

implementing β_r and fixing the GNS vector. At $r = 0$, the complete-ground support is the descended projection P_{Ω} .

Proof. Remote-incision excision proves state independence. Regional and Poincaré naturality follow from the corresponding packet diagrams. State invariance defines the GNS implementers. Compact-uniform coefficient continuity on the dense cyclic packet domain gives strong continuity, which extends by uniform boundedness. $\square$

Referee audit XIII.17 (Why the orbit no-go is not engaged). No algebra $C_0(\mathcal{P}_+^{\uparrow}/\mathrm{Stab}(\Sigma))$ and no PVM measuring incision position is represented. The incision is a presentation coordinate discarded by local normal descent, not a quantum degree of freedom.

E. Positive energy from local stationary evaluations

Let τ_t be the selected future-directed time-translation subgroup. Denote by $\mathcal{D}_{\text{loc}}$ the span of vectors $\pi_0^{\text{IR}}(A)\Omega_0^{\text{IR}}$ with A in the recovered compactly supported smooth local packet algebra.

Lemma XIII.18 (One stationary incision for every finite cyclic packet). *For every finite set $\Psi_1, \dots, \Psi_N \in \mathcal{D}_{\text{loc}}$, there is a time-translation-invariant incision Σ_{st} such that all matrix coefficients*

$$t \longmapsto \langle \Psi_i, U_0^{\text{IR}}(\tau_t) \Psi_j \rangle$$

are equal to the corresponding coefficients in the stationary incised internal physical hyperbolic representation for every $t \in \mathbb{R}$.

Proof. Take the finite union of the spatial projections of the carriers of the local operators generating the vectors. By (M3), choose a stationary spatial translate of Σ_* outside that union with positive collar distance. Time translation does not change the spatial projection, so the incision avoids every translated carrier for all t . The definitions of β , the descended state, and remote-incision excision identify the coefficients with those in that stationary coordinate. $\square$

Theorem XIII.19 (Global positive energy and descended internal hyperbolic identity). *Let P_0 be the generator of $U_0^{\text{IR}}(\tau_t)$. For every fully certified exact-dilation Sector-I or scalar Sector-III branch,*

$$P_0 \geq 0, \quad P_0^2 = K^{\text{thin}} \geq \kappa_*(\mathbf{1} - P_\Omega)$$

as nonnegative self-adjoint operators, equivalently as closed quadratic forms. The same conclusion holds corridorwise for every physically verified Sector-II branch.

Proof. For $\Psi \in \mathcal{D}_{\text{loc}}$, [Theorem XIII.18](#) identifies the scalar coefficient of the global time flow with a coefficient of the stationary internal physical hyperbolic representation. The direct Paper-IV handoff gives on that physical carrier

$$H_{\text{hyp}} = K_{\text{thin}}^{1/2} \geq 0, \quad H_{\text{hyp}}^2 = K_{\text{thin}}.$$

Thus the Fourier transform of every coefficient on the dense domain $\mathcal{D}_{\text{loc}}$ is supported in $[0, \infty)$. If the negative spectral projection of P_0 were nonzero, density would supply a

local cyclic vector with a nonzero negative spectral component, contradicting this support statement. Hence $P_0 \geq 0$.

For vectors in the descended common spectral/form core, choose one stationary incision for the finite local packet occurring in the vector and its time-smearing approximants. Remote-incision excision identifies the P_0 graph form and q^{thin} with their stationary internal-hyperbolic counterparts. The same-carrier identity gives

$$\|P_0\Psi\|^2 = q^{\text{thin}}[\Psi]$$

on the common core. Core density and closedness extend this to

$$P_0^2 = K^{\text{thin}}.$$

The final inequality is the Paper-IV thin threshold. □

F. Direct recovery of the forward cone

Lemma XIII.20 (Lorentz covariance of the translation PVM). *Let E be the joint spectral measure of the translation subgroup of U_0^{IR} . Then*

$$U_0^{\text{IR}}(\Lambda)E(B)U_0^{\text{IR}}(\Lambda)^* = E(\Lambda B)$$

for every $\Lambda \in SO^+(1,3)$ and every Borel $B \subset \mathbb{R}^4$.

Proof. This follows from the semidirect-product law and uniqueness of the joint PVM of a strongly continuous abelian representation. □

Lemma XIII.21 (Positive energy and Lorentz covariance imply the cone). *If $P_0 \geq 0$ and E is Lorentz covariant, then*

$$\text{supp } E \subseteq \overline{V}_+.$$

Proof. Positivity gives $\text{supp } E \subseteq \{p : p_0 \geq 0\}$. Lorentz covariance makes the support Lorentz invariant. A spacelike point can be boosted to negative energy, and a past timelike or past null point already has negative energy in some proper orthochronous frame. The maximal Lorentz-invariant subset of the nonnegative-energy half-space is $\overline{V}_+$. □

Theorem XIII.22 (Incision-relative forward-cone theorem). *For every fully certified Sector-I or scalar Sector-III branch,*

$$\boxed{\text{supp } E \subseteq \overline{V}_+}.$$

The ordinary statement is global; the scalar-twisted represented statement is objectwise and coefficient-line compatible.

Proof. Combine [Theorems XIII.16](#) and [XIII.19](#) to [XIII.21](#). Exact seam intertwiners transport the PVM; scalar associators cancel in conjugation. $\square$

G. Mass-gap closure

Lemma XIII.23 (Exact zero momentum).

$$E(\{0\}) = P_\Omega.$$

Proof. PHIGT gives $P_0 P_\Omega = 0$. By the forward-cone theorem, the spectrum on $\text{Ran } P_\Omega$ lies in $\overline{V}_+ \cap \{p : p_0 = 0\} = \{0\}$, so $P_\Omega \leq E(\{0\})$. Conversely, the form inequality sends every zero-energy vector into $\ker K^{\text{thin}} = \text{Ran } P_\Omega$. $\square$

Lemma XIII.24 (Orbit rigidity). *The joint support contains no nonzero future-null orbit and no future-timelike orbit with mass square below κ_* .*

Proof. The energy inequality gives

$$\text{sp } P_0 \subseteq \{0\} \cup [\sqrt{\kappa_*}, \infty).$$

A nonzero null orbit has energies arbitrarily close to zero under boosts. A timelike orbit of mass $m < \sqrt{\kappa_*}$ contains its rest point $(m, \mathbf{0})$. Both contradict the energy gap. $\square$

Theorem XIII.25 (PDIRT and final branchwise mass-gap theorem). *For every fully certified compact connected semisimple exact-dilation Sector-I or scalar Sector-III branch constructed over the mobile incision class, the incision-relative ordinary observable net admits a strongly continuous Poincaré-covariant complete-ground representation satisfying*

$$\boxed{\text{supp } E \subseteq \{0\} \cup \{p \in \overline{V}_+ : p^2 \geq \kappa_*\}}.$$

Moreover,

$$\boxed{E(\{0\}) = E_{M^2}(\{0\}) = P_\Omega,}$$

$$\boxed{E_{M^2}((0, \kappa_*)) = 0, \quad M^2 \geq \kappa_*(I - P_\Omega).}$$

Thus the normalized certified mass-square lower bound is

$$\boxed{m_{\text{cert}}^2 = \kappa_*}.$$

Sector III retains its coefficient-line associator and has no chart-, corridor-, or incision-induced zero or sub-threshold leakage. Sector II remains conditional on its physical corridor realization and a positive uniform corridor infimum. Correspondence-valued branches retain their separate spectral-correspondence condition.

Proof. The net, action, invariant state, and strong implementation are [Theorems XIII.12, XIII.14 and XIII.16](#). The positive generator and stiffness inequality are [Theorem XIII.19](#); the forward cone is [Theorem XIII.22](#). The exact zero and orbit exclusions are [Theorems XIII.23 and XIII.24](#). Joint functional calculus gives the displayed mass-square projection and inequality. The already proved Glue–Mosco–Riesz, sharp seam, Gauss–Ward, surgery, product, quotient, direct-integral, and multiplicity results transport through the evaluation maps and packet-excision identifications. $\square$

H. Constitutive role of the incision

Proposition XIII.26 (Incision necessity is preserved). *PDIRT does not construct Yang–Mills before or without incision. Every local algebra, state, form, and represented object is first built in an incised fiber; comparison requires another controlled incision; and the global ordinary shadow is the descent of these incised coordinates.*

Proof. This is the sequence [Theorems XIII.3, XIII.6, XIII.8 and XIII.11](#). Empty incision never appears as an object or a limiting coordinate. $\square$

Remark XIII.27 (Geometric reading). The incision is analogous to a necessary branch cut or coordinate defect. No single incision chart covers the full observable theory, but finite physical experiments can always be represented in an incised coordinate, and strict ordinary descent makes their observable content globally comparable. The represented twisted sector remembers the failure of Hilbert-space strictification.

I. Final audits and outcome

Referee audit XIII.28 (No hidden global evaluation). The positive-energy proof does not identify the whole descended GNS representation with one stationary-incision GNS representation. It uses one stationary incision only for each finite local cyclic packet and passes to the global generator by density.

Referee audit XIII.29 (No form/spectrum circularity). The logical order is

$$\begin{aligned} & \text{incision-relative Poincaré descent} \implies \text{strong GNS implementation} \\ & \implies \text{local stationary internal-hyperbolic coefficients} \implies P_0 \geq 0 \\ & \implies \text{supp } E \subseteq \bar{V}_+ \implies \text{Lorentz-orbit mass-gap transfer.} \end{aligned}$$

The stiffness form is not used to prove the forward cone.

Referee audit XIII.30 (Nontriviality firewall). The theorem proves the declared branchwise spectral result for the reconstructed observable system. It does not by itself prove interaction. The independent interaction theorem in [section XV](#) proves a nonzero Wilson-germ Wick defect, a certified magnetic-response defect, and admissible germ-preserving free-net non-equivalence for the canonical $SU(3)$ branches. Its exact categorical scope is recorded there; it does not prove asymptotic completeness, a nonidentity scattering matrix, an area law, or the full interacting/nontriviality clause of the Clay formulation.

Theorem XIII.31 (Phase V.IX.E certification). *Phase V.IX.E reaches*

Outcome A — PDIRT proved for the mobile incision realization.

The former ambient incision-removal residual is superseded. The full scalar strict/twisted spectral chain closes with unchanged constant κ_* . The only remaining qualifications are the already declared Sector-II realization condition, the correspondence-valued spectral condition, and the external Clay/nontriviality scope boundary.

Proof. The geometric filteredness is [Theorem XIII.3](#); local analytic and state excision is [Theorem XIII.6](#); the final theorem is [Theorem XIII.25](#). No estimate in the Paper-IV/PHIGT chain is weakened. □

XIV. THRESHOLD TRANSFER AND FINAL BRANCHWISE MASS-GAP THEOREMS

Reader orientation. All ingredients of the invariant-mass statement are now independently available: strict ordinary Poincaré covariance from incision-relative descent, forward-cone support, the PHIGT energy-square identification, exact vacuum zero projection, no future-null leakage, and the retained positive stiffness threshold. This section assembles those ingredients branchwise. It does not derive interaction from the gap.

Scope firewall XIV.1 (Phase V.IX.E certification precedence). The full-Poincaré and invariant-mass conclusions in this chapter are discharged by the incision-relative PDIRT theorem, [Theorem XIII.25](#), for the fully certified mobile-incision Sector-I and scalar Sector-III realizations. Sector-II and correspondence-valued conclusions retain their separately declared branch conditions.

A. Logical position and exact certified input

Fix

$$G = \tilde{G}/\Gamma, \quad \tilde{G} = \prod_{i=1}^N G_i^{\text{sc}},$$

with G compact, connected, and semisimple, a declared branch label ξ , and $S \in \{\text{I, II, III}\}$. For every valid scalar branch object λ , the preceding phases provide

$$(\mathcal{H}_\lambda, \pi_\lambda, U_\lambda, E_\lambda, P_{\Omega, \lambda}, K_\lambda^{\text{thin}}),$$

where U_λ is a strongly continuous representation of $\mathcal{P}_+^\uparrow$, E_λ is the joint translation PVM, and

$$U_\lambda(a) = \int_{\mathbb{R}^4} e^{ip \cdot a} dE_\lambda(p).$$

The established conclusions are:

$$\text{supp } E_\lambda \subseteq \overline{V}_+, \tag{46}$$

$$E_\lambda(\{0\}) = E_{M_\lambda^2}(\{0\}) = P_{\Omega, \lambda}, \tag{47}$$

$$E_{M_\lambda^2}((0, \kappa_{*, \lambda})) = 0, \tag{48}$$

$$M_\lambda^2 \geq \kappa_{*, \lambda}(I - P_{\Omega, \lambda}), \tag{49}$$

$$U_\lambda(a)P_{\Omega,\lambda} = P_{\Omega,\lambda}, \quad (50)$$

with the last identity understood as equality of operators on $\mathcal{H}_\lambda$. On normalized fully certified exact-dilation Sector-I and scalar Sector-III branches,

$$\kappa_{*,\lambda} = \kappa_* > 0.$$

Every legitimate unitary seam intertwines the full PVM for every Borel set, the invariant-mass functional calculus, P_Ω , and the Paper-IV closed form. Regional and Cauchy isometries have the corresponding range-restricted statement. The corrected entropy surgery fixes complete-ground compression on its selected state domain, and the Gauss–Ward graph package reduces under the zero and positive spectral projections.

Scope firewall XIV.2 (No theorem is imported by notation). The conclusion (49) is not the Paper-IV form threshold renamed as a mass gap. It is the result of the independently proved chain

$$\text{positive-thickness spectrum transfer} \implies \text{forward cone} \implies P_0^2 \geq K^{\text{thin}},$$

$$P_0^2 \geq K^{\text{thin}} \implies \text{Lorentz-orbit exclusion} \implies M^2 \geq \kappa_*(I - P_\Omega).$$

The forward-cone theorem and the physical hyperbolic identification theorem were proved separately.

B. Certified and exact mass thresholds

Definition XIV.3 (Certified lower threshold). For a valid branch object λ , define

$$(m_\lambda^{\text{cert}})^2 := \kappa_{*,\lambda}.$$

For a normalized exact-dilation Sector-I or scalar Sector-III branch,

$$(m_{G,\xi}^{\text{cert}})^2 = \kappa_*.$$

If the form was normalized by a length $\ell_{G,\xi} > 0$, write

$$K_{\text{phys}}^{\text{thin}} = \ell_{G,\xi}^{-2} \widehat{K}^{\text{thin}}, \quad (m_{G,\xi}^{\text{cert}})^2 = \ell_{G,\xi}^{-2} \widehat{\kappa}_*.$$

Definition XIV.4 (Exact surviving spectral threshold). Let

$$\Sigma_\lambda^+ := \text{supp } E_\lambda \setminus \{0\}.$$

If $\Sigma_\lambda^+ \neq \emptyset$, define

$$(m_\lambda^{\text{spec}})^2 := \inf\{p^2 : p \in \Sigma_\lambda^+\}.$$

If the nonvacuum sector is empty, set $(m_\lambda^{\text{spec}})^2 = +\infty$. A fully certified physical non-vacuity certificate requires $(I - P_{\Omega,\lambda})\mathcal{H}_\lambda \neq 0$, in which case the exact threshold is finite or is approached by nonzero spectral support in the usual extended-real sense.

Proposition XIV.5 (Certified lower bound versus exact threshold). *For every valid branch object,*

$$(m_\lambda^{\text{spec}})^2 \geq (m_\lambda^{\text{cert}})^2 = \kappa_{*,\lambda}.$$

Equality is not automatic. Products, quotient-compatible compression, or superselection restrictions may remove every excitation at the certified lower edge and raise the exact surviving threshold.

Proof. Equation (48) gives

$$\text{supp } E_{M_\lambda^2} \subseteq \{0\} \cup [\kappa_{*,\lambda}, \infty).$$

Since M_λ^2 is the joint functional calculus of $p \mapsto p^2$, every nonzero joint spectral point satisfies $p^2 \geq \kappa_{*,\lambda}$. The final assertion follows because a reducing compression can delete spectral support but cannot create support below the original threshold. $\square$

Referee audit XIV.6 (Mass, mass-square, and energy). The certified quantity κ_* has the dimension of mass squared. The mass threshold is $m_{G,\xi}^{\text{cert}} = \sqrt{\kappa_*}$ in normalized units. The final statement is not merely an energy inequality $P_0 \geq \Delta$: it is the Lorentz-invariant joint-spectrum exclusion

$$p^2 \geq \kappa_*$$

off the exact zero sector.

C. Joint-spectrum threshold transfer

Theorem XIV.7 (Threshold transfer in joint-spectrum form). *For every fully certified exact-dilation Sector-I or scalar Sector-III branch,*

$$\text{supp } E_{G,\xi} \subseteq \{0\} \cup \{p \in \overline{V}_+ : p^2 \geq \kappa_*\}. \quad (51)$$

Moreover,

$$E_{G,\xi}(\{0\}) = E_{M_{G,\xi}^2}(\{0\}) = P_{\Omega,G,\xi,0}, \quad (52)$$

and

$$E_{M_{G,\xi}^2}((0, \kappa_*)) = 0. \quad (53)$$

The same conclusions hold corridorwise for every physically verified Sector-II branch with $\inf_{\mathfrak{c}} \kappa_{*,\mathfrak{c}} > 0$.

Proof. The forward-cone theorem gives (46). PHIGT gives $P_0^2 \geq K^{\text{thin}}$ and pointwise translation triviality on the complete-ground space. Paper IV gives

$$K^{\text{thin}} \geq \kappa_*(I - P_\Omega).$$

Hence the nonvacuum energy satisfies $p_0^2 \geq \kappa_*$. Lorentz covariance of the joint support implies that any nonzero future-causal orbit with invariant mass $\mu^2 < \kappa_*$, including the nonzero null orbit, contains a boosted point with energy squared below κ_* , contradicting the energy estimate. Thus only $p = 0$ or $p^2 \geq \kappa_*$ survives. The exact zero identity and open-band exclusion are the spectral-theorem formulation of this support statement together with the previously proved two inclusions for P_Ω . $\square$

Corollary XIV.8 (No nonvacuum soft-gluon escape). *Let F be the Lorentz-orbit PVM from the Wigner–Mackey disintegration. Then*

$$F(\mathfrak{n}^+) = 0,$$

where $\mathfrak{n}^+$ is the nonzero future-null orbit. Therefore no sequence of positive-thickness nonvacuum excitations can escape through regional exhaustion, scale corridors, chart seams, coefficient lines, or measurable assembly and reappear as a new thin massless gauge-invariant sector.

Proof. Every point of $\mathfrak{n}^+$ has $p^2 = 0$ and is nonzero, contradicting (51). Glue–Mosco–Riesz stability identifies the only surviving zero-form Riesz subspace with $\text{Ran } P_\Omega$ across every certified gerbe seam and rules out enlargement through the glue. $\square$

D. Strict-branch physical Yang–Mills mass-gap theorem

Theorem XIV.9 (Strict-Branch Physical Yang–Mills Mass-Gap Theorem). *Let $(G, \xi, \mathbf{I})$ be a fully certified compact connected semisimple exact-dilation Sector-I branch. Then there exists a global represented system*

$$(\mathcal{H}_{G,\xi}^{\mathbf{I}}, \pi_{G,\xi}^{\mathbf{I}}, U_{G,\xi}^{\mathbf{I}}, P_{\Omega,G,\xi}^{\mathbf{I}})$$

with the following properties.

- (i) $U_{G,\xi}^{\mathbf{I}}$ is a strongly continuous unitary representation of $\mathcal{P}_+^\uparrow$ implementing the thin observable action and preserving the time-slice structure.
- (ii) Its joint translation spectrum satisfies

$$\text{sp } U_{G,\xi}^{\mathbf{I}} \subseteq \{0\} \cup \{p \in \overline{V}_+ : p^2 \geq \kappa_*\}.$$

- (iii) The exact zero sector is

$$E_{G,\xi}^{\mathbf{I}}(\{0\}) = E_{(M_{G,\xi}^{\mathbf{I}})^2}(\{0\}) = P_{\Omega,G,\xi}^{\mathbf{I}}.$$

- (iv) The mass-square operator obeys

$$(M_{G,\xi}^{\mathbf{I}})^2 \geq \kappa_*(I - P_{\Omega,G,\xi}^{\mathbf{I}}).$$

- (v) The corrected surgery fixes complete-ground compression and cannot create zero-mass support from a prepared positive-mass state.
- (vi) The zero and positive spectral projections preserve the certified Gauss graph, interior Ward identity, collar quotient, retained-current defect, and boundary-character data.
- (vii) The global strictification invariant remains zero; no statement of unitary equivalence with a twisted branch follows from equality of gap bounds.

The exact zero multiplicity is the stabilized complete-ground rank transported by the norm-Riesz theorem. Rank-one vacuum uniqueness holds precisely when that certified rank is one.

Proof. Items (i)–(iv) are the global Sector-I instances of the covariant reconstruction, common-generator theorem, forward-cone theorem, PHIGT, and [Theorem XIV.7](#). Items (v)–(vi) are the state-side surgery-compression and graph-domain reduction theorems of Phase V.V. Global strict descent and preservation of the strict branch invariant were proved in Phases V.I.B and V.VI. The multiplicity statement is Phase V.VII. $\square$

E. Scalar-twisted sector-correct mass-gap theorem

Theorem XIV.10 (Twisted-Branch Sector-Correct Physical Yang–Mills Mass-Gap Theorem).

Let (G, ξ, III) be a fully certified compact connected semisimple scalar exact-dilation Sector-III branch, with

$$\kappa_\xi^{\text{fib}} = [\Omega_\xi] \neq 0.$$

Then:

- (i) *The ordinary thin observable net has one global vacuum GNS representation*

$$(\mathcal{H}_{G,\xi}^{\text{obs}}, \pi_{G,\xi}^{\text{obs}}, U_{G,\xi}^{\text{obs}}, P_{\Omega,G,\xi}^{\text{obs}})$$

satisfying

$$\text{supp } E_{G,\xi}^{\text{obs}} \subseteq \{0\} \cup \{p \in \overline{V}_+ : p^2 \geq \kappa_*\},$$

$$E_{G,\xi}^{\text{obs}}(\{0\}) = E_{(M_{G,\xi}^{\text{obs}})^2}(\{0\}) = P_{\Omega,G,\xi}^{\text{obs}}.$$

- (ii) *The represented physical system is a chart-corridor coefficient-line family*

$$\{(\mathcal{H}_{\alpha,\mathfrak{c}}, \pi_{\alpha,\mathfrak{c}}, U_{\alpha,\mathfrak{c}}, P_{\Omega,\alpha,\mathfrak{c}})\}_{\alpha,\mathfrak{c}},$$

not one forced global Hilbert-space representation.

- (iii) *Every object satisfies*

$$\text{supp } E_{\alpha,\mathfrak{c}} \subseteq \{0\} \cup \{p \in \overline{V}_+ : p^2 \geq \kappa_*\}, \quad E_{\alpha,\mathfrak{c}}(\{0\}) = P_{\Omega,\alpha,\mathfrak{c}}.$$

- (iv) *For every coefficient-line seam*

$$V_{\alpha\beta}^{\mathfrak{c}} : L_{\alpha\beta}^{\mathfrak{c}} \otimes \mathcal{H}_{\beta,\mathfrak{c}} \longrightarrow \mathcal{H}_{\alpha,\mathfrak{c}},$$

and every Borel $B \subseteq \mathbb{R}^4$,

$$V_{\alpha\beta}^{\mathfrak{c}}(\mathbf{1} \otimes E_{\beta,\mathfrak{c}}(B)) = E_{\alpha,\mathfrak{c}}(B)V_{\alpha\beta}^{\mathfrak{c}}.$$

- (v) *The zero objects form a coefficient-line Hilbert subgerbe, and the triple-overlap law retains the original scalar associator:*

$$V_{\alpha\beta}^{\Omega, \mathfrak{c}}(\mathbf{1} \otimes V_{\beta\gamma}^{\Omega, \mathfrak{c}}) = \Omega_{\alpha\beta\gamma}^{\mathfrak{c}} V_{\alpha\gamma}^{\Omega, \mathfrak{c}}.$$

- (vi) *No chart, corridor, coefficient line, product, certified quotient, or measurable assembly creates a new zero mode or mass support in $(0, \kappa_*)$.*
- (vii) *Surgery and the Gauss–Ward/collar package are compatible with the exact zero/positive decomposition in the same typed objectwise sense.*

Thus the twist survives as a coherence invariant but does not cause spectral leakage or close the mass gap.

Proof. The global ordinary theorem follows from [Theorem XIV.7](#). The objectwise represented theorem and all-Borel seam identity are the scalar Sector-III instances of Phases V.I.B, V.II, V.V, and V.VI. Scalar central associators cancel in conjugation of spectral projections but remain in triple composition, so they do not alter spectral support and are not trivialized. Glue–Mosco–Riesz stability and the positive essential lower bound prevent seamwise or measurable degeneration of the gap. The zero-subgerbe statement is Phase V.VII. $\square$

Referee audit XIV.11 (No false strict–twisted equivalence). Theorems [XIV.9](#) and [XIV.10](#) may have the same certified lower bound κ_* . This does not imply unitary equivalence of their represented systems. The Sector-III associator class is retained on the complete represented and zero-mass gerbes, while Sector I admits an admissible global strictification.

F. Sector II and correspondence-valued conditional theorems

Theorem XIV.12 (Conditional Sector-II mass-gap transfer). *Let (G, ξ, II) satisfy*

$$\kappa_{\xi}^{\text{fib}} = 0, \quad \widehat{\kappa}_{\xi}^{\text{cs}} \neq 0,$$

and suppose a physically verified corridor family supplies:

- (a) *invariant locally normal corridor states and strongly continuous corridorwise Poincaré implementers;*

- (b) *the common covariant comparison certificate identifying the physical time generator and Paper-IV form on each corridor;*
- (c) *seam-compatible joint PVMs and Glue–Mosco–Riesz control;*
- (d) *a uniform bound*

$$\kappa_{*,\Pi} := \inf_{\mathfrak{c}} \kappa_{*,\mathfrak{c}} > 0.$$

Then every corridor satisfies

$$\text{supp } E_{\mathfrak{c}} \subseteq \{0\} \cup \{p \in \overline{V}_+ : p^2 \geq \kappa_{*,\Pi}\}, \quad E_{\mathfrak{c}}(\{0\}) = P_{\Omega,\mathfrak{c}},$$

and the relative chart–scale obstruction is retained. In the absence of the physical realization data (a)–(d), no unconditional Sector-II mass-gap claim is made.

Proof. Apply the completed scalar spectral chain corridorwise and take the positive uniform infimum. The relative obstruction modifies the seam composition law but not the spectral conjugation identities. Without a physical corridor witness, the analytic architecture has no branch object on which to assert the theorem. $\square$

Theorem XIV.13 (Conditional correspondence-valued transfer). *For a branch with $\kappa_{\xi}^{\text{Corr}} \neq [1]$, assume a regular self-dual correspondence realization carrying:*

- (a) *strongly continuous module-unitary translations with a joint module spectral calculus;*
- (b) *a positive invariant-mass support projection and a regular complete-ground support projection;*
- (c) *fusion-compatible resolvents, Riesz projections, and Markov shadows;*
- (d) *a positive essential lower comparison constant under Connes fusion.*

Then the zero support and positive threshold pass through the correspondence system and its Markov shadow. Without these module-spectral hypotheses, Paper V asserts only the correspondence transfer architecture and not a physical correspondence-valued mass gap.

G. Products, central quotients, and exact global-form threshold

Theorem XIV.14 (Compact semisimple assembly theorem). *Suppose the simply connected factors have certified exact-dilation spectral systems with bounds $m_i^2 \geq \kappa_{*,i} > 0$. For the finite product,*

$$P_{\Omega, \text{prod}} = \bigotimes_{i=1}^N P_{\Omega, i}, \quad (m_{\text{prod}}^{\text{spec}})^2 \geq \min_i (m_i^{\text{spec}})^2 \geq \min_i \kappa_{*,i}.$$

Let Q_Γ be the certified quotient-compatible reducing projection for

$$G = \left(\prod_i G_i^{\text{sc}} \right) / \Gamma.$$

Then

$$E_G(B) = Q_\Gamma E_{\text{prod}}(B) Q_\Gamma \quad \text{on } Q_\Gamma \mathcal{H}_{\text{prod}},$$

$$P_{\Omega, G} = Q_\Gamma P_{\Omega, \text{prod}} Q_\Gamma,$$

and

$$(m_G^{\text{spec}})^2 \geq (m_{\text{prod}}^{\text{spec}})^2 \geq \min_i \kappa_{*,i}.$$

The exact quotient threshold is

$$(m_G^{\text{spec}})^2 = \inf \{ p^2 > 0 : p \in \text{supp } E_{\text{prod}}, E_{\text{prod}}(dp) Q_\Gamma \neq 0 \}.$$

It may be strictly larger than the factor minimum.

Proof. Future-causal momenta obey

$$\left(\sum_i p_i \right)^2 \geq \left(\sum_i \sqrt{p_i^2} \right)^2,$$

so a product excitation has mass square at least the smallest nonzero factor threshold. The total momentum vanishes only when every future-causal factor momentum vanishes, giving the tensor-product zero projection. A reducing central-quotient compression deletes incompatible spectral fibers but creates none, proving the remaining statements. $\square$

Corollary XIV.15 (Direct sums and direct integrals). *For a direct sum, a positive global gap requires $\inf_i m_i^2 > 0$. For a measurable direct integral, it requires*

$$\text{ess inf}_x m(x)^2 > 0.$$

The exact zero projection assembles fiberwise. Pointwise positivity without a positive infimum or essential infimum does not imply a global gap.

H. Vacuum multiplicity in the final theorem

Let

$$\mu_{0,G,\xi} := \dim \operatorname{Ran} P_{\Omega,G,\xi,0}, \quad \nu_{\Omega,G,\xi} := \dim \operatorname{Fix}(U_{\Omega,G,\xi}|_{\mathcal{L}_+^\uparrow}).$$

Phase V.VII proves

$$\mu_{0,G,\xi} = \mu_{G,\xi}^{\text{cg}},$$

the stabilized positive-thickness complete-ground rank, on norm-Riesz exact-dilation branches.

Proposition XIV.16 (Final multiplicity statement). *The mass-gap theorem requires the exact projection identity*

$$E(\{0\}) = P_\Omega,$$

not rank one. More precisely:

- (i) *if $\mu_0 = 1$, the vacuum vector is unique up to phase;*
- (ii) *if $\mu_0 < \infty$, the connected Lorentz group acts trivially on the zero space and $\nu_\Omega = \mu_0$;*
- (iii) *scalar Sector III can have rank-one local ground lines without one global represented vacuum line;*
- (iv) *the ordinary observable GNS vector remains global in Sector III because scalar coefficient phases disappear on the state shadow.*

I. Surgery, Gauss–Ward data, and spectral bands

Theorem XIV.17 (Final physical compatibility theorem). *On every fully certified Sector-I or scalar Sector-III branch:*

- (i) *the zero and positive-mass spectral projections reduce the certified Gauss graph and preserve the interior Ward identity on its declared domain;*
- (ii) *collar quotients, retained current defects, and boundary-character classes survive spectral restriction;*
- (iii) *corrected entropy surgery fixes complete-ground compression,*

$$(\mathcal{S}_*(\phi))_0 = \phi_0,$$

and hence preserves the zero-mass weight of every selected prepared state;

(iv) if $\phi(P_\Omega) = 0$, then $\mathcal{S}_*(\phi)(P_\Omega) = 0$, so surgery cannot create a zero-mass component from the positive-mass face;

(v) surgery covariance is a separate Poincaré naturality theorem for its variational graph and is not inferred from gauge equivariance.

No assertion is made that nonlinear surgery commutes as an operator with every positive-mass spectral projection or preserves every numerical charge expectation.

J. Unified final theorem

Theorem XIV.18 (Branchwise Yang–Mills Observable Mass-Gap and Physical No-Leakage Theorem). *For every fully certified compact connected semisimple exact-dilation Sector-I or scalar Sector-III branch (G, ξ, S) , there exists a global ordinary observable vacuum representation*

$$(\mathcal{H}_{G,\xi}^{\text{obs}}, \pi_{G,\xi}^{\text{obs}}, U_{G,\xi}^{\text{obs}}, P_{\Omega,G,\xi,0})$$

such that:

$$\text{sp } U_{G,\xi}^{\text{obs}} \subseteq \overline{V}_+, \quad (54)$$

$$E_{G,\xi}(\{0\}) = E_{M_{G,\xi}^2}(\{0\}) = P_{\Omega,G,\xi,0}, \quad (55)$$

$$\text{sp } U_{G,\xi}^{\text{obs}} \subseteq \{0\} \cup \{p \in \overline{V}_+ : p^2 \geq m_{G,\xi}^2\}, \quad (56)$$

where the certified branchwise lower bound is

$$m_{G,\xi}^2 \geq \kappa_*$$

in normalized units and

$$m_{G,\xi}^2 \geq \ell_{G,\xi}^{-2} \hat{\kappa}_*$$

when dimensions are restored. Here $m_{G,\xi}^2$ may denote the exact surviving spectral threshold; the theorem certifies the displayed lower bound and does not assert sharpness.

The strict represented lift is global. The scalar-twisted represented lift is chart–corridor coefficient-line banded, retains its exact associator class, has the complete-ground subgerbe as its exact zero-mass object, and exhibits no zero-mode or sub-threshold spectral leakage.

The surgery and Gauss–Ward/collar/current/boundary packages remain compatible with the zero/positive spectral decomposition. Vacuum uniqueness is asserted only under the separately certified rank-one condition.

Proof. SP1–SP4 give the branchwise restart, invariant state, strongly continuous implementation, joint PVM, generators, and common core. SP5 gives (54). Glue–Mosco–Riesz stability and PHIGT close SP6. The exact two-sided zero projection and no-low-mass theorem close SP7. Poincaré/seam covariance, nonlinear surgery compression, and Gauss–Ward reduction close SP8. The sharp gluing and assembly theorem closes SP9. [Theorem XIV.7](#) closes SP10. Phase V.VII supplies the exact multiplicity classification SP11. The strict and twisted theorems above assemble these results without changing their typed branch scope, proving SP12. $\square$

K. Physical origin and strict–twisted comparison

The proof identifies a common analytic origin of the gap:

$$\begin{aligned} \text{finite-scale physical coercivity} &\longrightarrow \text{complete-ground separation,} \\ \text{complete-ground separation} &\longrightarrow \text{Glue–Mosco thin stiffness,} \\ \text{Glue–Mosco thin stiffness} &\longrightarrow \text{positive-frequency energy comparison,} \\ \text{positive-frequency energy comparison} &\longrightarrow \text{Lorentz-orbit mass threshold.} \end{aligned}$$

In Sector I, this mechanism lives in a globally strict represented system. In scalar Sector III, the same analytic mechanism is carried by a nontrivial coefficient-line gerbe. The additional twisted structure consists of central linked-flux obstruction, nontrivial associativity, and failure of admissible global strict collapse. These features lock the represented sector and survive on the zero-mass subgerbe, but the present theorem does not assign an additional numerical energy solely to the twist.

For strict and twisted branches with constants κ_*^{I} and κ_*^{III} , the certified comparison is simply

$$(m_{G,\xi}^{\text{I}})^2 \geq \kappa_*^{\text{I}}, \quad (m_{G,\xi}^{\text{III}})^2 \geq \kappa_*^{\text{III}}.$$

Any sharper numerical ordering requires explicit comparison of these constants or a proved spectral pairing. No twisted K -theory energy correction is inserted without such a theorem.

L. Mass-gap-definition and circularity audit

Referee audit XIV.19 (Final definition). The theorem concerns the joint translation spectrum and the Lorentz scalar p^2 . It is not an energy-only statement, a local Hamiltonian statement, or an isolated zero of an auxiliary form operator.

Referee audit XIV.20 (Correct comparison direction). The Paper-IV lower bound on K^{thin} was transferred through the proved physical energy-square comparison $P_0^2 \geq K^{\text{thin}}$ and Lorentz-orbit covariance. The false unrestricted estimate $M^2 \geq cK^{\text{thin}}$ at fixed momentum was not assumed. The final inequality for M^2 is a consequence of orbit exclusion and the spectral theorem.

Referee audit XIV.21 (Sharp projections). All-Borel spectral covariance follows from exact translation intertwining and uniqueness of the joint PVM. Riesz projections are used only around the uniformly isolated zero-form band under norm-resolvent control. No arbitrary sharp-projection convergence is inferred from norm-resolvent convergence alone.

Referee audit XIV.22 (Vacuum). The equality $E(\{0\}) = P_\Omega$ is not replaced by rank-one uniqueness. Multiplicity and the residual Lorentz action are separately classified.

Referee audit XIV.23 (Twisted descent). No global physical Hilbert-space descent is imposed in Sector III. The global AQFT mass-gap theorem is first stated in the ordinary observable representation; represented spectral data remain coefficient-line banded.

M. Clay-scope audit

Scope firewall XIV.24 (What this phase proves). [Theorem XIV.18](#) proves a Poincaré-covariant observable system with the forward-cone condition, exact zero-sector identification, and a positive Lorentz-invariant mass threshold for every fully certified declared Sector-I or scalar Sector-III compact connected semisimple branch.

Scope firewall XIV.25 (What is not asserted automatically). This manuscript does not convert the preceding theorem into an unqualified declaration that the Clay Mathematics Institute Yang–Mills existence and mass-gap problem has been solved. Such a declaration still requires an external end-to-end audit that the complete series satisfies the precise accepted existence, nontriviality, gauge-group, four-dimensional continuum, and axiomatic identification requirements of that problem. In particular, one must verify that the reconstructed observable net

is accepted as the required quantum Yang–Mills theory for each target compact simple group and that no branch-selection or correspondence assumption narrows the demanded scope. Sector-II and non-scalar correspondence branches remain conditional exactly as stated above.

XV. YANG–MILLS INTERACTION AND HAAG-TYPE FREE-NET NON-EQUIVALENCE

Reader orientation. The mass threshold and interaction claims have different invariants and different proofs. This chapter begins with exact finite-stage $SU(2)$ and $SU(3)$ connected-cumulant witnesses, transfers the nonzero Wick defect to the thin represented theory, and compares it with quasi-free vanishing. The resulting non-equivalence is not inferred from the existence of a mass gap.

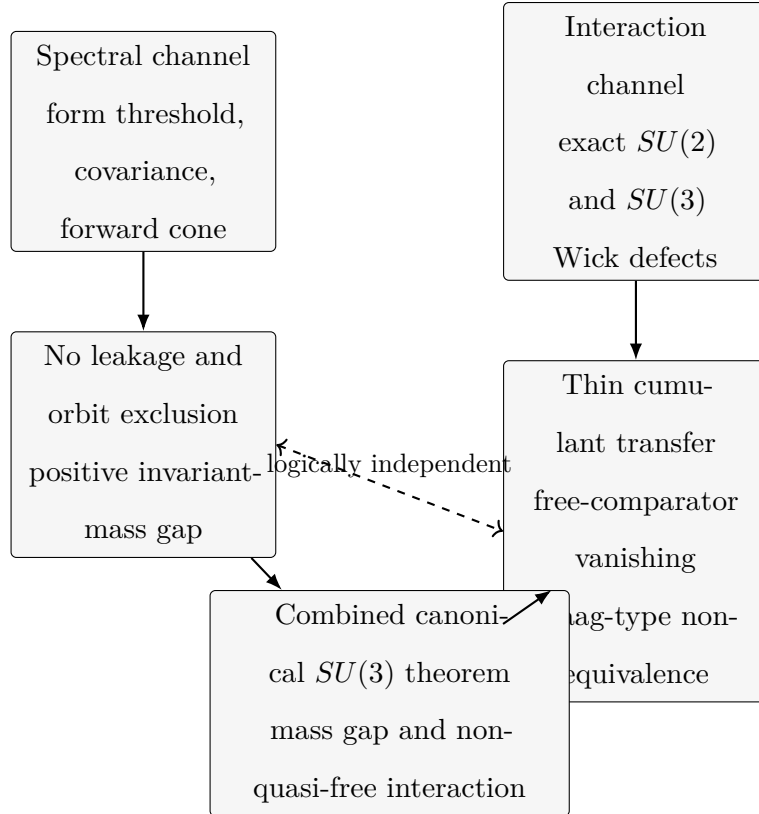


FIG. 4. The mass-gap and interaction claims have separate hypotheses and proofs. Their conjunction is a final theorem, not a circular argument.

A. Logical separation from the mass gap

Scope firewall XV.1 (Independent invariant). The mass-gap theorem concerns the joint translation spectrum and the Poincaré Casimir. The present theorem concerns non-Gaussian local vacuum correlations of selected physical Wilson germs. Neither conclusion is used to prove the other.

B. Packet-decorated free comparators

Wilson-loop and character observables enter through the standard Wilson formulation²³; the finite physical Wilson-packet realization used by this series is owned by Paper I¹.

Definition XV.2 (Selected analytic Wilson germ). Let O be an admissible bounded region. A selected analytic Wilson germ is a norm-analytic family

$$W_X(s) = e^{isX} \in \mathcal{A}^W(O), \quad s \in \mathbb{C} \text{ near } 0,$$

where $X = X^*$ is a bounded gauge-invariant Wilson-packet generator with the declared representation-theoretic normalization. Its centered generator in a state ω is

$$Y_X := X - \omega(X)I.$$

Definition XV.3 (Admissible free-comparator class). $\mathbf{Free}_{G,\xi}^{\text{adm}}$ consists of Poincaré-covariant packet-decorated observable nets with the same selected spacetime/regional category, corresponding analytic packet germs, a centered quasi-free or generalized-free vacuum on the declared linear germ space, the same one-particle covariance type where meaningful, and no imported coefficient-line associator unless separately declared. An admissible equivalence must preserve localization, vacuum, Poincaré action, the selected germ, and whether that germ is declared linear or composite.

Remark XV.4 (Why the germ type is part of the structure). A nonlinear composite of a Gaussian field can have nonzero raw connected cumulants. Therefore freeness cannot be tested by arbitrary Wilson exponential moments after forgetting how the packet was embedded. The invariant below is attached to a declared *linearized Wilson germ*; the comparator must preserve that declaration.

C. Connected cumulants and Wick defect

The comparison polynomial is the Wick pairing rule²⁴; the nonzero defect computed below is a new invariant of the selected reconstructed germ.

Definition XV.5 (Ordered connected cumulant). For analytic germs $W_{X_j}(s_j)$ define

$$Z_\omega(\mathbf{s}) = \omega(W_{X_1}(s_1) \cdots W_{X_k}(s_k))$$

and

$$\text{Cum}_k^\omega(X_1, \dots, X_k) = \partial_{s_1} \cdots \partial_{s_k} \log Z_\omega(\mathbf{s})|_{\mathbf{s}=0}.$$

For one centered self-adjoint generator Y ,

$$\text{Cum}_4^\omega(Y, Y, Y, Y) = \omega(Y^4) - 3\omega(Y^2)^2.$$

Definition XV.6 (Dynamical Wick defect). Let C_2 be the two-point covariance of the corresponding admissible free comparator. Define

$$\Delta_k^{G,\xi}(X_1, \dots, X_k) = \text{Cum}_k^{\omega_{\Omega,G,\xi,0}}(X_1, \dots, X_k) - \text{WickCum}_k^{C_2}(X_1, \dots, X_k).$$

For a centered linear Gaussian germ, $\text{WickCum}_k^{C_2} = 0$ for all $k \geq 3$.

D. Exact strict $SU(2)$ finite-stage witness

Let

$$\mathcal{K}_2 = \text{span}\{\chi_0, \chi_{1/2}, \chi_1, \chi_{3/2}\} \cong \mathbb{C}^4$$

with the Haar-orthonormal character basis. Fundamental-character multiplication compresses to

$$X_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and the certified finite Hamiltonian is

$$H_2 = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 4 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 9 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 16 \end{pmatrix}.$$

Its characteristic polynomial, multiplied by 16, is

$$p_2(\lambda) = 16\lambda^4 - 480\lambda^3 + 4356\lambda^2 - 12932\lambda + 8561.$$

For a simple root λ , an adjugate ground vector is

$$u_2(\lambda) = \begin{pmatrix} -4(2\lambda^3 - 58\lambda^2 + 487\lambda - 1142) \\ 4\lambda^2 - 100\lambda + 575 \\ -2(\lambda - 16) \\ 1 \end{pmatrix}.$$

Theorem XV.7 (Exact interval-certified $SU(2)$ Wick defect). *Let $E_{0,2}$ be the lowest root of p_2 and*

$$\Omega_2 = \frac{u_2(E_{0,2})}{\|u_2(E_{0,2})\|}.$$

Set $\omega_2(A) = \langle \Omega_2, A\Omega_2 \rangle$ and

$$Y_2 = X_2 - \omega_2(X_2)I.$$

Then

$$0.9421975143186003517 < \omega_2(Y_2^2) < 0.9421975143186003520,$$

so $Y_2\Omega_2 \neq 0$ and $Y_2\Omega_2 \perp \Omega_2$. Moreover,

$$\boxed{-0.75 < \text{Cum}_4^{\omega_2}(Y_2, Y_2, Y_2, Y_2) < -0.74.}$$

In particular the strict finite $SU(2)$ physicalized seed has a nonzero fourth-order linear-germ Wick defect.

Proof. The lowest root is isolated by an exact rational Sturm interval of width below 10^{-70} .

The displayed adjugate column is nonzero there and spans the simple ground eigenspace.

Each raw moment

$$m_j(\lambda) = \frac{u_2(\lambda)^T X_2^j u_2(\lambda)}{u_2(\lambda)^T u_2(\lambda)}$$

is a rational function of λ . Substitution into

$$\kappa_4 = m_4 - 4m_3m_1 - 3m_2^2 + 12m_2m_1^2 - 6m_1^4$$

and exact rational interval evaluation gives

$$-0.741329279691708958 < \kappa_4 < -0.741329279691708952.$$

The variance interval is positive. Centering gives $\langle \Omega_2, Y_2\Omega_2 \rangle = 0$, hence complete-ground orthogonality. $\square$

Scope firewall XV.8 ($SU(2)$ role). This proves a concrete finite positive-thickness strict interaction witness. The existing $SU(2)$ strict-control package is not, by itself, a thickness-cofinal incised-spacetime reconstruction. No thin $SU(2)$ interaction theorem is inferred here without that additional realization.

Proposition XV.9 (Canonical electric-reference response for $SU(2)$). *Let*

$$H_2(g) = \text{diag}(0, 3, 8, 15) + I - \frac{g}{2}X_2, \quad 0 \leq g \leq 1.$$

At $g = 0$ the normalized ground vector is $e_0 = (1, 0, 0, 0)^T$, and the centered fourth cumulant of the same character germ is exactly

$$\text{Cum}_4^{\omega_2, \text{el}}(X_2, X_2, X_2, X_2) = -1.$$

Consequently switching on the exact Wilson magnetic multiplier changes the fourth cumulant by the certified nonzero amount

$$0.25 < \text{Cum}_4^{\omega_2}(Y_2, Y_2, Y_2, Y_2) - \text{Cum}_4^{\omega_2, \text{el}}(X_2, X_2, X_2, X_2) < 0.26.$$

E. Exact $SU(3)$ witness

On

$$\mathcal{K}_3 = \text{span}\{\chi_1, \chi_3, \chi_{\bar{3}}, \chi_8\} \cong \mathbb{C}^4,$$

let

$$M_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad X_3 = M_3 + M_3^*.$$

The real fundamental Wilson germ is X_3 , and

$$H_3 = \text{diag}(0, 4, 4, 9) + I - \frac{1}{6}X_3.$$

The lowest eigenvalue is the lowest root of

$$p_3(\lambda) = 18\lambda^3 - 285\lambda^2 + 1135\lambda - 859.$$

An adjugate ground vector is

$$u_3(\lambda) = \begin{pmatrix} -(18\lambda^2 - 267\lambda + 869) \\ 3(\lambda - 10) \\ 3(\lambda - 10) \\ -1 \end{pmatrix}.$$

Theorem XV.10 (Exact interval-certified $SU(3)$ Wick defect). *Let $E_{0,3}$ be the lowest root of p_3 , let Ω_3 be the normalized $u_3(E_{0,3})$, and set*

$$Y_3 = X_3 - \langle \Omega_3, X_3 \Omega_3 \rangle I.$$

Then

$$2.1595087316958641831 < \|Y_3 \Omega_3\|^2 < 2.1595087316958641832$$

and

$$\boxed{-3.98 < \text{Cum}_4^{\omega_3}(Y_3, Y_3, Y_3, Y_3) < -3.97.}$$

Thus one may take the canonical seed interaction constant

$$c_{\text{int}}^{SU(3)} = 3.97$$

in the declared character normalization.

Proof. Exact rational Sturm isolation gives a simple lowest root

$$E_{0,3} \in (0.98553855293366932014155096090553698247103273943879824374809704597087835, \\ 0.98553855293366932014155096090553698247103273943879824374809704597087842).$$

The same rational-moment computation as above gives

$$-3.971379297144000549 < \kappa_4 < -3.971379297144000548.$$

The positive variance proves nontrivial complete-ground-complement creation. □

Proposition XV.11 (Canonical electric-reference response for $SU(3)$). *Let*

$$H_3(g) = \text{diag}(0, 4, 4, 9) + I - \frac{g}{6} X_3, \quad 0 \leq g \leq 1.$$

At $g = 0$ the normalized ground vector is $e_0 = (1, 0, 0, 0)^T$, and

$$\text{Cum}_4^{\omega_3, \text{el}}(X_3, X_3, X_3, X_3) = -2.$$

Hence the exact magnetic character term changes the selected fourth cumulant by

$$\boxed{-1.98 < \text{Cum}_4^{\omega_3}(Y_3, Y_3, Y_3, Y_3) - \text{Cum}_4^{\omega_3, \text{el}}(X_3, X_3, X_3, X_3) < -1.97.}$$

Proposition XV.12 (Non-Abelian curvature origin of the magnetic response). *The multiplier $I - \frac{1}{6}X_3$ is the retained real fundamental plaquette character term. For a small plaquette of area a^2 and anti-Hermitian curvature F_A ,*

$$1 - \frac{1}{d_R} \operatorname{Re} \chi_R(e^{a^2 F_A}) = \frac{a^4}{2d_R} \|F_A\|_R^2 + O(a^6),$$

where

$$F_A = dA + \frac{1}{2}[A \wedge A].$$

Thus the leading polarization contains the cubic term

$$\langle dA, [A \wedge A] \rangle$$

and the quartic term

$$\frac{1}{4} \|[A \wedge A]\|^2.$$

The two preceding electric-reference calculations prove that the selected vacuum statistic responds nontrivially when this exact compact-group magnetic interaction is switched on. They do not separately identify a scattering amplitude or assign the entire response to only the cubic or only the quartic polarization.

F. Physical survival of the witness

Proposition XV.13 (Gauge recompression and Gauss–Ward preparation). *The operators X_2 and X_3 are class-function Wilson multipliers and hence gauge invariant. Their centered versions remain gauge invariant. Therefore the selected germs survive interior Gauss reduction and satisfy the zero interior Ward variation. Since*

$$P_\Omega Y_j \Omega_j = 0, \quad \|(I - P_\Omega) Y_j \Omega_j\|^2 = \omega_j(Y_j^2) > 0,$$

complete-ground preparation does not annihilate either witness.

Proposition XV.14 (Chart, corridor, and twisted-line invariance). *Let $V_{\alpha\beta}$ be an exact chart/corridor implementer, possibly coefficient-line valued, and suppose*

$$V_{\alpha\beta}(I \otimes X_\beta) V_{\alpha\beta}^* = X_\alpha, \quad V_{\alpha\beta}(I \otimes \Omega_\beta) = \Omega_\alpha.$$

Then every ordered moment and cumulant is preserved. On triple overlaps a scalar associator cancels from adjoint observable transport. Hence the $SU(3)$ dynamical Wick defect is

independent of the linked-flux label $m \in \mathbb{Z}_3$, whereas the coefficient class $m\mathbf{u}_3$ remains an independent represented invariant.

Proof. Unitary conjugation preserves products and vacuum matrix coefficients. An external scalar multiplies both sides of a represented comparison and disappears under conjugation. Therefore it affects neither the ordinary observable germ nor its vacuum cumulants. $\square$

Proposition XV.15 (Entropy-surgery preservation on the selected ground state). *At every finite stage where the complete-ground projection is rank one and the surgery selector preserves the complete-ground face, the selected vacuum state is fixed. Consequently the Wilson moments and the Wick defect are unchanged by entropy surgery. In the thin system the VSR/SLSP selected state is the unique compatible recovery of these ground states, so the same conclusion follows after state descent.*

G. Uniform thin cumulant transfer

Lemma XV.16 (Bounded moment continuity). *Let ω_n, ω be normal states and X_n, X bounded operators with $\sup_n \|X_n\| \leq L$. Suppose, in a common comparison carrier,*

$$\omega_n(X_n^j) \longrightarrow \omega(X^j), \quad 1 \leq j \leq 4.$$

Then

$$\text{Cum}_4^{\omega_n}(X_n, X_n, X_n, X_n) \longrightarrow \text{Cum}_4^{\omega}(X, X, X, X).$$

The conclusion is uniform on any packet family for which the four moment errors converge uniformly.

Proof. The fourth cumulant is the universal polynomial

$$m_4 - 4m_3m_1 - 3m_2^2 + 12m_2m_1^2 - 6m_1^4.$$

Polynomial continuity on the compact moment box $|m_j| \leq L^j$ proves the assertion. $\square$

Lemma XV.17 (Analytic packet version). *Suppose the generating functions*

$$Z_n(\mathbf{s}) = \omega_n(W_{1,n}(s_1) \cdots W_{4,n}(s_4))$$

are holomorphic and uniformly bounded on one common polydisc, converge uniformly on compact sub-polydiscs, and remain uniformly separated from zero near the origin. Then their logarithmic fourth derivatives converge uniformly on smaller polydiscs.

Proof. Uniform convergence and the nonzero neighborhood select one logarithm branch. Cauchy's integral formula gives convergence of all derivatives through order four. $\square$

Theorem XV.18 (Paper-IV thin survival of a nonzero Wick defect). *Let (X_r, ω_r) be a coherent bounded physical Wilson packet along a certified Paper-IV branch. Assume its observable-section and state-section components belong to the comparison certificate $\mathcal{C}_{\varepsilon \rightarrow 0}$ and that its finite-stage defect obeys*

$$|\Delta_{4,r}| \geq c_{\text{seed}} > 0$$

on a cofinal tail. Then

$$\Delta_{4,r} \longrightarrow \Delta_{4,0}.$$

If the comparison error is eventually less than $c_{\text{seed}}/2$, then

$$|\Delta_{4,0}| \geq c_{\text{seed}}/2 > 0.$$

For an exact dilation reference whose cumulant is constant and a physical branch converging to it, the thin cumulant equals the seed cumulant exactly.

Proof. The observable comparison gives convergence of finite products of the selected bounded packet germs; the state comparison gives convergence of their normal matrix coefficients. The preceding lemmas transfer the moment polynomial, or equivalently the logarithmic derivatives. Exact Gauss–Ward, surgery, and coefficient-line compatibility prevents an additional defect. The final lower bound is the triangle inequality. $\square$

H. Canonical $SU(3)$ branchwise interaction theorem

Theorem XV.19 (Canonical $SU(3)$ thin interaction theorem). *For the canonical dyadic exact-dilation family with corridor label $r = 0$, the primitive form, state, observable, ground, Gauss–Ward, surgery, and decorated-shell comparison estimates are uniform in $m \in \mathbb{Z}_3$. The real fundamental Wilson germ therefore has the same thin fourth Wick defect in all three center labels:*

$$\boxed{\Delta_{4,0}^{SU(3),m} = \kappa_4^{SU(3)} \in (-3.98, -3.97), \quad \left| \Delta_{4,0}^{SU(3),m} \right| > 3.97, \quad m = 0, 1, 2.}$$

The negative sign records the signed sub-Gaussian fourth connected response of the selected real-character germ. It is unrelated to the positive invariant-mass threshold; interaction is certified here by the nonvanishing magnitude $\left| \Delta_{4,0}^{SU(3),m} \right| > 3.97$. Consequently:

- (i) *the $m = 0$ branch is a strict Sector-I interacting branch;*
- (ii) *the $m = 1, 2$ branches are scalar Sector-III interacting branches;*
- (iii) *for $m = 1, 2$, the independent invariants are*

$$\mathfrak{I}_{SU(3),m}^{\text{dyn}} > 0, \quad [\Omega_m^{\text{fib}}] = m\mathbf{u}_3 \neq 0.$$

Proof. The exact dilation reference transports H_3 , P_Ω , X_3 , and the ground state unitarily, so the reference cumulant is independent of scale. The canonical physical shell converges to that reference with the declared vanishing master defect; the Gauss–Ward and spectral-surgery components are exact. The thin-transfer theorem therefore identifies the thin cumulant with the interval-certified seed value. The coefficient label m occurs on the external line system and cancels from ordinary observable moments. The fixed-thickness detector separately gives $m\mathbf{u}_3$ for $m = 1, 2$ and zero for $m = 0$. $\square$

Referee audit XV.20 (Strict $m = 0$ extension). The analytic comparison estimates in the canonical physical branch are stated uniformly over $m \in \mathbb{Z}_3$. The $m = 0$ branch removes, rather than adds, the nontrivial coefficient-line obstruction. Thus the Paper-IV canonical reconstruction argument applies with simpler represented descent. This is the strict thin witness used here. The independent $SU(2)$ seed remains a separate finite-stage strict control and journal witness.

I. Quasi-free vanishing

Theorem XV.21 (Admissible free-comparator vanishing theorem). *For every comparator in $\mathbf{Free}_{G,\xi}^{\text{adm}}$ and every selected centered linear packet germ,*

$$\text{Cum}_k^{\text{fr}}(X_1, \dots, X_k) = 0, \quad k \geq 3.$$

Equivalently,

$$\Delta_k^{\text{fr}} = 0.$$

For nonlinear composites, the raw cumulant need not vanish; the defect is still zero because the raw cumulant equals its complete Wick contraction polynomial.

Proof. A centered quasi-free state is determined by its two-point function. Wick's theorem expresses every moment as a sum over pairings^{24–26}. The moment–cumulant relation then kills connected linear cumulants above order two and reproduces the declared Wick polynomial for composites. $\square$

J. Haag-type non-equivalence

Theorem XV.22 (Haag–Yang–Mills interaction and free-net non-equivalence). *Let $\mathcal{A}_{G,\xi,0}^{\text{YM}}$ be a reconstructed branch carrying a nonzero physical Wick defect. There is no vacuum-preserving Poincaré-covariant regional net isomorphism*

$$\Theta : \mathcal{A}_{G,\xi,0}^{\text{YM}} \longrightarrow \mathcal{A}_{G,\xi}^{\text{fr}}$$

to any $\mathcal{A}_{G,\xi}^{\text{fr}} \in \mathbf{Free}_{G,\xi}^{\text{adm}}$ that intertwines the selected analytic packet germs and their declared linear/composite type. Hence

$\mathcal{A}_{G,\xi,0}^{\text{YM}}$ is neither free nor generalized free in the admissible packet-decorated category.

Proof. Such an isomorphism preserves the vacuum moments of every selected germ and therefore preserves its connected cumulants. Preservation of the germ declaration and two-point covariance preserves the free Wick prediction. Hence it preserves Δ_4 . The Yang–Mills witness has $\Delta_4 \neq 0$, whereas every admissible free comparator has $\Delta_4 = 0$, a contradiction. $\square$

Remark XV.23 (Haag-type, not the point-field Haag theorem). The obstruction is a physical net invariant, not an invocation of the traditional interaction-picture Haag theorem^{16,27}. Abstract isomorphisms of isolated von Neumann factors after forgetting localization, vacuum, dynamics, and selected germs are irrelevant to the conclusion.

Scope firewall XV.24 (Exact categorical scope of the non-equivalence theorem). The theorem concerns the admissible packet-decorated category in which the selected primitive Wilson germ and its linear/composite declaration are part of the physical structure. It does not assert that every nonlinear observable in a free field theory has Gaussian statistics. Nor does it exclude a bare algebra isomorphism that forgets the selected germ or maps it to an unrelated nonlinear composite. The claim is precisely that no vacuum-preserving Poincaré-covariant equivalence can preserve the declared physical Wilson germ while replacing its nonzero Wick defect by a quasi-free factorization law.

K. PHIGT firewall

Proposition XV.25 (Internal hyperbolic identity is not a free-net equivalence). *The internal PHIGT theorem identifies the selected time generator and stiffness operator on the same physical carrier. It does not identify the selected physical Wilson germ with a linear Gaussian local field germ and does not provide a packet-preserving regional free-net isomorphism. Therefore the exact physical generator/form identity is compatible with the nonzero physical Wick defect.*

L. Combined mass-gap and interaction theorem

Theorem XV.26 (Combined canonical $SU(3)$ theorem). *For the Poincaré-mobile incision realization of the canonical dyadic exact-dilation $SU(3)$ branches $(m, r) = (0, 0), (1, 0), (2, 0)$, the reconstructed ordinary observable theory satisfies independently*

$$\text{supp } E_m \subseteq \{0\} \cup \{p \in \overline{V}_+ : p^2 \geq \kappa_*\},$$

$$E_m(\{0\}) = E_{M_m^2}(\{0\}) = P_{\Omega, m}, \quad E_{M_m^2}((0, \kappa_*)) = 0,$$

and

$$\Delta_{4,0}^{SU(3),m} \in (-3.98, -3.97).$$

Thus every such branch has a positive Lorentz-invariant mass gap and is Haag-type non-equivalent to every admissible free or generalized-free comparator. For $m = 1, 2$, the scalar coefficient-line class supplies an additional independent coherence obstruction.

Scope firewall XV.27 (Scattering boundary). No nonidentity S -matrix is claimed. Such a result requires an isolated mass hyperboloid, Haag–Ruelle or comparable asymptotic states, clustering or phase-space estimates, and an independently nonzero scattering amplitude.

M. Certification and source ledger

Item	Status	Basis
$SU(2)$ Hamiltonian and rank-one ground	inherited/exact	Certified strict $SU(2)$ Wilson-character realization
$SU(2)$ fourth cumulant and electric-reference response	proved/certified	Exact rational root isolation and exact rational interval arithmetic in the accompanying script

$SU(3)$ Hamiltonian and rank-one ground	inherited/exact	Certified four-character $SU(3)$ Wilson-tube seed and recursive branch
$SU(3)$ fourth cumulant and electric-reference response	proved/certified	Exact rational root isolation and exact rational interval arithmetic
Gauss–Ward survival	proved	Class-function gauge invariance and exact branch compatibility
Complete-ground survival	proved	Centering plus positive variance
Twisted-line independence	proved	Scalar cancellation in adjoint observable transport
Thin cumulant transfer	proved	Paper-IV observable/state comparison certificate plus bounded moment continuity
Free-comparator vanishing	proved	Quasi-free Wick factorization
Free-net non-equivalence	proved	Preservation of packet-decorated vacuum cumulants
Nontrivial scattering	not claimed	Requires separate asymptotic theorem

Referee audit XV.28 (Machine certificate). The accompanying machine certificate reconstructs both exact rational matrices, isolates the lowest roots with exact Sturm intervals, and evaluates the variance and cumulants using exact rational interval arithmetic. Its JSON report records that every declared sign, positivity, and nonvanishing check passes.

XVI. TRANSPORT-GROUPOID INTERPRETATION AND CONJECTURAL OUTLOOK

A. Logical firewall

This chapter is entirely non-load-bearing. None of its conjectures is used in the construction of the thin observable net, the Poincaré implementation, the proof of the forward-cone spectrum condition, the Glue–Mosco–Riesz theorem, the Physical Hyperbolic Identification and Ground-Triviality theorem, the exact zero-mode theorem, or the branchwise mass-gap theorem. The rigorous conclusion is already contained in [Theorem XIV.18](#).

B. The natural global object is a transport groupoid

Let $\mathbf{Bkg}_{\text{YM}}$ denote the admissible background groupoid whose objects contain spacetime, incision, gauge bundle, branch, chart, collar, and scale data,

$$\mathfrak{b} = (M, \Sigma, \mathcal{P}, \xi, \mathfrak{c}, \dots),$$

and whose arrows include admissible spacetime transformations, gauge transformations, Cauchy transports, chart refinements, corridor transitions, scale-bordism maps, and coefficient-line or correspondence seams. The natural global Yang–Mills object is an assignment

$$\mathfrak{b} \longmapsto (\mathcal{A}_{\mathfrak{b}}, \omega_{\mathfrak{b}}, \mathcal{H}_{\mathfrak{b}}, \pi_{\mathfrak{b}}, U_{\mathfrak{b}})$$

with coherent transport along the arrows.

Conjecture XVI.1 (Transport-groupoid ontology). The natural nonperturbative state of Yang–Mills theory is a coherent state section over an admissible transport groupoid rather than a single globally preferred vector in one fixed Hilbert space. The ordinary vacuum representation is the global observable shadow of this higher transport object.

C. Gluons as Hilbert-gerbe sections

In scalar Sector III the represented system consists of Hilbert spaces $\mathcal{H}_{\alpha, \mathfrak{c}}$, coefficient lines $L_{\alpha\beta}^{\mathfrak{c}}$, and seam maps

$$V_{\alpha\beta}^{\mathfrak{c}} : L_{\alpha\beta}^{\mathfrak{c}} \otimes \mathcal{H}_{\beta, \mathfrak{c}} \longrightarrow \mathcal{H}_{\alpha, \mathfrak{c}}$$

with scalar-twisted triple-overlap law

$$V_{\alpha\beta}^{\mathfrak{c}}(1 \otimes V_{\beta\gamma}^{\mathfrak{c}}) = \Omega_{\alpha\beta\gamma}^{\mathfrak{c}} V_{\alpha\gamma}^{\mathfrak{c}}.$$

Conjecture XVI.2 (Gerbe-sectional gluons). A nonperturbative colored gluonic excitation is naturally a local or sectional object of the represented Yang–Mills Hilbert gerbe. Globally descended physical vectors arise only after local color, dressing, and transport data combine into a descent-compatible neutral composite.

Conjecture XVI.3 (No pointlike infrared color). There is no compactly localized, gauge-invariant, pointlike infrared one-gluon state in the global observable Hilbert space. Every nontrivial infrared color excitation requires noncompact dressing, cone or corridor localization, boundary attachment, or gerbe-sectional realization.

D. Scattering as dynamical stackification

The stack need not be a passive background container. It may be generated by local excitations and their admissible compositions. Local scattering maps

$$\mathcal{S}_\alpha : \mathfrak{H}_{\text{in},\alpha} \longrightarrow \mathfrak{H}_{\text{out},\alpha}$$

should satisfy

$$V_{\alpha\beta}^{\text{out}}(1 \otimes \mathcal{S}_\beta) = \mathcal{S}_\alpha V_{\alpha\beta}^{\text{in}}.$$

Conjecture XVI.4 (Dynamical stackification by scattering). The physical Yang–Mills stack is the coherence completion of local gluonic excitation, transport, and scattering data. Local excitations define objects; scattering and transport define morphisms; comparisons among multi-channel factorizations define two-morphisms; and residual associativity defects define the cocycle data required for stackification.

Conjecture XVI.5 (Descent-compatible scattering). The exact asymptotic scattering theory of a nonperturbative Yang–Mills branch is a transformation between asymptotic Hilbert gerbes. In Sector I it strictifies to an ordinary global scattering operator whenever the usual asymptotic hypotheses hold. In Sector III it remains a coefficient-line-compatible family whose compositions retain the scalar associator.

Conjecture XVI.6 (Infrared cancellation as descent). An infrared-safe observable is an asymptotic functional that descends through the relevant soft-transport groupoid. Virtual corrections, unresolved radiation, and dressing terms are not separately descended objects; their inclusive combination is finite because it is the global shadow of one coherent sectional process.

E. Hadronization, confinement, and effective strings

Conjecture XVI.7 (Hadronization as descent completion). Hadronization is the process by which locally defined colored sections become globally admissible neutral objects. A hadron or glueball is a minimal stable global section obtained by completing the dressing, transport, and coherence data required for physical descent.

Conjecture XVI.8 (Space-filling confinement). The microscopic support of confining Yang–Mills transport is a percolating or effectively space-filling network of local obstruction data. The observed flux tube is a mesoscopic compression of this network rather than a fundamental one-dimensional object.

$$\begin{aligned} \text{gerbe transport network} &\longrightarrow \text{space-filling confining structure} \\ &\longrightarrow \text{coarse flux tube} \longrightarrow \text{effective worldsheet.} \end{aligned}$$

Conjecture XVI.9 (Effective string as a secondary shadow). A confining string is the long-distance worldsheet shadow of a descent-compatible gluonic transport network. Its worldsheet degrees of freedom are collective coordinates of a coarse flux tube, which is itself an emergent compression of a more distributed microscopic transport obstruction.

Strings are not fundamental; coherence is fundamental.

F. Critical spectral-stack emergence in three spatial dimensions

A companion spectral-action analysis derives, under its declared regularity and emergence hypotheses,

$$(d, s, \lambda) = (d, 1, d - 2),$$

for first-order Dirac matter coupled to its own second-order static gauge mediator. The critical diagonal $s = \lambda$ is attained only at $d = 3$. This exponent-level result is not used in the mass-gap proof.

Conjecture XVI.10 (Critical spectral-stack emergence). A stable relativistic stack of self-generated matter and gauge excitations is naturally realized at the critical spectral balance $s = \lambda$. For first-order Dirac matter whose gauge mediator is generated by a second-order spectral-action sector, this balance selects three spatial dimensions.

G. Stacky duality and the landscape as an atlas

Conjecture XVI.11 (Stacky sectional duality). The familiar S -, T -, and U -dualities are equivalences between higher stacks of local physical presentations rather than pointwise

identifications of fundamental objects. They act on local sections, coefficient lines, defects, scattering morphisms, and boundary data, and their compositions close only up to coherent natural transformations.

Conjecture XVI.12 (Spectrum-preserving coherence permutation). A duality may preserve the global observable spectrum while permuting represented coefficient lines, boundary characters, associator classes, and defect-fusion data. Equality of mass spectra does not imply equality of represented transport structures.

Conjecture XVI.13 (Landscape-as-atlas). A substantial portion of the apparent theory landscape is an atlas of local presentations, sectors, phases, compactifications, or descent realizations of one higher transport stack. Dualities provide transition equivalences; anomalies and obstruction classes provide higher overlap data.

H. A topological theory controlling gluing bordisms

Conjecture XVI.14 (Gluing-bordism topological field theory). There exists an extended topological field theory $\mathfrak{T}_{\text{glue}}$ whose bordism assignments encode admissible coefficient lines, center sectors, associators, boundary characters, defect junctions, and transport compositions. The physical Yang–Mills theory is a module, boundary condition, or dynamical fibration over $\mathfrak{T}_{\text{glue}}$.

Possible low-energy models may resemble a center-valued BF theory, a finite Dijkgraaf–Witten theory, a higher-form symmetry TFT, or a correspondence-valued extended theory. No such identification is proved here.

Conjecture XVI.15 (Boundary inflow for incision data). The retained collar-current and boundary-character data are governed by an induced lower-dimensional topological theory on or near Σ , with a bulk-boundary inflow relation

$$\delta S_{\text{bulk}} + \delta S_{\Sigma} = 0.$$

I. Strict-versus-twisted observables

Conjecture XVI.16 (Twisted-sector interferometry). There exist linked extended-operator configurations for which the normalized ratio between twisted and strict expectation values

detects the scalar associator class,

$$\frac{\langle W_1 W_2 W_3 \rangle_{\text{III}}}{\langle W_1 W_2 W_3 \rangle_{\text{I}}} = \exp(2\pi i \langle [\Omega], [C] \rangle),$$

for a suitable linked compositional cycle $[C]$.

Conjecture XVI.17 (Higher Berry two-holonomy). Adiabatic transport of an incision, defect, background, or scale corridor around a nontrivial two-parameter cycle produces a measurable two-holonomy whose local Čech representative is the associator class retained by the scalar Sector-III Hilbert gerbe.

J. The mass gap as a coherence threshold

The rigorous theorem gives

$$\text{supp } E \subseteq \{0\} \cup \{p \in \overline{V}_+ : p^2 \geq \kappa_*\}, \quad E(\{0\}) = E_{M^2}(\{0\}) = P_\Omega,$$

and

$$E_{M^2}((0, \kappa_*)) = 0.$$

Conjecture XVI.18 (Mass gap as a coherence threshold). The Yang–Mills mass gap is the minimum invariant spectral cost required to deform the complete-ground transport section into a nontrivial globally admissible physical section. The vacuum transport object is isolated from every nontrivial descended excitation by a positive coherence threshold.

The mass gap is the price of becoming globally physical.

Conjecture XVI.19 (Gap-controlled stack rigidity). The inverse mass gap determines the characteristic length scale beyond which the physical transport prestack behaves rigidly for gauge-invariant observables. Below this scale local dressing and defect data retain substantial sectional dependence; above it the observable content organizes into stable descended neutral composites.

Conjecture XVI.20 (Gap–confinement coherence relation). The mass gap measures the minimum local spectral cost of nontrivial transport deformation, while the confining string tension measures the accumulated spatial cost of maintaining a nontrivial color-transport obstruction over macroscopic separation. They are distinct observables of one coercive transport geometry.

Conjecture XVI.21 (Twist as sector locking above a common gap). Strict and scalar-twisted theories may share the same certified mass threshold while differing in multiplicities, extended-operator selection rules, scattering composition, finite-volume splitting, and higher interference phases. The twist organizes the spectrum above the gap rather than necessarily generating the numerical gap itself.

K. Operational mass realization and the meaning of $E = mc^2$

The invariant mass is the Poincaré Casimir,

$$M^2 = P_\mu P^\mu.$$

The familiar relation $E = mc^2$ is a rest-frame statement following from

$$E^2 = |\mathbf{p}|^2 c^2 + m^2 c^4$$

when $\mathbf{p} = 0$. The vacuum has $p = 0$ but is not a material excitation. A nonzero null excitation has energy and momentum but no rest frame. The theorem eliminates every nonzero future-null physical sector, so the first nonvacuum globally physical orbit is timelike.

Conjecture XVI.22 (Operational Mass-Realization Conjecture).

The Yang–Mills mass gap is the threshold at which a nontrivial quantum transport section first admits a stable, frame-independent classical measurement as a massive object.

The claim is operational, not definitional: M^2 is already a quantum spectral observable. The first nonvacuum descended sector is the first sector admitting a rest frame, proper-time parametrization, invariant rest energy, and stable apparatus correlation as a massive object.

local quantum section $\longrightarrow$ descent-compatible excitation
 $\longrightarrow$ isolated timelike Poincaré orbit
 $\longrightarrow$ stable apparatus correlation
 $\longrightarrow$ classical rest-mass value.

The mass gap is where energy first becomes rest energy.

Mass begins as a Casimir; at the gap, it becomes an object.

L. Criticality stabilized by a dynamical gap

Conjecture XVI.23 (Criticality stabilized by a dynamical gap). The $3 + 1$ -dimensional Yang–Mills mass gap is the nonperturbative rigidity mechanism stabilizing the critical spectral balance $s = \lambda = 1$. Criticality selects the distinguished dimension; the dynamically generated gap prevents the critical theory from degenerating into a continuum of arbitrarily soft globally descended excitations.

spectral-action emergence $\longrightarrow d = 3$ critical scaling
 $\longrightarrow$ Yang–Mills coercivity
 $\longrightarrow$ positive invariant-mass gap
 $\longrightarrow$ no soft spectral leakage
 $\longrightarrow$ rigid causal and scattering transport
 $\longrightarrow$ stable critical spectral stack.

M. Conjectural synthesis

Local gluonic excitations are sections. Their transports and scatterings generate morphisms. Their competing compositions generate higher coherence data. Stackification produces the represented physical object. Globally neutral states and observables descend to the ordinary observable shadow. Confinement is the energetic obstruction to global colored descent. Flux tubes are coarse compressions of that obstruction. Effective strings are long-distance worldsheet descriptions. Dualities are equivalences among sectional presentations. The landscape is the atlas of the resulting stack. A topological gluing theory controls admissible bordisms and defect compositions.

The infrared physical content of Yang–Mills theory is encoded not by fundamental pointlike colored particles or fundamental strings, but by descent-compatible sections, scattering morphisms, transport holonomies, and globally neutral composites. Confinement, effective strings, duality, and sector structure are distinct observable shadows of one critical spectral transport stack.

**There is no unique fundamental object; there is a globally
consistent hierarchy of coherence.**

XVII. ADMISSIBILITY BOUNDARY, NONCOERCIVE DEGENERATION, AND CATEGORICAL COMPLETION

Scope firewall XVII.1 (The domain of the final theorem). The spectral theorem proved in this manuscript is a theorem on the admissible coercive reconstruction class. It is not a classification theorem asserting that every finite-scale compact semisimple gauge datum automatically belongs to that class. The word “every” in the final theorem quantifies over every branch for which the declared analytic, geometric, descent, and representation certificates have been verified.

A. Complete reconstruction data and the certified class

Because the symbol Γ already denotes the central subgroup in a global form $G = \tilde{G}/\Gamma$, write $\mathfrak{D}$ for a complete reconstruction datum. It includes the local Lie-theoretic and global-form data, center and line sectors, bundle and topological sectors, branch and coefficient system, admissible transport, incision and Cauchy data, finite packet, complete-ground projection, comparison and restart maps, thick and thin forms, represented descent data, and the physical covariance package.

Definition XVII.2 (Admissible coercive reconstruction class). A complete datum $\mathfrak{D}$ belongs to

$$\mathbf{Adm}_{\text{coer}}$$

when the following certificates hold at the branch scope asserted in the manuscript:

- (A1) the finite-scale complete-ground coercive floor is positive and uniform along the selected admissible cofinal system;
- (A2) branch selection, entropy surgery, Gauss–Ward compatibility, and admissible transport preserve the hypotheses required for thick-to-thin descent;

(A3) the comparison–Mosco–Riesz package produces the closed thin form $\bar{\mathfrak{q}}_{\text{thin}}$, its represented operator K_{thin} , the exact complete-ground kernel, and a positive thin floor;

(A4) the selected physical hyperbolic realization satisfies

$$H_{\text{hyp}} = K_{\text{thin}}^{1/2}, \quad P_0 = H_{\text{hyp}}, \quad P_0^2 = K_{\text{thin}};$$

(A5) the incision-relative Poincaré construction supplies the strongly continuous joint translation representation, the forward-cone condition, Lorentz covariance of the joint spectral measure, and the exact-zero identity;

(A6) every product, quotient, measurable assembly, strict seam, or scalar coefficient-line operation used in the branch is covered by the corresponding closure theorem; correspondence-valued branches retain their separately stated module-spectral condition.

Membership in $\mathbf{Adm}_{\text{coer}}$ is therefore a proved property of a complete datum, not a convention attached only to the name of its gauge group.

Proposition XVII.3 (Quantifier boundary for spectral closure). *For every $\mathfrak{D} \in \mathbf{Adm}_{\text{coer}}$ in a fully certified exact-dilation Sector-I or scalar Sector-III branch, the conclusions of Theorem XIV.18 hold. In particular,*

$$\text{supp } E_{\mathfrak{D}} \subseteq \{0\} \cup \{p \in \bar{V}_+ : p^2 \geq \kappa_{*,\mathfrak{D}}\},$$

$$E_{\mathfrak{D}}(\{0\}) = E_{M_{\mathfrak{D}}^2}(\{0\}) = P_{\Omega,\mathfrak{D}}, \quad M_{\mathfrak{D}}^2 \geq \kappa_{*,\mathfrak{D}}(I - P_{\Omega,\mathfrak{D}}).$$

The corridorwise Sector-II conclusion requires its declared physical realization and a positive uniform corridor infimum. No conclusion is asserted for a datum outside $\mathbf{Adm}_{\text{coer}}$.

Proof. This is the exact domain restriction already present in the hypotheses of Theorems XIV.7 and XIV.18. The present proposition records the quantifier explicitly and introduces no new analytic input. $\square$

B. Noncoercive degeneration

A reconstruction can leave the certified class in several inequivalent ways. For example, the finite-scale lower floor may collapse,

$$\inf_r \kappa_r = 0;$$

the corridor infimum may vanish; comparison or restart constants may lose uniformity; the complete-ground projections may fail to converge strongly; the thin form may fail to retain the required exact kernel; or represented descent may fail to exist at the claimed branch type. These failures are not interchangeable, and Paper V does not identify them with one universal mechanism.

Scope firewall XVII.4 (Failure of a hypothesis is not a counterexample). If $\mathfrak{D} \notin \mathbf{Adm}_{\text{coer}}$, the mass-gap theorem is silent. Such a datum may describe a genuinely gapless regime, a degenerate or empty physical branch, a reconstruction requiring a different functional space, or simply a case for which the present method supplies no certificate. Failure of coercivity or descent is therefore not a counterexample to the spectral theorem; it is failure of membership in the theorem’s proved domain.

In particular, the selected-frame identity

$$P_0^2 = K_{\text{thin}}$$

does not create positivity when the thin floor has collapsed. Likewise, positive finite-scale data do not yield a continuum threshold unless the uniform comparison, descent, and exact-kernel hypotheses survive the limiting process. The theorem closes the coercive branch; it does not redefine a noncoercive branch as coercive.

C. Non-vacuity and the role of the group witnesses

The class $\mathbf{Adm}_{\text{coer}}$ is not introduced merely as a formal conditional. The selected $SU(3)$ strict and scalar-twisted realizations provide explicit certified members, and [Theorem XV.26](#) supplies the independent interaction witness for the canonical $SU(3)$ branches. The finite $SU(2)$ calculations verify the abstract local mechanism and its conventions, but they neither transfer their constants into the $SU(3)$ branch nor prove a thin $SU(2)$ theorem without the remaining reconstruction certificates.

Products and finite central quotients remain inside the final theorem only when their compatibility, reducing projections, surviving nonvacuum support, and positive threshold are established by the corresponding closure results. The group name alone does not substitute for those proofs.

D. Categorical completion

The phrase *admissibility boundary* is logical here, not yet topological: no topology on the class of complete reconstruction data is fixed in this paper. A later categorical completion may seek to classify the ways in which coercivity, exact descent, or physical realization fails, and may organize noncoercive degeneration by obstruction, boundary, or no-go data. Such a classification would require additional objects and theorems; it is not silently contained in the present universal quantifier.

Scope firewall XVII.5 (Categorical completion is downstream). Paper V proves spectral closure after admissibility and coercivity have been certified. It does not prove that all compact semisimple global forms admit the same selected packet, that every topological sector lies on a coercive branch, or that every failure of coercivity has one categorical cause. Those are classification questions beyond the theorem proved here.

The boundary is therefore exact:

$$\boxed{\mathfrak{D} \in \mathbf{Adm}_{\text{coer}} \implies \text{spectral closure and a positive invariant-mass threshold;}}$$

$$\boxed{\mathfrak{D} \notin \mathbf{Adm}_{\text{coer}} \implies \text{no conclusion from this theorem alone.}}$$

This is not a retreat from the result. It is the statement of the result at its correct level of generality.

XVIII. PAPER-SERIES CONCLUSION

Papers I–IV construct and control the finite-scale, positive-thickness, spacetime, and thin-limit Yang–Mills observable systems^{1–4}. The present manuscript supplies the missing spectral closure and a separate dynamical interaction certificate. It constructs the invariant complete-ground state, implements the Poincaré action strongly continuously, builds the joint translation generators and spectral measure, decomposes the representation over Lorentz orbits, transfers the positive-thickness spectrum condition, proves Hilbert-gerbe Glue–Mosco–Riesz stability, identifies the physical hyperbolic generator, eliminates zero-mode and future-null leakage, and assembles the result through strict, twisted, product, quotient, and measurable branch operations. Independently, it computes a nonzero fourth-order Wick defect for the centered fundamental Wilson germ, certifies its nonzero response to switching

on the exact compact-group magnetic character term, transfers the witness through the thin comparison certificate, and proves Haag-type non-equivalence to every admissible free or generalized-free packet-decorated net that preserves the declared germ structure.

The rigorous endpoint is a branch-certified continuum Yang–Mills observable theory on incised $3 + 1$ -dimensional Minkowski spacetime for every compact connected semisimple gauge datum belonging to the admissible coercive reconstruction class, with a positive Lorentz-invariant mass threshold in each fully certified strict and scalar-twisted branch. For the canonical $SU(3)$ strict and scalar-twisted witnesses, the same continuum theory is also certified non-quasi-free in the explicitly selected physical Wilson-germ category; no stronger bare-net or scattering claim is silently inferred. The ordinary observable system is global. The represented physical system remembers whether descent is strict, relatively twisted, coefficient-line twisted, or correspondence valued.

The mathematical lesson is that spectral equivalence is weaker than physical equivalence. The spectrum supplies the gap; the physical Wilson-germ embedding supplies an independent interaction invariant. Two branches may share the same certified gap while differing in associators, boundary characters, defect transport, and scattering coherence. The mass gap isolates the complete-ground transport object from every nontrivial descended excitation; it does not erase the higher geometry carried by the represented theory.

Scope reminder. The formal endpoint is the represented thin Yang–Mills system with incision-relative Poincaré covariance, forward-cone spectrum, exact complete-ground zero mode, positive invariant-mass threshold, and a separately certified Wilson-germ interaction witness. Scattering, confinement, stackification, and operational measurement interpretations remain downstream research questions.

The conjectural lesson is more ambitious. Yang–Mills theory may be naturally transport-groupoid valued. Confining strings may be long-distance shadows of a distributed transport obstruction rather than fundamental objects. Dualities may be stack equivalences. The apparent landscape may be an atlas. The mass gap may be the threshold at which quantum transport coherence first admits a stable classical measurement as a massive object. These interpretations are not part of the proof. They are the research program made visible by the proof.

SUPPLEMENTARY MATERIAL

The supplementary material contains the transport-groupoid and operational mass-realization outlook, the branch and obligation matrices, the complete notation and normalization ledger, supersession rules, the final scope and circularity firewall, and the machine-readable interaction certificate used to verify the exact compact-group cumulant intervals.

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The author used OpenAI’s ChatGPT, including GPT-4.5 through GPT-5.6, as an interactive research and adversarial editorial tool during the development of this manuscript and the broader five-paper series. Its use included assistance with organizing mathematical material, testing formulations and proof strategies, improving exposition, checking notation and cross-references, preparing and reviewing LaTeX source, and supporting computational and document-audit workflows.

The conceptual framework, ontologies, definitions, conjectures, identification of the principal obstructions, load-bearing components of the proofs, and overall proof strategies originated with and remain solely those of the author.

All mathematical claims, definitions, theorems, proofs, computations, interpretations, citations, and final wording were reviewed and approved by the author, who assumes full responsibility for the content of the manuscript. No generative artificial-intelligence system is credited as an author.

AUTHOR DECLARATIONS

Conflict of Interest

The author has no conflicts to disclose.

Author Contributions

Jeffrey S Little: Conceptualization (lead); Formal analysis (lead); Investigation (lead); Methodology (lead); Software (lead); Validation (lead); Visualization (lead); Writing–original draft (lead); Writing–review and editing (lead).

DATA AVAILABILITY

Data sharing is not applicable to this article because no new experimental data were created or analyzed. The exact finite compact-group certificate and its machine-readable output are included in the supplementary material and accompanying computational reproducibility archive.

Appendix A: Definitive Branch and Obligation Matrices

1. Branch matrix

Branch	Ordinary observable object	Represented object	Spectral theorem	Status
Sector I	One global observable vacuum GNS system	One global strict represented system	Forward cone, exact zero sector, and positive invariant-mass gap	Closed
Sector II	Global ordinary system after physical state-section verification	Strict in each corridor, relatively twisted across corridor germs	Corridorwise theorem requiring a positive uniform corridor infimum	Conditional
Scalar Sector III	One global observable vacuum GNS system	Chart-corridor coefficient-line Hilbert gerbe	Objectwise and all-seam forward cone, exact zero subgerbe, and mass gap	Closed
Corr.	Ordinary or Markov shadow where available	Normal correspondences with Connes fusion	Requires module generators, support calculus, and fusion-stable essential infimum	Conditional

2. SP1–SP12 definitive closure and branch certification

ID	Obligation	Definitive closure
SP1	Spectral restart	Closed for certified scalar branches; declared conditional data retained.
SP2	Covariant vacuum state	Global ordinary invariant state; represented object typed by sector.

SP3	Strong implementation	Global, corridorwise, or chart-corridor as required.
SP4	Generators	Joint PVM, strongly commuting momenta, invariant mass, and common core.
SP5	Forward cone	Past and spacelike Lorentz-orbit fibers vanish.
SP6	Physical bridge	Glue–Mosco–Riesz plus $P_0^2 \geq K^{\text{thin}}$; orbit rigidity gives the invariant bound.
SP7	Exact zero sector	$E(\{0\}) = E_{M^2}(\{0\}) = P_\Omega$; nonzero null support vanishes.
SP8	Covariance and compatibility	All-Borel seam covariance; nonlinear surgery compression; Gauss–Ward graph reduction.
SP9	Gluing stability	Strict, corridor, coefficient-line, product, quotient, direct sum, and direct integral in scalar scope.
SP10	Threshold transfer	Certified normalized mass-square lower bound κ_* .
SP11	Vacuum multiplicity	Stabilized complete-ground rank and sector-refined zero object.
SP12	Final theorem	Outcome A for fully certified mobile-incision Sector-I and scalar Sector-III branches; declared Sector-II/correspondence conditions retained.

Appendix B: Definitive Notation, Normalization, and Supersession

1. Notation and constants

Symbol	Definitive meaning
$M_{\text{inc}} = \mathbb{R}^{3,1} \setminus \Sigma$	Incised spacetime; never a quotient by Σ .
$\mathcal{A}_{G,\xi,0}^W$	Ordinary thin observable net.
$(\mathcal{H}^{\text{obs}}, \pi^{\text{obs}}, U^{\text{obs}})$	Global ordinary observable vacuum representation.
$E(B)$	Joint translation PVM on momentum space.
P_μ	Strongly commuting translation generators.
$M^2 = P_\mu P^\mu$	Invariant-mass-square operator after the forward-cone theorem.

$\bar{q}_0, K^{\text{thin}}$	Paper-IV stiffness form and associated operator; not physical mass by notation.
P_Ω	Complete-ground projection and exact zero-mass projection.
κ_*	Inherited normalized threshold and certified mass-square lower bound.
$m_{\text{cert}} = \sqrt{\kappa_*}$	Certified normalized mass lower bound.
m_{spec}	Exact lowest surviving nonzero mass shell; only $(m_{\text{spec}})^2 \geq \kappa_*$ is universal.
$L_{\alpha\beta}^\epsilon, V_{\alpha\beta}^\epsilon$	Sector-III coefficient lines and seam unitaries.
$\Omega_{\alpha\beta\gamma}^\epsilon$	Scalar associator retained by the represented theory.
$\mathcal{S}_*$	Generally nonlinear entropy-surgery map on normal states.

2. Dimensional restoration

If the form was normalized using a reference length $\ell_{G,\xi}$, then

$$(m_{G,\xi}^{\text{cert}})^2 = \ell_{G,\xi}^{-2} \hat{\kappa}_*.$$

The certified lower bound is distinguished from the exact lowest surviving shell.

3. Supersession rules

The invariant-state residual of Phase V.0 is superseded by the invariant complete-ground state theorem. The preliminary PHIGT residual is superseded by the physical hyperbolic identification theorem. The primitive candidate inequality $M^2 \geq cK^{\text{thin}}$ is rejected as a general starting theorem; the authoritative bridge is

$$P_0^2 \geq K^{\text{thin}} \geq \kappa_*(I - P_\Omega),$$

followed by the orbit argument. Norm-resolvent convergence is not used to infer arbitrary sharp physical projections. Sector-III represented data are never collapsed to one global Hilbert space. Surgery is treated as a nonlinear state map, not as an operator commuting with every spectral projection.

Appendix C: Final Scope and Circularity Firewall

1. Mass gap versus energy gap

The final theorem concerns the joint translation spectrum and the invariant Casimir M^2 , not merely a positive lower bound for P_0 in one frame. The forward-cone theorem is independent of stiffness. Lorentz-orbit rigidity upgrades the energy-square estimate to the invariant-mass statement.

2. Vacuum multiplicity

The identity $E(\{0\}) = P_\Omega$ does not imply rank one. Zero-momentum Hilbert multiplicity, multiplicity of Lorentz-fixed vacuum vectors, and the dimension of the normal ground-state face are distinct.

3. Clay-scope statement

The manuscript proves a branch-certified Haag–Kastler existence-and-gap theorem inside the constructed mobile-incision, incision-relative framework. It does not declare an unqualified resolution of the Clay Yang–Mills existence and mass-gap problem⁷. Such a claim requires an external end-to-end verification of the complete series against the exact accepted axiomatic and nontriviality requirements.

4. Conjecture firewall

The transport ontology, sectional gluons, scattering stack, space-filling confinement, emergent strings, stacky dualities, landscape-as-atlas, gluing TFT, and operational mass-realization statements are conjectures only. They introduce no premise into the proof of [Theorem XIV.18](#).

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