Coarse Transport, Central Twist, and Finite-Scale Coercivity for Yang–Mills Wilson-Tube Hamiltonians

Physical Wilson-channel reduction and exact ultraviolet–infrared leakage

J. Scott Little

Abstract

We prove a resolved finite-scale coercivity criterion for canonical gauge-invariant Yang–Mills Wilson-tube Hamiltonians. The ground sector is treated through the complete spectral projection $P^{\varepsilon}_{n,\mathrm{vac}}$, with no distinguished vacuum vector or rank-one assumption. Starting from a finite center-sensitive Wilson graph, we remove every raw graph-excitation direction lying in the exact interacting ground sector, capture the surviving channel inside a finite spectral subspace of the vacuum-orthogonal electric form operator, and Gram-normalize it. Vacuum orthogonality and electric admissibility are therefore conclusions rather than hypotheses. Recompression of the canonical magnetic cells yields a physical infrared constant. The remaining ultraviolet–infrared magnetic block decomposes exactly into reference-band mixing, localized-transport associator leakage, and collar leakage. Strict subcriticality of this mixed block gives

$$H^{\varepsilon}_{n} \geq E^{\varepsilon}_{0,n} I + \widehat{\kappa}_{n,\varepsilon,\theta}(b)(I - P^{\varepsilon}_{n,\mathrm{vac}}), \qquad \widehat{\kappa}_{n,\varepsilon,\theta}(b) > 0.$$

A complete $j \leq 1$ $SU(2)$ Peter–Weyl Galerkin realization computes the ground projection, Gram operator, physical recompression, exact reference leakage, and the explicit bound $\widehat{\kappa}_G = 0.657490679870\ldots$. The result is finite-scale and time-zero; no continuum or relativistic mass-gap claim is made.

Keywords: lattice gauge Hamiltonians; Wilson observables; central twist; magnetic graph Laplacian; quadratic-form coercivity; spectral projection; coarse transport.
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1 Introduction

1.1 The finite-scale spectral problem

Following the finite-lattice Hamiltonian framework of Wilson and Kogut–Susskind [1, 2, 5, 3], for a compact gauge group on a finite spatial complex the regulated electric Hamiltonian has discrete representation spectrum, the Wilson magnetic potential is bounded, and the complete Hamiltonian has compact resolvent. Abstract discreteness alone, however, does not identify the mechanisms producing separation above the ground sector. The aim of this article is to resolve the finite-scale lower bound into three visible pieces:

$$\boxed{\begin{array}{c}\text{electric exclusion} + \text{physicalized magnetic separation}\\ > \text{reference, transport, and collar leakage.}\end{array}} \tag{1.1}$$

The resulting estimate is relative to the complete interacting ground projection,

$$H_n^\varepsilon - E_{0,n}^\varepsilon I \geq \widehat{\kappa}_{n,\varepsilon,\theta}(b)(I - P_{n,\mathrm{vac}}^\varepsilon). \tag{1.2}$$

The ground eigenspace may be degenerate; no normalized vector in it is selected.

1.2 Why graph minima are not physical vacua

Detected center phases, in the spirit of center-flux data [4], define a finite magnetic graph operator $L_\theta = d_\theta^* d_\theta$. Its complete lowest spectral projection $P_{\theta,\min}$ acts on a coefficient space, whereas $P_{n,\mathrm{vac}}^\varepsilon$ acts on the physical gauge-invariant Hilbert space and is determined by the full electric-plus-magnetic Hamiltonian. We therefore do not assume

$$P_{n,\mathrm{vac}}^\varepsilon \mathsf{F}_{n,\theta}^{\mathrm{W}} = \mathsf{F}_{n,\theta}^{\mathrm{W}} P_{\theta,\min}.$$

The lowest graph space is used only to define a raw excitation complement. The exact physical vacuum is then removed independently.

1.3 Physicalization of a raw Wilson channel

Let $\mathsf{F}^{\mathrm{W}}_{n,\theta} : \mathscr{C}^0_n \to \mathcal{H}^{\varepsilon}_n(\Omega)$ be an isometric raw Wilson frame and set

$$\mathscr{E}^{\mathrm{W}}_{n,\theta} = \mathrm{Ran}(I - P_{\theta,\min}).$$

The vacuum-removed map is

$$T_{n,\theta} = (I - P^{\varepsilon}_{n,\mathrm{vac}})\mathsf{F}^{\mathrm{W}}_{n,\theta}|_{\mathscr{E}^{\mathrm{W}}_{n,\theta}}.$$

After deleting its kernel, the surviving coefficient space is

$$\mathscr{C}^{\mathrm{phys}}_{n,\theta} = \mathscr{E}^{\mathrm{W}}_{n,\theta} \ominus \mathrm{Ker}\, T_{n,\theta}.$$

A finite spectral projection of the vacuum-orthogonal electric operator captures the image injectively. If J is the captured map and $G = J^*J$, then

$$\mathsf{F}^{\mathrm{phys}}_{n,\theta} = JG^{-1/2}$$

is an isometry satisfying

$$P^{\varepsilon}_{n,\mathrm{vac}}\mathsf{F}^{\mathrm{phys}}_{n,\theta} = 0, \qquad P^{\varepsilon}_{n,\mathrm{IR}}\mathsf{F}^{\mathrm{phys}}_{n,\theta} = \mathsf{F}^{\mathrm{phys}}_{n,\theta}.$$

1.4 Main contributions

The article makes four principal contributions.

1. It replaces vacuum–graph alignment by an exact physicalization procedure based on vacuum removal, finite electric capture, and Gram normalization.

2. It proves that the graph separation survives physicalization while requiring the canonical magnetic cells to be recompressed on the resulting physical channel.

3. It removes exact electric block closure from the theorem and retains the complete mixed operator as the sum of reference, transport, and collar contributions.

4. It supplies a complete finite Peter–Weyl Galerkin witness in which every comparison datum and the final positive coercivity constant are computed explicitly.

1.5 Exact mixed-block decomposition

Wilson multiplication need not preserve the physical electric cutoff. We therefore retain ordinary resolved band mixing rather than imposing block closure. With an interior–transported–collar partition, the magnetic mixed block is decomposed exactly as

$$P^{\varepsilon}_{n,\mathrm{UV}}H^{\varepsilon}_{n,\mathrm{mag}}P^{\varepsilon}_{n,\mathrm{IR}} = K^{\varepsilon}_{n,\mathrm{ref}} + K^{\varepsilon}_{n,\mathrm{tr}} + K^{\varepsilon}_{n,\mathrm{col}}.$$

The transport term measures only the difference between two-step localized transport and a direct comparator; it is not the entire mixed block.

1.6 Principal result

If $\widehat{c}^{\varepsilon}_{n,\mathrm{UV}} > 0$, $\widehat{c}^{\theta}_{n,\mathrm{IR}}(b) > 0$, and

$$(\widehat{\varepsilon}^{\varepsilon}_{n,\mathrm{mix}})^2 < \widehat{c}^{\varepsilon}_{n,\mathrm{UV}}\widehat{c}^{\theta}_{n,\mathrm{IR}}(b), \qquad \widehat{\varepsilon}^{\varepsilon}_{n,\mathrm{mix}} = \|P^{\varepsilon}_{n,\mathrm{UV}}H^{\varepsilon}_{n,\mathrm{mag}}P^{\varepsilon}_{n,\mathrm{IR}}\|,$$

then the lower bound Equation (1.2) holds with

$$\widehat{\kappa}_{n,\varepsilon,\theta}(b) = \frac{\widehat{c}^{\varepsilon}_{n,\mathrm{UV}} + \widehat{c}^{\theta}_{n,\mathrm{IR}}(b) - \sqrt{(\widehat{c}^{\varepsilon}_{n,\mathrm{UV}} - \widehat{c}^{\theta}_{n,\mathrm{IR}}(b))^2 + 4(\widehat{\varepsilon}^{\varepsilon}_{n,\mathrm{mix}})^2}}{2}. \tag{1.3}$$

1.7 Notation guide

The principal layers of notation are summarized in Table 1.

Table 1: Principal notation used in the physicalization and block-coercivity argument.

Symbol	Meaning
$P^{\varepsilon}_{n,\mathrm{vac}}$, $E^{\varepsilon}_{0,n}$	Complete interacting ground projection and ground energy.
L_θ, $P_{\theta,\min}$, $\delta_{\mathrm{sec}}(\theta)$	Detected graph operator, complete lowest graph projection, and separation above its lowest eigenspace.
$\mathsf{F}^{\mathrm{W}}_{n,\theta}$, $T_{n,\theta}$	Raw Wilson frame and its restriction after exact vacuum removal.
$J^{\varepsilon}_{n,\theta}$, $G^{\varepsilon}_{n,\theta}$, $\mathsf{F}^{\mathrm{phys}}_{n,\theta}$	Electrically captured map, Gram operator, and normalized physical channel.
$c^{\mathrm{phys}}_{n,\mathrm{ch}}$, $c_{n,\mathrm{res}}$, $\widehat{c}^{\theta}_{n,\mathrm{IR}}$	Physical channel, residual infrared, and complete infrared coercivity constants.
$\widehat{c}^{\varepsilon}_{n,\mathrm{UV}}$	Vacuum-renormalized electric exclusion constant on the ultraviolet block.
$K^{\varepsilon}_{n,\mathrm{ref}}$, $K^{\varepsilon}_{n,\mathrm{tr}}$, $K^{\varepsilon}_{n,\mathrm{col}}$	Reference-band, transport-associator, and collar mixed blocks.
$\widehat{\varepsilon}^{\varepsilon}_{n,\mathrm{mix}}$, $\varepsilon^{\varepsilon}_{n,\mathrm{hyb}}$	Exact mixed-block norm and its hybrid sufficient majorant.
$\widehat{\kappa}_{n,\varepsilon,\theta}(b)$	Final finite-scale coercivity constant.

1.8 Scope and organization

The theorem is a finite-scale coercivity criterion, and it does not resolve the continuum Yang–Mills problem formulated by Jaffe and Witten [16]. It does not construct a continuum measure, a Haag–Kastler net, Poincaré covariance, a continuum vacuum, confinement, or a relativistic mass gap. Sections 2–5 construct the canonical finite system, coarse transport, central graph data, and raw Wilson channels. Section 6 constructs the vacuum-orthogonal electric operator. Section 7 performs physical channel reduction. Section 8 decomposes the exact mixed block. Sections 9 and 10 prove the vacuum reduction and principal theorem. Section 11 records uniform familywise criteria, and Section 12 states interpretation and scope. The appendices provide raw $SU(2)$ and $SU(3)$ witnesses, a complete physicalized $SU(2)$ Galerkin verification, and the technical finite-dimensional lemmas.

2 Canonical finite-scale Wilson-tube systems

2.1 Spatial complexes and configuration space

Let $\Omega \subset \mathbb{R}^3$ be bounded and let $\mathcal{L}_n(\Omega)$ be a finite oriented cell complex with positively oriented edge set $E_n^+(\Omega)$, vertex set $V_n(\Omega)$, and oriented magnetic cell set $F_n(\Omega)$. Let G be a compact connected Lie group with normalized Haar measure dg. The configuration space and product measure are

$$X_n(\Omega) = G^{E_n^+(\Omega)}, \qquad \mu_n = \bigotimes_{e \in E_n^+(\Omega)} dg_e.$$

Orientation reversal is represented by $U_{\bar{e}} = U_e^{-1}$.

2.2 Gauge action and the physical Hilbert space

The finite gauge group is $\mathcal{G}_n(\Omega) = G^{V_n^{\mathrm{int}}(\Omega)}$, or the corresponding boundary-fixed subgroup. It acts unitarily on

$$\mathcal{H}_{n,\mathrm{kin}} = L^2(X_n, \mu_n)$$

by

$$(h \cdot U)_e = h_{s(e)} U_e h_{t(e)}^{-1}.$$

Haar averaging gives the orthogonal Gauss projection

$$\Gamma_n = \int_{\mathcal{G}_n(\Omega)} U_n(h)\, dh,$$

and the physical Hilbert space is

$$\mathcal{H}_n(\Omega) = \mathrm{Ran}\, \Gamma_n = \mathcal{H}_{n,\mathrm{kin}}^{\mathcal{G}_n(\Omega)}. \tag{2.1}$$

The Hilbert space does not depend on the tube thickness ε; the notation $\mathcal{H}_n^\varepsilon(\Omega)$ used later is retained only to keep the scale labels aligned with the magnetic Hamiltonian.

2.3 The kinematic and physical electric Hamiltonians

For each edge let $\Delta_e \geq 0$ be the positive Laplace–Casimir operator on the corresponding copy of G. Fix weights $\eta_{n,e}^{\mathrm{el}} > 0$ and coupling $g_n > 0$. On the kinematic Hilbert space define

$$\widetilde{H}_{n,\mathrm{el}} = \frac{g_n^2}{2} \sum_{e \in E_n^+(\Omega)} \eta_{n,e}^{\mathrm{el}} \Delta_e. \tag{2.2}$$

Peter–Weyl theory decomposes $L^2(G^{E_n^+})$ into finite-dimensional representation sectors. On a sector labelled by irreducible representations $(R_e)_e$, the eigenvalue of $\widetilde{H}_{n,\mathrm{el}}$ is

$$\frac{g_n^2}{2} \sum_e \eta_{n,e}^{\mathrm{el}} C_2(R_e).$$

Because every edge weight is positive and the Casimir eigenvalues diverge along irreducible representations, $\widetilde{H}_{n,\mathrm{el}}$ is nonnegative, self-adjoint, and has compact resolvent.

The Casimir commutes with left and right translations. Hence

$$[\widetilde{H}_{n,\mathrm{el}}, \Gamma_n] = 0. \tag{2.3}$$

The physical Hilbert space is therefore reducing for the electric operator. Define

$$H_{n,\mathrm{el}} = \widetilde{H}_{n,\mathrm{el}}|_{\mathcal{H}_n(\Omega)}. \tag{2.4}$$

Its resolvent is

$$(H_{n,\mathrm{el}} + 1)^{-1} = \Gamma_n (\widetilde{H}_{n,\mathrm{el}} + 1)^{-1}|_{\mathcal{H}_n(\Omega)},$$

so it is compact. Its closed quadratic form is denoted by $Q_{n,\mathrm{el}}$, with sesquilinear form $q_{n,\mathrm{el}}$.

2.4 Wilson tubes and canonical magnetic cells

A Wilson tube is a finite tubular thickening of a closed spatial loop or loop family, with fixed $\varepsilon > 0$. For a unitary representation R, its trace observable is

$$W_{T,R}(U) = \mathrm{Tr}_R(\mathrm{Hol}_T(U)).$$

For each magnetic cell f, choose a bounded gauge-invariant self-adjoint potential $V^{\varepsilon}_{n,f} \geq 0$. Define

$$h^{\varepsilon}_{n,f} = \omega^{\mathrm{mag}}_{n,f} M_{V^{\varepsilon}_{n,f}}, \qquad \omega^{\mathrm{mag}}_{n,f} = \frac{\eta^{\mathrm{mag}}_{n,f}}{2g_n^2} > 0.$$

The magnetic Hamiltonian is

$$H^{\varepsilon}_{n,\mathrm{mag}} = \sum_{f \in F_n(\Omega)} h^{\varepsilon}_{n,f}. \tag{2.5}$$

It is bounded, self-adjoint, gauge invariant, and nonnegative.

2.5 Canonical Hamiltonian

Set

$$H^{\varepsilon}_n = H_{n,\mathrm{el}} + H^{\varepsilon}_{n,\mathrm{mag}}, \qquad D(H^{\varepsilon}_n) = D(H_{n,\mathrm{el}}). \tag{2.6}$$

The associated form domain is $D(Q_{n,\mathrm{el}})$. Bounded self-adjoint perturbation preserves compact resolvent.

Theorem 2.1 (Finite-scale canonical Hamiltonian)**.** *The operator H^{ε}_n is nonnegative, self-adjoint, gauge invariant, and has compact resolvent on $\mathcal{H}_n(\Omega)$. Its lowest spectral value*

$$E^{\varepsilon}_{0,n} = \inf \mathrm{spec}(H^{\varepsilon}_n)$$

is an eigenvalue of finite multiplicity, and

$$P^{\varepsilon}_{n,\mathrm{vac}} = \mathbf{1}_{\{E^{\varepsilon}_{0,n}\}}(H^{\varepsilon}_n)$$

is a finite-rank projection.

Proof. The electric assertions follow from the Peter–Weyl decomposition and Equation (2.3). The magnetic term is bounded and self-adjoint, so the Kato–Rellich bounded-perturbation theorem gives self-adjointness and domain equality. The resolvent identity shows that compactness of the electric resolvent is preserved under a bounded perturbation. The spectral assertions follow from the compact-resolvent spectral theorem. □

2.6 Finite Wilson windows

Let $\mathfrak{O}_n^{\mathrm{gi}}(\Omega) \subset L^\infty(X_n)^{\mathcal{G}_n}$ be an enlarged gauge-invariant observable space. A finite Wilson window is a finite-dimensional unital involution-invariant subspace

$$\mathscr{W}_n^\varepsilon(\Omega) \subset \mathfrak{O}_n^{\mathrm{gi}}(\Omega).$$

The localized Wilson algebra generated by the selected windows is denoted $\mathcal{A}_n^\varepsilon(\Omega)$. These spaces are comparison data; they do not modify the canonical Hamiltonian.

Because $\mathscr{W}_n^\varepsilon(\Omega)$ is finite-dimensional and contained in L^∞, every linear map into the window is bounded from L^2 to L^∞ on its finite-dimensional range. In particular, the L^2-orthogonal projection used below produces bounded functions. Involution invariance makes that projection adjoint preserving, but it is not assumed to be positive.

3 Exact coarse transport and finite Wilson localization

3.1 Haar-compatible microscopic refinement

For $m \preceq n$, a Haar-compatible refinement map is a measurable coarse-holonomy map

$$p_{mn} : X_n(\Omega) \longrightarrow X_m(\Omega) \tag{3.1}$$

satisfying:

1. $p_{\ell n} = p_{\ell m} \circ p_{mn}$ for $\ell \preceq m \preceq n$;
2. $(p_{mn})_* \mu_n = \mu_m$;
3. p_{mn} is equivariant for the corresponding coarse and fine gauge actions.

For ordinary edge subdivision, p_{mn} is obtained by taking the ordered product of fine edge holonomies along each coarse edge. Haar convolution and integration over unused fine variables give the measure-preserving property.

Define the cylindrical pullback

$$\iota_{mn} : L^p(X_m, \mu_m) \longrightarrow L^p(X_n, \mu_n), \qquad \iota_{mn} A = A \circ p_{mn}. \tag{3.2}$$

It is an isometric unital $*$-monomorphism for $1 \le p \le \infty$, and

$$\iota_{\ell n} = \iota_{mn} \iota_{\ell m}. \tag{3.3}$$

Let

$$\mathbb{E}_{mn} = \iota_{mn}^* \tag{3.4}$$

be the L^2-adjoint, equivalently the fiberwise Haar conditional expectation [8]. Gauge equivariance implies that ι_{mn} and $\mathbb{E}_{mn}$ preserve gauge-invariant observables.

Theorem 3.1 (Exact microscopic coarse transport)**.** *For every Haar-compatible refinement system, the maps $\mathbb{E}_{mn}$ are unital, positive, adjoint preserving, and contractive in L^2 and L^∞. They satisfy the exact contravariant composition law*

$$\boxed{\mathbb{E}_{\ell m} \mathbb{E}_{mn} = \mathbb{E}_{\ell n}.} \tag{3.5}$$

Proof. The conditional-expectation properties follow from fiberwise Haar integration. By Equation (3.3),

$$\mathbb{E}_{\ell m} \mathbb{E}_{mn} = \iota_{\ell m}^* \iota_{mn}^* = (\iota_{mn} \iota_{\ell m})^* = \iota_{\ell n}^* = \mathbb{E}_{\ell n}.$$

□

3.2 Finite Wilson localization

Let

$$\jmath_n^\Omega : \mathscr{W}_n^\varepsilon(\Omega) \hookrightarrow \mathfrak{O}_n^{\mathrm{gi}}(\Omega)$$

denote the inclusion of the finite Wilson window, and let

$$L_n^\Omega : L^2(X_n, \mu_n)^{\mathcal{G}_n} \longrightarrow \mathscr{W}_n^\varepsilon(\Omega)$$

be the L^2-orthogonal projection. Then

$$L_n^\Omega \jmath_n^\Omega = I, \qquad L_n^\Omega(A^*) = L_n^\Omega(A)^*.$$

The second identity follows because the window is invariant under the antiunitary involution $A \mapsto A^*$. Define

$$\Lambda_n^\Omega = \jmath_n^\Omega L_n^\Omega - I. \tag{3.6}$$

The full norm of a proper orthogonal-projection defect is generally one; all smallness statements below are source restricted.

3.3 Wilson-side transport and associator

Define

$$T_{mn}^\Omega = L_m^\Omega \mathbb{E}_{mn} \jmath_n^\Omega. \tag{3.7}$$

For $\ell \preceq m \preceq n$, the coherence defect [14] is

$$\alpha_{\ell mn}^\Omega = T_{\ell m}^\Omega T_{mn}^\Omega - T_{\ell n}^\Omega. \tag{3.8}$$

Proposition 3.2 (Exact Wilson localization-defect identity)**.** *For every composable triple,*

$$\boxed{\alpha_{\ell mn}^\Omega = L_\ell^\Omega \mathbb{E}_{\ell m} \Lambda_m^\Omega \mathbb{E}_{mn} \jmath_n^\Omega.} \tag{3.9}$$

Proof. Insert $\jmath_m^\Omega L_m^\Omega = I + \Lambda_m^\Omega$ into the two-step transport and use Equation (3.5):

$$\begin{aligned} T_{\ell m}^\Omega T_{mn}^\Omega &= L_\ell^\Omega \mathbb{E}_{\ell m} \jmath_m^\Omega L_m^\Omega \mathbb{E}_{mn} \jmath_n^\Omega \\ &= L_\ell^\Omega \mathbb{E}_{\ell n} \jmath_n^\Omega + L_\ell^\Omega \mathbb{E}_{\ell m} \Lambda_m^\Omega \mathbb{E}_{mn} \jmath_n^\Omega. \end{aligned}$$

The first term is $T_{\ell n}^\Omega$. □

3.4 Source-restricted laxity and multiplier bound

For any nonzero $A \in \mathscr{W}_n^\varepsilon$, write

$$D_{\ell mn}^{\mathrm{lax}}(A) = \frac{\|\alpha_{\ell mn}^\Omega A\|_2}{\|A\|_2}.$$

For a selected source $A_{n,f}^\varepsilon$, define

$$D_{\ell mn,f}^{\mathrm{lax}} = \frac{\|\alpha_{\ell mn}^\Omega A_{n,f}^\varepsilon\|_2}{\|A_{n,f}^\varepsilon\|_2}. \tag{3.10}$$

For a finite transported-cell family set

$$D^{\varepsilon}_{n,\mathrm{lax}} = \begin{cases} \max\limits_{f \in F^{\varepsilon}_{n,\mathrm{tr}}} D^{\mathrm{lax}}_{\ell_f m_f n, f}, & F^{\varepsilon}_{n,\mathrm{tr}} \neq \varnothing, \\ 0, & F^{\varepsilon}_{n,\mathrm{tr}} = \varnothing. \end{cases} \tag{3.11}$$

The finite-window norm-conversion constant is

$$C^{\varepsilon}_{m,\infty\leftarrow 2} = \sup_{0 \neq A \in \mathscr{W}^{\varepsilon}_{m}} \frac{\|A\|_{\infty}}{\|A\|_{2}} < \infty. \tag{3.12}$$

Theorem 3.3 (Explicit localized-transport multiplier estimate)**.** *Let $A \in \mathscr{W}^{\varepsilon}_{n}$, and suppose the associator output lies in $\mathscr{W}^{\varepsilon}_{\ell}$. Then*

$$\boxed{\left\| M_{\iota_{\ell n} \jmath^{\Omega}_{\ell} \alpha^{\Omega}_{\ell m n} A} \right\|_{\mathrm{op}} \leq C^{\varepsilon}_{\ell,\infty\leftarrow 2} D^{\mathrm{lax}}_{\ell m n}(A) \|A\|_{2}.} \tag{3.13}$$

Consequently, a transported magnetic cell of weight $\omega^{\mathrm{mag}}_{n,f}$ satisfies

$$\|\Delta h^{\mathrm{tr}}_{n,f}\|_{\mathrm{op}} \leq \omega^{\mathrm{mag}}_{n,f} C^{\varepsilon}_{\ell_f,\infty\leftarrow 2} D^{\mathrm{lax}}_{\ell_f m_f n, f} \|A^{\varepsilon}_{n,f}\|_{2}. \tag{3.14}$$

The associator and its multiplier correction are adjoint preserving on self-adjoint sources.

Proof. Cylindrical pullback preserves the L^{∞} norm, multiplication has operator norm equal to the multiplier sup norm, and finite-dimensional norm conversion gives

$$\|M_{\iota_{\ell n} \jmath^{\Omega}_{\ell} \alpha A}\|_{\mathrm{op}} = \|\alpha A\|_{\infty} \leq C^{\varepsilon}_{\ell,\infty\leftarrow 2} \|\alpha A\|_{2}.$$

Insert the definition of the source-restricted ratio. Adjoint preservation follows from the corresponding properties of L, $\mathbb{E}$, $\jmath$, and ι. □

Remark 3.4. The identity Equation (3.9) shows that nonassociativity is precisely the intermediate finite-window localization error transported to the coarse window. It does not identify the associator with the complete ultraviolet–infrared mixed block.

4 Central sectors and magnetic graph separation

4.1 Corrected-cell covers and center cocycles

A corrected-cell cover is a finite cover $\{U_v\}_{v \in V_\theta}$ of the selected Wilson region together with local Wilson trivializations and overlap maps $g_{vw} : U_v \cap U_w \to G$ whose failure of the ordinary cocycle identity is central. On a nonempty triple overlap,

$$g_{uv} g_{vw} g_{wu} = \zeta_{uvw} \in Z(G). \tag{4.1}$$

Associativity on quadruple overlaps gives $\delta\zeta = 1$. Thus ζ defines a simplicial two-cocycle on the nerve $N_n(\Omega)$, with coefficients in the abelian group $Z(G)$, and

$$[\zeta_n] \in H^2(N_n(\Omega); Z(G)).$$

Lemma 4.1 (Independence from central lifts)**.** *If $g'_{vw} = c_{vw} g_{vw}$ with a central one-cochain $c_{vw} \in Z(G)$, then*

$$\zeta' = \zeta\, \delta c.$$

Hence $[\zeta_n]$ is independent of the chosen central lifts. If R is a unitary representation with center character $\zeta_R : Z(G) \to \mathbb{T}$, then $[\zeta_R(\zeta_n)]$ is likewise independent of the lifts.

Proof. Centrality permits the factors c_{uv}, c_{vw}, c_{wu} to be moved through the overlap maps. Multiplying the three modified transitions gives

$$g'_{uv} g'_{vw} g'_{wu} = (c_{uv} c_{vw} c_{wu}) \zeta_{uvw} = (\delta c)_{uvw} \zeta_{uvw}.$$

Applying the homomorphism ζ_R sends coboundaries to coboundaries. □

4.2 Refinement compatibility

A refinement map of covers induces a simplicial map $r_{mn} : N_n \to N_m$ and pullback on cochains. A compatible finite sector family satisfies

$$[\zeta_n] = r_{mn}^* [\zeta_m].$$

Proposition 4.2 (Detection commutes with refinement)**.** *For every center character ζ_R,*

$$\zeta_R(r_{mn}^* \zeta_m) = r_{mn}^* \zeta_R(\zeta_m),$$

and therefore the diagram

$$\begin{array}{ccc} H^2(N_m; Z(G)) & \xrightarrow{(\zeta_R)_*} & H^2(N_m; \mathbb{T}) \\ \downarrow{\scriptstyle r_{mn}^*} & & \downarrow{\scriptstyle r_{mn}^*} \\ H^2(N_n; Z(G)) & \xrightarrow[(\zeta_R)_*]{} & H^2(N_n; \mathbb{T}) \end{array}$$

commutes.

Proof. Both operations act pointwise on simplicial cochains, and a group homomorphism commutes with pullback. □

4.3 Detected graph realization

A detected graph realization of the central sector consists of:

1. a finite connected weighted graph $\mathcal{G}_\theta = (V_\theta, E_\theta, a)$;
2. a specified detection or transgression rule
$$\mathfrak{T}_{n,R} : Z^2(N_n; Z(G)) \longrightarrow C^1(\mathcal{G}_\theta; \mathbb{T});$$
3. the edge phase cochain $\theta = \mathfrak{T}_{n,R}(\zeta_n)$, with $\theta_{wv} = \overline{\theta_{vw}}$.

The required covariance is that replacing ζ_n by a cohomologous cocycle changes θ only by a vertex rephasing:

$$\theta'_{vw} = u_v \theta_{vw} \overline{u_w} \quad \text{for some } u : V_\theta \to \mathbb{T}. \tag{4.2}$$

No canonical transgression is asserted for every cover. The fixed-scale theorem uses the resulting finite graph connection θ; construction and refinement transport of $\mathfrak{T}_{n,R}$ are separate geometric data.

4.4 Magnetic graph operator

The finite magnetic Laplacian below follows the standard spectral-graph framework [11, 12]. Identify the raw coefficient space with

$$\mathscr{C}_n^0 = \mathbb{C}^{V_\theta}. \tag{4.3}$$

For an oriented edge (v, w), define the unweighted covariant edge vector and rank-one form

$$r_{vw}^\theta = e_v - \overline{\theta_{vw}} e_w, \qquad \ell_{vw}^\theta = |r_{vw}^\theta\rangle\langle r_{vw}^\theta|. \tag{4.4}$$

Define

$$(d_\theta c)_{vw} = a_{vw}^{1/2}(c_v - \theta_{vw} c_w), \qquad L_\theta = d_\theta^* d_\theta.$$

Then

$$\boxed{L_\theta = \sum_{\{v,w\}\in E_\theta} a_{vw}\ell_{vw}^\theta} \tag{4.5}$$

and

$$\langle c, L_\theta c\rangle = \sum_{\{v,w\}\in E_\theta} a_{vw}|c_v - \theta_{vw} c_w|^2. \tag{4.6}$$

4.5 Vertex rephasing and strict sectors

For $u : V_\theta \to \mathbb{T}$, let $(U_u c)_v = u_v c_v$. If θ' is given by Equation (4.2), then

$$L_{\theta'} = U_u L_\theta U_u^*.$$

Thus the spectrum depends only on graph holonomy.

A connected graph sector is called strict when its connection has trivial holonomy around every graph cycle, equivalently when it is vertex-rephasing equivalent to the trivial connection. Otherwise it is frustrated.

4.6 Lowest eigenspace and separation

Let

$$\lambda_1(\theta) \le \lambda_2(\theta) \le \cdots$$

be the graph eigenvalues with multiplicity, and let $m_1(\theta)$ be the multiplicity of λ_1. Define

$$P_{\theta,\min} = \mathbf{1}_{\{\lambda_1(\theta)\}}(L_\theta), \qquad \delta_{\sec}(\theta) = \lambda_{m_1(\theta)+1}(\theta) - \lambda_1(\theta),$$

whenever the excitation complement is nonzero.

Theorem 4.3 (Magnetic graph separation)**.** *On* $\operatorname{Ran}(I - P_{\theta,\min})$*,*

$$L_\theta - \lambda_1(\theta) I \ge \delta_{\sec}(\theta) I. \tag{4.7}$$

In a connected strict sector, $\lambda_1 = 0$ *and* $\operatorname{Ran} P_{\theta,\min} = \operatorname{Ker} d_\theta$ *is one-dimensional. In a connected frustrated sector,* $\operatorname{Ker} d_\theta = \{0\}$ *and* $\lambda_1 > 0$*, although* $P_{\theta,\min} \ne 0$*.*

Proof. The inequality follows from finite-dimensional spectral calculus. In a connected strict sector, vertex rephasing reduces the connection to the trivial one, whose kernel consists of constant vectors. Conversely, a nonzero covariantly constant section forces every cycle holonomy to be trivial. Hence a frustrated connected graph has trivial kernel and positive lowest eigenvalue. □

Remark 4.4. The relevant coercive graph scale is the separation above the complete lowest eigenspace, not the frustrated baseline λ_1 alone. Neither graph-minimum space is identified with the interacting physical vacuum.

5 Raw Wilson channels and canonical magnetic compression

5.1 Vertex-labelled raw channel frame

With $\mathscr{C}_n^0 = \mathbb{C}^{V_\theta}$, choose gauge-invariant Wilson-tube vectors $\Phi_{n,v}^{\mathrm{W}} \in \mathcal{H}_n(\Omega)$, indexed by graph vertices, and define

$$\mathsf{F}_{n,\theta}^{\mathrm{W}} e_v = \Phi_{n,v}^{\mathrm{W}}. \tag{5.1}$$

The isometric-frame condition

$$(\mathsf{F}_{n,\theta}^{\mathrm{W}})^* \mathsf{F}_{n,\theta}^{\mathrm{W}} = I_{\mathscr{C}_n^0} \tag{5.2}$$

is part of the finite comparison data; it is not automatic for an arbitrary Wilson family. Define

$$\mathscr{E}_{n,\theta}^{\mathrm{W}} = \mathrm{Ran}(I - P_{\theta,\min}).$$

The graph estimate Equation (4.7) holds on this coefficient space before any claim about the physical vacuum is made.

5.2 Canonical cell compressions

For each canonical magnetic cell define

$$M_{n,f}^{\mathrm{W}} = (\mathsf{F}_{n,\theta}^{\mathrm{W}})^* h_{n,f}^{\varepsilon} \mathsf{F}_{n,\theta}^{\mathrm{W}} \geq 0. \tag{5.3}$$

5.3 Disjoint positive edge extraction

Choose an active edge set $E_{\mathrm{act}} \subset E_\theta$ and a disjoint allocation of canonical cells

$$F_{n,\mathrm{ch}}^{\varepsilon} = \dot{\bigcup}_{(v,w)\in E_{\mathrm{act}}} F_{n;vw}^{\varepsilon} \subset F_n(\Omega). \tag{5.4}$$

A cell is used in at most one edge extraction. Unallocated cells remain as a nonnegative remainder.

For $f \in F_{n;vw}^{\varepsilon}$, define

$$\beta_{n,f;vw}^{\mathrm{W}} = \sup\{\beta \geq 0 : M_{n,f}^{\mathrm{W}} - \beta \ell_{vw}^{\theta} \geq 0\}. \tag{5.5}$$

Set

$$\beta_{n;vw}^{\mathrm{W}} = \sum_{f\in F_{n;vw}^{\varepsilon}} \beta_{n,f;vw}^{\mathrm{W}}, \qquad b_{n,\mathrm{ch}}^{\mathrm{W,max}} = \min_{(v,w)\in E_{\mathrm{act}}} \frac{\beta_{n;vw}^{\mathrm{W}}}{a_{vw}}. \tag{5.6}$$

Proposition 5.1 (Raw magnetic domination)**.** *Assume every graph edge required by the comparison lies in* E_{act} *and* $b_{n,\mathrm{ch}}^{\mathrm{W,max}} > 0$*. Then, for every* $0 \leq b \leq b_{n,\mathrm{ch}}^{\mathrm{W,max}}$*,*

$$\sum_{f\in F_n(\Omega)} M_{n,f}^{\mathrm{W}} \geq b L_\theta. \tag{5.7}$$

Proof. For each active edge, summing the cellwise inequalities over the disjoint set $F_{n;vw}^{\varepsilon}$ gives

$$\sum_{f\in F_{n;vw}^{\varepsilon}} M_{n,f}^{\mathrm{W}} \geq \beta_{n;vw}^{\mathrm{W}} \ell_{vw}^{\theta} \geq b a_{vw} \ell_{vw}^{\theta}.$$

The allocations are disjoint, so the inequalities may be summed over edges without reusing a cell. The unallocated canonical cell compressions are positive. Apply Equation (4.5). □

Scope Note 5.2. Positive raw extraction does not imply positive extraction after exact vacuum removal and finite electric capture. The canonical cell operators must be recompressed on the physicalized frame constructed in Section 7.

6 Vacuum-orthogonal electric spectral reduction

6.1 Restricted electric form

Define the complete ground sector and its orthogonal complement by

$$\mathcal{V}^{\varepsilon}_{n,\mathrm{vac}} = \operatorname{Ran} P^{\varepsilon}_{n,\mathrm{vac}}, \qquad \mathcal{H}^{\varepsilon}_{n,\perp} = \operatorname{Ran}(I - P^{\varepsilon}_{n,\mathrm{vac}}).$$

The vacuum projection need not commute with $H_{n,\mathrm{el}}$. Define instead

$$D(Q^{\perp}_{n,\mathrm{el}}) = D(Q_{n,\mathrm{el}}) \cap \mathcal{H}^{\varepsilon}_{n,\perp}, \qquad Q^{\perp}_{n,\mathrm{el}}[\psi] = Q_{n,\mathrm{el}}[\psi].$$

Lemma 6.1 (Form-domain preservation)**.** *The finite-rank projection $P^{\varepsilon}_{n,\mathrm{vac}}$ maps $D(Q_{n,\mathrm{el}})$ into itself.*

Proof. Every ground eigenvector of H^{ε}_{n} lies in $D(H^{\varepsilon}_{n}) = D(H_{n,\mathrm{el}})$. Expanding the finite-rank projection in an orthonormal ground basis gives the claim. □

Theorem 6.2 (Vacuum-orthogonal electric operator)**.** *The restricted form $Q^{\perp}_{n,\mathrm{el}}$ is densely defined, closed, and nonnegative on $\mathcal{H}^{\varepsilon}_{n,\perp}$. Its associated operator $H^{\perp}_{n,\mathrm{el}}$ is nonnegative, self-adjoint, and has compact resolvent, by the standard representation and compact-form embedding theory [6, 7].*

Proof. For $x \in \mathcal{H}^{\varepsilon}_{n,\perp}$, approximate x by $x_k \in D(Q_{n,\mathrm{el}})$ and apply $I - P^{\varepsilon}_{n,\mathrm{vac}}$; Theorem 6.1 gives density. Closedness follows because the intersection with the Hilbert-closed subspace $\mathcal{H}^{\varepsilon}_{n,\perp}$ is closed in the form norm. Compact resolvent of $H_{n,\mathrm{el}}$ is equivalent to compactness of its form-domain embedding; the restriction to a form-closed subspace remains compact. □

Remark 6.3. The notation $H^{\perp}_{n,\mathrm{el}}$ denotes a form restriction. It is not asserted to equal $(I - P^{\varepsilon}_{n,\mathrm{vac}})H_{n,\mathrm{el}}(I - P^{\varepsilon}_{n,\mathrm{vac}})$.

6.2 Whole-eigenspace cutoffs

Let

$$0 \le \mu^{\varepsilon}_{n,0} < \mu^{\varepsilon}_{n,1} < \mu^{\varepsilon}_{n,2} < \cdots$$

be the distinct eigenvalues of $H^{\perp}_{n,\mathrm{el}}$. For $\Lambda \ge 0$, define

$$\Pi^{\varepsilon}_{n,\perp}(\Lambda) = \mathbf{1}_{[0,\Lambda]}(H^{\perp}_{n,\mathrm{el}}).$$

These projections have finite rank and converge strongly to the identity on $\mathcal{H}^{\varepsilon}_{n,\perp}$.

Choose J and a cutoff satisfying

$$\mu^{\varepsilon}_{n,J} \le \Lambda^{\varepsilon}_{n,\mathrm{IR}} < \mu^{\varepsilon}_{n,J+1}.$$

Set

$$P^{\varepsilon}_{n,\mathrm{IR}} = \Pi^{\varepsilon}_{n,\perp}(\Lambda^{\varepsilon}_{n,\mathrm{IR}}), \qquad P^{\varepsilon}_{n,\mathrm{UV}} = I - P^{\varepsilon}_{n,\mathrm{vac}} - P^{\varepsilon}_{n,\mathrm{IR}},$$

and define

$$\Lambda^{\varepsilon}_{n,\mathrm{ret}} = \mu^{\varepsilon}_{n,J}, \qquad \Lambda^{\varepsilon}_{n,\mathrm{excl}} = \mu^{\varepsilon}_{n,J+1}, \qquad \delta^{\varepsilon}_{n,\mathrm{el}} = \mu^{\varepsilon}_{n,J+1} - \mu^{\varepsilon}_{n,J} > 0.$$

The cutoff contains complete eigenspaces and never splits a degeneracy.

Proposition 6.4 (Electric block orthogonality and exclusion)**.** *For $u \in D(Q_{n,\mathrm{el}}) \cap \operatorname{Ran} P^{\varepsilon}_{n,\mathrm{UV}}$ and $w \in \operatorname{Ran} P^{\varepsilon}_{n,\mathrm{IR}}$,*

$$q_{n,\mathrm{el}}(u, w) = 0, \qquad Q_{n,\mathrm{el}}[u] \geq \Lambda^{\varepsilon}_{n,\mathrm{excl}} \|u\|^2.$$

Consequently, if $\Lambda^{\varepsilon}_{n,\mathrm{excl}} > E^{\varepsilon}_{0,n}$, then

$$\widehat{Q}^{\varepsilon}_{n}[u] \geq \widehat{c}^{\varepsilon}_{n,\mathrm{UV}} \|u\|^2, \qquad \widehat{c}^{\varepsilon}_{n,\mathrm{UV}} = \Lambda^{\varepsilon}_{n,\mathrm{excl}} - E^{\varepsilon}_{0,n} > 0. \tag{6.1}$$

Proof. The two vectors belong to disjoint spectral subspaces of $H^{\perp}_{n,\mathrm{el}}$, giving form orthogonality and the electric lower bound. Since $H^{\varepsilon}_{n,\mathrm{mag}} \geq 0$, subtracting $E^{\varepsilon}_{0,n}\|u\|^2$ yields Equation (6.1). □

Corollary 6.5 (Joint capture and exclusion)**.** *For every finite-dimensional subspace $S \subset \mathcal{H}^{\varepsilon}_{n,\perp}$, there is a finite whole-eigenspace cutoff such that $P^{\varepsilon}_{n,\mathrm{IR}}|_S$ is injective and $\Lambda^{\varepsilon}_{n,\mathrm{excl}} > E^{\varepsilon}_{0,n}$.*

Proof. Strong convergence of the spectral projections is norm-uniform on the unit sphere of S, so sufficiently large projections are injective on S. The distinct eigenvalues diverge to infinity. □

7 Vacuum–electric reduction and physical Wilson-channel coercivity

7.1 Exact vacuum removal

Let

$$T_{n,\theta} = (I - P^{\varepsilon}_{n,\mathrm{vac}})\mathsf{F}^{\mathrm{W}}_{n,\theta}|_{\mathscr{E}^{\mathrm{W}}_{n,\theta}}, \qquad \mathscr{N}^{\mathrm{vac}}_{n,\theta} = \operatorname{Ker} T_{n,\theta}.$$

Define

$$\mathscr{C}^{\mathrm{phys}}_{n,\theta} = \mathscr{E}^{\mathrm{W}}_{n,\theta} \ominus \mathscr{N}^{\mathrm{vac}}_{n,\theta} \tag{7.1}$$

and let $\mathsf{P}^{\mathrm{phys}}_{n,\theta}$ be its coefficient-space projection. The substantive survival condition is

$$\mathscr{C}^{\mathrm{phys}}_{n,\theta} \neq \{0\}. \tag{7.2}$$

On this finite-dimensional space, define

$$\sigma^{\mathrm{vac}}_{n,\theta} = \inf_{\substack{c \in \mathscr{C}^{\mathrm{phys}}_{n,\theta} \\ \|c\|=1}} \|T_{n,\theta} c\| > 0.$$

7.2 Finite electric capture

For a variable cutoff, set

$$J^{\varepsilon}_{n,\theta}(\Lambda) = \Pi^{\varepsilon}_{n,\perp}(\Lambda) T_{n,\theta}|_{\mathscr{C}^{\mathrm{phys}}_{n,\theta}}, \qquad \tau^{\varepsilon}_{n,\theta}(\Lambda) = \|(I - \Pi^{\varepsilon}_{n,\perp}(\Lambda)) T_{n,\theta}|_{\mathscr{C}^{\mathrm{phys}}_{n,\theta}}\|.$$

Theorem 7.1 (Finite capture of the surviving channel)**.** *There is a finite whole-eigenspace cutoff such that*

$$\tau^{\varepsilon}_{n,\theta}(\Lambda) < \sigma^{\mathrm{vac}}_{n,\theta},$$

so $J^{\varepsilon}_{n,\theta}(\Lambda)$ is injective. The same cutoff may be chosen with $\Lambda^{\varepsilon}_{n,\mathrm{excl}} > E^{\varepsilon}_{0,n}$.

Proof. Strong spectral exhaustion is uniform on the finite-dimensional range of $T_{n,\theta}|_{\mathscr{C}^{\mathrm{phys}}_{n,\theta}}$, hence the tail norm tends to zero. If $Jc = 0$, then

$$\sigma^{\mathrm{vac}}_{n,\theta}\|c\| \le \|Tc\| = \|(I-\Pi)Tc\| \le \tau\|c\|,$$

forcing $c = 0$. Enlarge the cutoff further and apply Theorem 6.5. □

Fix such a cutoff and abbreviate

$$J^{\varepsilon}_{n,\theta} = J^{\varepsilon}_{n,\theta}(\Lambda^{\varepsilon}_{n,\mathrm{IR}}), \qquad G^{\varepsilon}_{n,\theta} = (J^{\varepsilon}_{n,\theta})^{*}J^{\varepsilon}_{n,\theta}.$$

Then $0 < G^{\varepsilon}_{n,\theta} \le I$. Define

$$\boxed{\mathsf{F}^{\mathrm{phys}}_{n,\theta} = J^{\varepsilon}_{n,\theta}(G^{\varepsilon}_{n,\theta})^{-1/2}.} \tag{7.3}$$

Theorem 7.2 (Vacuum–electric channel reduction)**.** *The map* $\mathsf{F}^{\mathrm{phys}}_{n,\theta} : \mathscr{C}^{\mathrm{phys}}_{n,\theta} \to \mathcal{H}_n(\Omega)$ *is an isometry and satisfies*

$$P^{\varepsilon}_{n,\mathrm{vac}}\mathsf{F}^{\mathrm{phys}}_{n,\theta} = 0, \qquad P^{\varepsilon}_{n,\mathrm{IR}}\mathsf{F}^{\mathrm{phys}}_{n,\theta} = \mathsf{F}^{\mathrm{phys}}_{n,\theta}. \tag{7.4}$$

Thus exact vacuum orthogonality and finite electric admissibility are conclusions, not assumptions.

Proof. By definition,

$$(\mathsf{F}^{\mathrm{phys}}_{n,\theta})^{*}\mathsf{F}^{\mathrm{phys}}_{n,\theta} = G^{-1/2}J^{*}JG^{-1/2} = I.$$

The range of J lies in the vacuum complement and in the selected electric spectral subspace, giving Equation (7.4). □

7.3 Physicalized graph operator

Set

$$A^{\mathrm{raw}}_{n,\theta} = \mathsf{P}^{\mathrm{phys}}_{n,\theta}(L_\theta - \lambda_1(\theta)I)\mathsf{P}^{\mathrm{phys}}_{n,\theta}|_{\mathscr{C}^{\mathrm{phys}}_{n,\theta}}.$$

Since $\mathscr{C}^{\mathrm{phys}}_{n,\theta} \subset \mathscr{E}^{\mathrm{W}}_{n,\theta}$,

$$A^{\mathrm{raw}}_{n,\theta} \ge \delta_{\mathrm{sec}}(\theta)I.$$

Define

$$\widetilde{A}^{\varepsilon}_{n,\theta} = (G^{\varepsilon}_{n,\theta})^{-1/2}A^{\mathrm{raw}}_{n,\theta}(G^{\varepsilon}_{n,\theta})^{-1/2}. \tag{7.5}$$

Then

$$\widetilde{A}^{\varepsilon}_{n,\theta} \ge \delta_{\mathrm{sec}}(\theta)(G^{\varepsilon}_{n,\theta})^{-1} \ge \delta_{\mathrm{sec}}(\theta)I. \tag{7.6}$$

For each active edge define

$$\widetilde{\ell}^{\theta,\varepsilon}_{n;vw} = (G^{\varepsilon}_{n,\theta})^{-1/2}\mathsf{P}^{\mathrm{phys}}_{n,\theta}\ell^{\theta}_{vw}\mathsf{P}^{\mathrm{phys}}_{n,\theta}(G^{\varepsilon}_{n,\theta})^{-1/2}. \tag{7.7}$$

The full physicalized graph operator is

$$\widetilde{L}^{\varepsilon}_{n,\theta} = (G^{\varepsilon}_{n,\theta})^{-1/2}\mathsf{P}^{\mathrm{phys}}_{n,\theta}L_\theta\mathsf{P}^{\mathrm{phys}}_{n,\theta}(G^{\varepsilon}_{n,\theta})^{-1/2} = \sum_{\{v,w\}} a_{vw}\widetilde{\ell}^{\theta,\varepsilon}_{n;vw}. \tag{7.8}$$

Moreover,

$$\widetilde{L}^{\varepsilon}_{n,\theta} = \widetilde{A}^{\varepsilon}_{n,\theta} + \lambda_1(\theta)(G^{\varepsilon}_{n,\theta})^{-1}. \tag{7.9}$$

7.4 Physical magnetic compression and disjoint extraction

Recompute every canonical cell on the physical frame:

$$M_{n,f}^{\mathrm{phys}} = (\mathsf{F}_{n,\theta}^{\mathrm{phys}})^{*} h_{n,f}^{\varepsilon} \mathsf{F}_{n,\theta}^{\mathrm{phys}} \geq 0. \tag{7.10}$$

Choose a disjoint physical cell allocation

$$F_{n,\mathrm{ch}}^{\mathrm{phys}} = \dot{\bigcup}_{(v,w)\in E_{\mathrm{act}}} F_{n;vw}^{\mathrm{phys}} \subset F_n(\Omega). \tag{7.11}$$

For $f \in F_{n;vw}^{\mathrm{phys}}$, define

$$\beta_{n,f;vw}^{\mathrm{phys}} = \sup\{\beta \geq 0 : M_{n,f}^{\mathrm{phys}} - \beta \widetilde{\ell}_{n;vw}^{\theta,\varepsilon} \geq 0\}.$$

Set

$$\beta_{n;vw}^{\mathrm{phys}} = \sum_{f\in F_{n;vw}^{\mathrm{phys}}} \beta_{n,f;vw}^{\mathrm{phys}}, \qquad b_{n,\mathrm{ch}}^{\mathrm{phys,max}}(\theta) = \min_{(v,w)\in E_{\mathrm{act}}} \frac{\beta_{n;vw}^{\mathrm{phys}}}{a_{vw}}.$$

Positive physical extraction,

$$b_{n,\mathrm{ch}}^{\mathrm{phys,max}}(\theta) > 0, \tag{7.12}$$

is a substantive verification condition.

Let

$$M_{n,\mathrm{ch}}^{\mathrm{phys}} = \sum_{f\in F_n(\Omega)} M_{n,f}^{\mathrm{phys}}.$$

For $0 < b \leq b_{n,\mathrm{ch}}^{\mathrm{phys,max}}$, disjoint allocation and positivity of unallocated cells give

$$M_{n,\mathrm{ch}}^{\mathrm{phys}} \geq b \widetilde{L}_{n,\theta}^{\varepsilon}. \tag{7.13}$$

7.5 Form compression and exact physical calibration

The physical channel lies in the form domain of the electric Hamiltonian, but no claim is made that it lies in $D(H_{n,\mathrm{el}})$. Define the finite-dimensional Hermitian electric matrix by

$$\langle c, E_{n,\mathrm{ch}}^{\mathrm{phys}} d\rangle = q_{n,\mathrm{el}}(\mathsf{F}_{n,\theta}^{\mathrm{phys}} c, \mathsf{F}_{n,\theta}^{\mathrm{phys}} d). \tag{7.14}$$

Likewise define the vacuum-renormalized channel matrix by

$$\langle c, \widehat{\mathcal{H}}_{n,\mathrm{ch}}^{\mathrm{phys}} d\rangle = \widehat{q}_{n}^{\varepsilon}(\mathsf{F}_{n,\theta}^{\mathrm{phys}} c, \mathsf{F}_{n,\theta}^{\mathrm{phys}} d). \tag{7.15}$$

Because the coefficient space is finite-dimensional, these forms are represented by bounded Hermitian matrices.

Set

$$D_{n,\mathrm{mag}}^{\mathrm{phys}}(b) = M_{n,\mathrm{ch}}^{\mathrm{phys}} - b \widetilde{L}_{n,\theta}^{\varepsilon} \geq 0. \tag{7.16}$$

Then the exact finite-dimensional form identity is

$$\widehat{\mathcal{H}}_{n,\mathrm{ch}}^{\mathrm{phys}} = b \widetilde{A}_{n,\theta}^{\varepsilon} + \mathcal{R}_{n,\mathrm{sec}}^{\mathrm{phys}}(b), \tag{7.17}$$

where

$$\mathcal{R}_{n,\mathrm{sec}}^{\mathrm{phys}}(b) = E_{n,\mathrm{ch}}^{\mathrm{phys}} + D_{n,\mathrm{mag}}^{\mathrm{phys}}(b) + b\lambda_1(\theta)(G_{n,\theta}^{\varepsilon})^{-1} - E_{0,n}^{\varepsilon} I. \tag{7.18}$$

Set

$$r_{n,\mathrm{sec}}^{\mathrm{phys}}(b) = \|[\mathcal{R}_{n,\mathrm{sec}}^{\mathrm{phys}}(b)]_{-}\|, \qquad c_{n,\mathrm{ch}}^{\mathrm{phys}}(b) = b\delta_{\mathrm{sec}}(\theta) - r_{n,\mathrm{sec}}^{\mathrm{phys}}(b).$$

Theorem 7.3 (Physical Wilson-channel coercivity)**.** *If $c^{\mathrm{phys}}_{n,\mathrm{ch}}(b) > 0$, then for every $s \in \operatorname{Ran} \mathsf{F}^{\mathrm{phys}}_{n,\theta}$,*

$$\widehat{Q}^{\varepsilon}_{n}[s] \geq c^{\mathrm{phys}}_{n,\mathrm{ch}}(b)\|s\|^2. \tag{7.19}$$

Proof. Write $s = \mathsf{F}^{\mathrm{phys}}_{n,\theta} c$. By Equations (7.6) and (7.17),

$$\widehat{Q}^{\varepsilon}_{n}[s] = \langle c, \widehat{\mathcal{H}}^{\mathrm{phys}}_{n,\mathrm{ch}} c\rangle \geq \big(b\delta_{\mathrm{sec}}(\theta) - r^{\mathrm{phys}}_{n,\mathrm{sec}}(b)\big)\|c\|^2.$$

The physical frame is isometric, so $\|c\| = \|s\|$. □

7.6 Residual infrared sector

Let

$$\mathcal{S}^{\mathrm{phys}}_{n,\theta} = \operatorname{Ran} \mathsf{F}^{\mathrm{phys}}_{n,\theta}, \qquad \mathcal{R}^{\mathrm{IR}}_{n,\theta} = \operatorname{Ran} P^{\varepsilon}_{n,\mathrm{IR}} \ominus \mathcal{S}^{\mathrm{phys}}_{n,\theta}.$$

If the residual space is nonzero, define

$$c_{n,\mathrm{res}} = \inf_{\substack{r \in \mathcal{R}^{\mathrm{IR}}_{n,\theta} \\ \|r\|=1}} \widehat{Q}^{\varepsilon}_{n}[r],$$

and

$$\varepsilon_{n,\mathrm{sr}} = \sup_{\substack{s \in \mathcal{S}^{\mathrm{phys}}_{n,\theta},\ r \in \mathcal{R}^{\mathrm{IR}}_{n,\theta} \\ \|s\|=\|r\|=1}} |\widehat{q}^{\varepsilon}_{n}(s,r)|.$$

If $c_{n,\mathrm{res}} > 0$ and

$$\varepsilon^2_{n,\mathrm{sr}} < c^{\mathrm{phys}}_{n,\mathrm{ch}}(b) c_{n,\mathrm{res}},$$

define

$$\widehat{c}^{\theta}_{n,\mathrm{IR}}(b) = \frac{c^{\mathrm{phys}}_{n,\mathrm{ch}}(b) + c_{n,\mathrm{res}} - \sqrt{(c^{\mathrm{phys}}_{n,\mathrm{ch}}(b) - c_{n,\mathrm{res}})^2 + 4\varepsilon^2_{n,\mathrm{sr}}}}{2} > 0. \tag{7.20}$$

If the residual space is zero, set $\widehat{c}^{\theta}_{n,\mathrm{IR}}(b) = c^{\mathrm{phys}}_{n,\mathrm{ch}}(b)$.

Theorem 7.4 (Complete infrared coercivity)**.** *Under the preceding conditions,*

$$\widehat{Q}^{\varepsilon}_{n}[w] \geq \widehat{c}^{\theta}_{n,\mathrm{IR}}(b)\|w\|^2, \qquad w \in \operatorname{Ran} P^{\varepsilon}_{n,\mathrm{IR}}. \tag{7.21}$$

Proof. Decompose $w = s + r$ according to $\operatorname{Ran} P^{\varepsilon}_{n,\mathrm{IR}} = \mathcal{S}^{\mathrm{phys}}_{n,\theta} \oplus \mathcal{R}^{\mathrm{IR}}_{n,\theta}$. Apply the channel and residual diagonal estimates and the definition of $\varepsilon_{n,\mathrm{sr}}$, then use Theorem 10.1. If the residual block vanishes, the claim is Theorem 7.3. □

8 Exact mixed-block decomposition without block closure

8.1 Cell partition and reference operator

Choose a disjoint partition

$$F_n(\Omega) = F^{\varepsilon}_{n,\mathrm{int}} \dot{\cup} F^{\varepsilon}_{n,\mathrm{tr}} \dot{\cup} F^{\varepsilon}_{n,\mathrm{col}}.$$

For each transported cell f, choose $\ell_f \preceq m_f \preceq n$ and a self-adjoint fine seed $A^{\varepsilon}_{n,f}$ such that the actual potential and direct comparator are

$$V^{\varepsilon}_{n,f} = \iota_{\ell_f n} j^{\Omega}_{\ell_f} T^{\Omega}_{\ell_f m_f} T^{\Omega}_{m_f n} A^{\varepsilon}_{n,f},$$

$$V_{n,f}^{\mathrm{dir}} = \iota_{\ell_f n} J_{\ell_f}^{\Omega} T_{\ell_f n}^{\Omega} A_{n,f}^{\varepsilon}.$$

The comparator is bounded and self-adjoint; positivity is not asserted because orthogonal Wilson localization need not preserve positivity. Let

$$h_{n,f}^{\mathrm{dir}} = \omega_{n,f}^{\mathrm{mag}} M_{V_{n,f}^{\mathrm{dir}}},$$

$$H_{n,\mathrm{int}}^{\varepsilon} = \sum_{f \in F_{n,\mathrm{int}}^{\varepsilon}} h_{n,f}^{\varepsilon}, \quad H_{n,\mathrm{dir}}^{\varepsilon} = \sum_{f \in F_{n,\mathrm{tr}}^{\varepsilon}} h_{n,f}^{\mathrm{dir}}, \quad H_{n,\mathrm{ref}}^{\varepsilon} = H_{n,\mathrm{int}}^{\varepsilon} + H_{n,\mathrm{dir}}^{\varepsilon}.$$

Thus $H_{n,\mathrm{ref}}^{\varepsilon}$ is a bounded self-adjoint reference operator, not an additional canonical Hamiltonian.

8.2 Transported correction and exact decomposition

By Theorem 3.2,

$$V_{n,f}^{\varepsilon} - V_{n,f}^{\mathrm{dir}} = \iota_{\ell_f n} J_{\ell_f}^{\Omega} \alpha_{\ell_f m_f n}^{\Omega} A_{n,f}^{\varepsilon}.$$

Define

$$R_{n,\mathrm{tr}}^{\varepsilon} = \sum_{f \in F_{n,\mathrm{tr}}^{\varepsilon}} \omega_{n,f}^{\mathrm{mag}} M_{V_{n,f}^{\varepsilon} - V_{n,f}^{\mathrm{dir}}}, \qquad H_{n,\mathrm{col}}^{\varepsilon} = \sum_{f \in F_{n,\mathrm{col}}^{\varepsilon}} h_{n,f}^{\varepsilon}.$$

Then

$$\boxed{H_{n,\mathrm{mag}}^{\varepsilon} = H_{n,\mathrm{ref}}^{\varepsilon} + R_{n,\mathrm{tr}}^{\varepsilon} + H_{n,\mathrm{col}}^{\varepsilon}.} \tag{8.1}$$

8.3 Exact three-part mixed block

Set

$$K_n^{\varepsilon} = P_{n,\mathrm{UV}}^{\varepsilon} H_{n,\mathrm{mag}}^{\varepsilon} P_{n,\mathrm{IR}}^{\varepsilon},$$

$$K_{n,\mathrm{ref}}^{\varepsilon} = P_{n,\mathrm{UV}}^{\varepsilon} H_{n,\mathrm{ref}}^{\varepsilon} P_{n,\mathrm{IR}}^{\varepsilon}, \quad K_{n,\mathrm{tr}}^{\varepsilon} = P_{n,\mathrm{UV}}^{\varepsilon} R_{n,\mathrm{tr}}^{\varepsilon} P_{n,\mathrm{IR}}^{\varepsilon}, \quad K_{n,\mathrm{col}}^{\varepsilon} = P_{n,\mathrm{UV}}^{\varepsilon} H_{n,\mathrm{col}}^{\varepsilon} P_{n,\mathrm{IR}}^{\varepsilon}.$$

Applying the block projection to Equation (8.1) yields

$$\boxed{K_n^{\varepsilon} = K_{n,\mathrm{ref}}^{\varepsilon} + K_{n,\mathrm{tr}}^{\varepsilon} + K_{n,\mathrm{col}}^{\varepsilon}.} \tag{8.2}$$

Define

$$\widehat{\varepsilon}_{n,\mathrm{mix}}^{\varepsilon} = \|K_n^{\varepsilon}\|_{\mathrm{op}}, \qquad \varepsilon_{n,\mathrm{ref}}^{\varepsilon} = \|K_{n,\mathrm{ref}}^{\varepsilon}\|_{\mathrm{op}}. \tag{8.3}$$

Exact reference block closure is the special case $\varepsilon_{n,\mathrm{ref}}^{\varepsilon} = 0$; it is not assumed.

8.4 Transport and collar majorants

The finite-window estimate gives

$$\|K_{n,\mathrm{tr}}^{\varepsilon}\| \le C_{n,\mathrm{tr}}^{\varepsilon} D_{n,\mathrm{lax}}^{\varepsilon},$$

where

$$C_{n,\mathrm{tr}}^{\varepsilon} = \sum_{f \in F_{n,\mathrm{tr}}^{\varepsilon}} \omega_{n,f}^{\mathrm{mag}} C_{\ell_f,\infty\leftarrow 2}^{\varepsilon} \|A_{n,f}^{\varepsilon}\|_2.$$

Define

$$\Delta_{n,\mathrm{col}}^{\varepsilon} = \sum_{f \in F_{n,\mathrm{col}}^{\varepsilon}} \|V_{n,f}^{\varepsilon}\|_{\infty}, \qquad C_{n,\mathrm{col}}^{\varepsilon} = \max_{f \in F_{n,\mathrm{col}}^{\varepsilon}} \omega_{n,f}^{\mathrm{mag}}.$$

Then

$$\|K^{\varepsilon}_{n,\mathrm{col}}\| \le C^{\varepsilon}_{n,\mathrm{col}}\Delta^{\varepsilon}_{n,\mathrm{col}}.$$

Therefore the hybrid majorant

$$\boxed{\varepsilon^{\varepsilon}_{n,\mathrm{hyb}} = \varepsilon^{\varepsilon}_{n,\mathrm{ref}} + C^{\varepsilon}_{n,\mathrm{tr}}D^{\varepsilon}_{n,\mathrm{lax}} + C^{\varepsilon}_{n,\mathrm{col}}\Delta^{\varepsilon}_{n,\mathrm{col}}} \tag{8.4}$$

satisfies

$$\widehat{\varepsilon}^{\varepsilon}_{n,\mathrm{mix}} \le \varepsilon^{\varepsilon}_{n,\mathrm{hyb}}. \tag{8.5}$$

8.5 Optional commutator control

Let $A = H^{\perp}_{n,\mathrm{el}}$, $P = P^{\varepsilon}_{n,\mathrm{IR}}$, and $Q = P^{\varepsilon}_{n,\mathrm{UV}}$. If a bounded operator B satisfies $B^{\perp}P\mathcal{H} \subset D(A)$, where $B^{\perp} = (I - P^{\varepsilon}_{n,\mathrm{vac}})B(I - P^{\varepsilon}_{n,\mathrm{vac}})$, then the Sylvester equation gives

$$\|QBP\| \le \frac{\|Q[A, B^{\perp}]P\|}{\delta^{\varepsilon}_{n,\mathrm{el}}}. \tag{8.6}$$

This may be applied to $B = H^{\varepsilon}_{n,\mathrm{ref}}$ when its domain condition is verified. Finite Peter–Weyl bandwidth alone does not imply exact closure of the physical electric cutoff.

Theorem 8.1 (Exact mixed-block estimate)**.** *Under the cell-partition and comparator hypotheses,*

$$K^{\varepsilon}_{n} = K^{\varepsilon}_{n,\mathrm{ref}} + K^{\varepsilon}_{n,\mathrm{tr}} + K^{\varepsilon}_{n,\mathrm{col}}$$

exactly, and Equation (8.5) holds. No interior or direct-comparator block-closure hypothesis is required.

9 Complete vacuum reduction and physical excitation blocks

9.1 Vacuum-renormalized form

Set

$$\widehat{H}^{\varepsilon}_{n} = H^{\varepsilon}_{n} - E^{\varepsilon}_{0,n}I \ge 0$$

and denote its form by $\widehat{Q}^{\varepsilon}_{n}$ and sesquilinear form by $\widehat{q}^{\varepsilon}_{n}$. Since

$$\operatorname{Ker}\widehat{H}^{\varepsilon}_{n} = \mathcal{V}^{\varepsilon}_{n,\mathrm{vac}},$$

one has

$$\widehat{q}^{\varepsilon}_{n}(v, \psi) = 0 \tag{9.1}$$

for every $v \in \mathcal{V}^{\varepsilon}_{n,\mathrm{vac}}$ and $\psi \in D(\widehat{Q}^{\varepsilon}_{n})$.

9.2 Exact three-space decomposition

The projections satisfy

$$I = P^{\varepsilon}_{n,\mathrm{UV}} + P^{\varepsilon}_{n,\mathrm{IR}} + P^{\varepsilon}_{n,\mathrm{vac}}$$

orthogonally and preserve the form domain. For $\psi \in D(\widehat{Q}^{\varepsilon}_{n})$, set

$$u = P^{\varepsilon}_{n,\mathrm{UV}}\psi, \qquad w = P^{\varepsilon}_{n,\mathrm{IR}}\psi, \qquad v = P^{\varepsilon}_{n,\mathrm{vac}}\psi.$$

By Equation (9.1),

$$\widehat{Q}^{\varepsilon}_{n}[\psi] = \widehat{Q}^{\varepsilon}_{n}[u] + \widehat{Q}^{\varepsilon}_{n}[w] + 2\Re\widehat{q}^{\varepsilon}_{n}(u, w). \tag{9.2}$$

The electric cross term vanishes by Theorem 6.4; hence

$$\widehat{q}^{\varepsilon}_{n}(u, w) = \langle u, H^{\varepsilon}_{n,\mathrm{mag}}w\rangle = \langle u, K^{\varepsilon}_{n}w\rangle. \tag{9.3}$$

9.3 Physical channel inside the retained block

By Theorem 7.2,

$$\mathcal{S}^{\mathrm{phys}}_{n,\theta} \subset \operatorname{Ran} P^{\varepsilon}_{n,\mathrm{IR}} \subset \operatorname{Ran}(I - P^{\varepsilon}_{n,\mathrm{vac}}),$$

and

$$\operatorname{Ran} P^{\varepsilon}_{n,\mathrm{IR}} = \mathcal{S}^{\mathrm{phys}}_{n,\theta} \oplus \mathcal{R}^{\mathrm{IR}}_{n,\theta}.$$

The graph-minimum projection $P_{\theta,\min}$, surviving coefficient projection $\mathsf{P}^{\mathrm{phys}}_{n,\theta}$, retained electric projection $P^{\varepsilon}_{n,\mathrm{IR}}$, and physical vacuum projection $P^{\varepsilon}_{n,\mathrm{vac}}$ act on different spaces or encode different structures and are never identified.

Theorem 9.1 (Exact excitation reduction)**.** *For every $\psi \in D(\widehat{Q}^{\varepsilon}_{n})$, the complete ground component drops out and the form reduces exactly to Equation (9.2), with mixed term Equation (9.3). No vacuum vector, vacuum–graph alignment, or electric block closure is used.*

10 Abstract block coercivity and the principal theorem

10.1 Sharp two-block estimate

Theorem 10.1 (Two-block coercivity)**.** *Let Q be a nonnegative closed quadratic form on $\mathcal{H}_1 \oplus \mathcal{H}_2$. Suppose*

$$Q[u] \geq c_1\|u\|^2, \qquad Q[w] \geq c_2\|w\|^2, \qquad |q(u,w)| \leq \eta\|u\|\|w\|,$$

with $c_1, c_2 > 0$. Then

$$Q[u+w] \geq \kappa(\|u\|^2 + \|w\|^2),$$

where

$$\kappa = \frac{c_1 + c_2 - \sqrt{(c_1 - c_2)^2 + 4\eta^2}}{2}. \tag{10.1}$$

Moreover, $\kappa > 0$ if and only if $\eta^2 < c_1 c_2$, and the constant is sharp relative to the three scalar estimates.

Proof. For $x = \|u\|$ and $y = \|w\|$,

$$Q[u+w] \geq c_1 x^2 + c_2 y^2 - 2\eta xy.$$

The right-hand side is the quadratic form of the symmetric matrix $\left(\begin{smallmatrix} c_1 & -\eta \\ -\eta & c_2 \end{smallmatrix}\right)$, whose smaller eigenvalue is Equation (10.1). □

Corollary 10.2 (Relative leakage bound)**.** *If $\eta^2 \leq \rho c_1 c_2$ with $0 \leq \rho < 1$, then*

$$\kappa \geq \frac{1-\rho}{2}\min\{c_1, c_2\}.$$

10.2 Reduced physical comparison package

A verified fixed-scale package consists of:

(A1) a canonical finite Hamiltonian H^{ε}_{n} and complete ground projection;

(A2) a raw Wilson frame and a graph with nonzero excitation complement;

(A3) nontrivial physical channel survival Equation (7.2);

(A4) a finite electric capture producing $\mathsf{F}_{n,\theta}^{\mathrm{phys}}$ and positive $\widehat{c}_{n,\mathrm{UV}}^{\varepsilon}$;

(A5) positive physical magnetic extraction Equation (7.12);

(A6) positive physical channel calibration and complete residual infrared control, yielding $\widehat{c}_{n,\mathrm{IR}}^{\theta}(b) > 0$;

(A7) an interior–transported–collar partition;

(A8) a quantified exact or estimated reference leakage $\varepsilon_{n,\mathrm{ref}}^{\varepsilon}$;

(A9) exact transported-cell comparator identities;

(A10) unital involution-invariant finite Wilson windows.

10.3 Principal finite-scale theorem

Theorem 10.3 (Finite-scale Wilson-tube coercivity)**.** *Assume the reduced physical comparison package and*

$$(\widehat{\varepsilon}_{n,\mathrm{mix}}^{\varepsilon})^2 < \widehat{c}_{n,\mathrm{UV}}^{\varepsilon}\widehat{c}_{n,\mathrm{IR}}^{\theta}(b). \tag{10.2}$$

Then

$$\boxed{H_n^{\varepsilon} \geq E_{0,n}^{\varepsilon} I + \widehat{\kappa}_{n,\varepsilon,\theta}(b)(I - P_{n,\mathrm{vac}}^{\varepsilon}),} \tag{10.3}$$

where

$$\widehat{\kappa}_{n,\varepsilon,\theta}(b) = \frac{\widehat{c}_{n,\mathrm{UV}}^{\varepsilon} + \widehat{c}_{n,\mathrm{IR}}^{\theta}(b) - \sqrt{(\widehat{c}_{n,\mathrm{UV}}^{\varepsilon} - \widehat{c}_{n,\mathrm{IR}}^{\theta}(b))^2 + 4(\widehat{\varepsilon}_{n,\mathrm{mix}}^{\varepsilon})^2}}{2} > 0.$$

Equivalently,

$$\widehat{Q}_n^{\varepsilon}[\psi] \geq \widehat{\kappa}_{n,\varepsilon,\theta}(b)\|(I - P_{n,\mathrm{vac}}^{\varepsilon})\psi\|^2.$$

Proof. Use the exact reduction Theorem 9.1, the ultraviolet estimate Equation (6.1), complete infrared estimate Equation (7.21), and mixed norm Equation (8.3). Apply Theorem 10.1. □

Corollary 10.4 (Hybrid sufficient condition)**.** *It is sufficient that*

$$(\varepsilon_{n,\mathrm{hyb}}^{\varepsilon})^2 < \widehat{c}_{n,\mathrm{UV}}^{\varepsilon}\widehat{c}_{n,\mathrm{IR}}^{\theta}(b).$$

Replacing the exact mixed norm in the smaller-eigenvalue formula by $\varepsilon_{n,\mathrm{hyb}}^{\varepsilon}$ gives a valid, possibly weaker, coercivity constant.

10.4 Spectral consequence

Since H_n^{ε} has compact resolvent, Equation (10.3) implies

$$\mathrm{spec}(H_n^{\varepsilon}) \cap (E_{0,n}^{\varepsilon}, E_{0,n}^{\varepsilon} + \widehat{\kappa}_{n,\varepsilon,\theta}(b)) = \varnothing.$$

This is a resolved finite-scale gap above the complete ground spectral value.

11 Uniform physicalized coercivity and refinement handoff

11.1 Directed finite-scale families

Let $(\mathfrak{G}, \preceq)$ be directed and let

$$\gamma = (n_\gamma, \varepsilon_\gamma, \theta_\gamma, \Lambda_\gamma, b_\gamma, \mathscr{D}_\gamma^{\mathrm{phys}})$$

index verified finite-scale packages. Write H_γ, $E_{0,\gamma}$, $P_{\mathrm{vac},\gamma}$, and

$$\widehat{\kappa}_\gamma = \frac{\widehat{c}_{\mathrm{UV},\gamma} + \widehat{c}_{\mathrm{IR},\gamma} - \sqrt{(\widehat{c}_{\mathrm{UV},\gamma} - \widehat{c}_{\mathrm{IR},\gamma})^2 + 4\widehat{\varepsilon}_{\mathrm{mix},\gamma}^2}}{2}.$$

No common ambient Hilbert space is required for the numerical statement.

Theorem 11.1 (Uniform finite-scale coercivity)**.** *Suppose*

$$\widehat{c}_{\mathrm{UV},\gamma} \geq c_{\mathrm{UV},*} > 0, \qquad \widehat{c}_{\mathrm{IR},\gamma} \geq c_{\mathrm{IR},*} > 0,$$

and

$$\widehat{\varepsilon}_{\mathrm{mix},\gamma}^2 \leq \rho \widehat{c}_{\mathrm{UV},\gamma} \widehat{c}_{\mathrm{IR},\gamma}, \qquad 0 \leq \rho < 1.$$

Then

$$\inf_\gamma \widehat{\kappa}_\gamma \geq \frac{1-\rho}{2} \min\{c_{\mathrm{UV},*}, c_{\mathrm{IR},*}\} > 0. \tag{11.1}$$

Proof. Apply Theorem D.6 with $c_1 = \widehat{c}_{\mathrm{UV},\gamma}$, $c_2 = \widehat{c}_{\mathrm{IR},\gamma}$, and $\eta = \widehat{\varepsilon}_{\mathrm{mix},\gamma}$, then take the infimum over γ. □

11.2 Uniform physical channel survival

For each γ, let

$$T_\gamma = (I - P_{\mathrm{vac},\gamma}) \mathsf{F}_\gamma^{\mathrm{W}}|_{\mathrm{Ran}(I - P_{\theta_\gamma,\min})}$$

and define $\mathscr{C}_\gamma^{\mathrm{phys}}$ by deleting $\operatorname{Ker} T_\gamma$. Quantitative survival is expressed by

$$\dim \mathscr{C}_\gamma^{\mathrm{phys}} \geq m_* > 0, \qquad \sigma_{\mathrm{vac},\gamma} = \inf_{\substack{c \in \mathscr{C}_\gamma^{\mathrm{phys}} \\ \|c\|=1}} \|T_\gamma c\| \geq \sigma_{\mathrm{vac},*} > 0.$$

The second inequality is the least positive principal-angle condition between the surviving raw channel and the complete vacuum sector.

Let

$$\tau_{\mathrm{cap},\gamma} = \|(I - \Pi_{\perp,\gamma}(\Lambda_\gamma)) T_\gamma|_{\mathscr{C}_\gamma^{\mathrm{phys}}}\|.$$

A uniform capture margin is

$$\tau_{\mathrm{cap},\gamma} \leq \eta_{\mathrm{cap}} \sigma_{\mathrm{vac},\gamma}, \qquad \eta_{\mathrm{cap}} < 1.$$

Then the Gram operators satisfy

$$(1 - \eta_{\mathrm{cap}})^2 \sigma_{\mathrm{vac},*}^2 I \leq G_\gamma \leq I. \tag{11.2}$$

Thus physical normalization does not become singular.

11.3 Uniform infrared data

The required graph, extraction, and calibration conditions may be stated as

$$\delta_{\mathrm{sec},\gamma} \geq \delta_* > 0, \qquad b_\gamma \geq b_* > 0,$$

$$c^{\mathrm{phys}}_{\mathrm{ch},\gamma}(b_\gamma) = b_\gamma \delta_{\mathrm{sec},\gamma} - r^{\mathrm{phys}}_{\mathrm{sec},\gamma}(b_\gamma) \geq c_{\mathrm{ch},*} > 0.$$

When residual infrared spaces are nonzero, assume

$$c_{\mathrm{res},\gamma} \geq c_{\mathrm{res},*} > 0, \qquad \varepsilon^2_{\mathrm{sr},\gamma} \leq \rho_{\mathrm{sr}} c^{\mathrm{phys}}_{\mathrm{ch},\gamma} c_{\mathrm{res},\gamma}, \quad \rho_{\mathrm{sr}} < 1.$$

Then

$$\widehat{c}_{\mathrm{IR},\gamma} \geq \frac{1-\rho_{\mathrm{sr}}}{2} \min\{c_{\mathrm{ch},*}, c_{\mathrm{res},*}\}$$

for nonzero residual blocks, while $\widehat{c}_{\mathrm{IR},\gamma} = c^{\mathrm{phys}}_{\mathrm{ch},\gamma}$ if the residual block vanishes.

11.4 Uniform three-part leakage

The exact mixed block is

$$K_\gamma = K_{\mathrm{ref},\gamma} + K_{\mathrm{tr},\gamma} + K_{\mathrm{col},\gamma}.$$

Define

$$\varepsilon_{\mathrm{hyb},\gamma} = \varepsilon_{\mathrm{ref},\gamma} + C_{\mathrm{tr},\gamma} D_{\mathrm{lax},\gamma} + C_{\mathrm{col},\gamma} \Delta_{\mathrm{col},\gamma}.$$

A sufficient uniform condition is

$$\varepsilon^2_{\mathrm{hyb},\gamma} \leq \rho \widehat{c}_{\mathrm{UV},\gamma} \widehat{c}_{\mathrm{IR},\gamma}, \qquad \rho < 1. \tag{11.3}$$

A stronger separated condition is

$$(\varepsilon_{\mathrm{ref},*} + C_{\mathrm{tr},*} D_{\mathrm{lax},*} + C_{\mathrm{col},*} \Delta_{\mathrm{col},*})^2 < c_{\mathrm{UV},*} c_{\mathrm{IR},*}.$$

Failure of this separated estimate does not imply failure of the exact norm condition, because it discards cancellation.

Theorem 11.2 (Uniform physicalized comparison criterion)**.** *Uniform graph separation, rank survival, vacuum-angle control, electric capture, Gram conditioning, physical extraction and calibration, residual control, ultraviolet exclusion, and Equation (11.3) imply Equation (11.1).*

Proof. The survival and capture hypotheses give the uniform Gram lower bound. Physical extraction, calibration, and residual control give a uniform positive infrared constant. The hybrid estimate dominates the exact mixed norm, so Theorem 11.1 applies. □

11.5 Failure modes and handoff

The revised architecture separates the possible failures:

1. collapse of graph separation;
2. loss of all surviving physical rank;
3. asymptotic tangency to the vacuum sector;

4. failure of uniform electric capture or Gram conditioning;
5. collapse of physical magnetic extraction or calibration;
6. residual infrared collapse;
7. growth of ordinary reference-band mixing;
8. accumulation of transport-associator or collar leakage;
9. criticality of the final mixed-block ratio.

Uniform coercivity does not construct a limiting Hilbert space or generator. A later exhaustion theorem must also supply comparison maps, convergence of forms and vacuum projections, and no-leakage or norm-recovery for normalized excitation sequences; comparison–Mosco convergence is one natural framework [15].

12 Interpretation and scope

12.1 What has been resolved

The theorem resolves the finite-scale spectral lower bound into

$$\widehat{c}^{\varepsilon}_{n,\mathrm{UV}} = \text{vacuum-orthogonal electric exclusion},$$

$$\widehat{c}^{\theta}_{n,\mathrm{IR}}(b) = \text{physical channel plus residual infrared separation},$$

and

$$\widehat{\varepsilon}^{\varepsilon}_{n,\mathrm{mix}} = \|K^{\varepsilon}_{n,\mathrm{ref}} + K^{\varepsilon}_{n,\mathrm{tr}} + K^{\varepsilon}_{n,\mathrm{col}}\|.$$

This is stronger in explanatory content than merely invoking compact resolvent, although the final theorem remains conditional on the explicit verification package.

12.2 Strict and twisted detected sectors

In a strict connected graph sector, $\lambda_1 = 0$ and the graph minimum consists of covariantly constant sections. In a frustrated detected sector, $\lambda_1 > 0$ and $\operatorname{Ker} d_\theta = \{0\}$, but the finite graph still has a nonzero lowest eigenspace. The coercive comparison uses

$$\delta_{\mathrm{sec}}(\theta) = \lambda_{m_1+1}(\theta) - \lambda_1(\theta),$$

not the baseline alone. There is no universal ordering between final strict and twisted coercivity constants: vacuum survival, Gram conditioning, physical extraction, calibration, residual modes, and total leakage all contribute.

12.3 The physical role of the associator

The transport associator satisfies

$$\alpha^{\Omega}_{\ell mn} = L^{\Omega}_{\ell}\mathbb{E}_{\ell m}\Lambda^{\Omega}_{m}\mathbb{E}_{mn}\jmath^{\Omega}_{n}.$$

It therefore measures intermediate localization information loss. Its Hamiltonian contribution is the additional correction $K^{\varepsilon}_{n,\mathrm{tr}}$ between actual two-step transport and a direct comparator. Even perfectly associative Wilson transport would leave ordinary reference-band mixing $K^{\varepsilon}_{n,\mathrm{ref}}$, and an unresolved geometry would leave $K^{\varepsilon}_{n,\mathrm{col}}$.

12.4 Vacuum sector rather than vacuum vector

All statements are formulated with

$$P^{\varepsilon}_{n,\mathrm{vac}} = \mathbf{1}_{\{E^{\varepsilon}_{0,n}\}}(H^{\varepsilon}_{n}).$$

A basis of the finite-dimensional ground sector may be chosen for computation, but no basis vector is physically distinguished. Uniqueness of a continuum vacuum, or uniqueness up to a later admissible gerbe equivalence, is not a result of the present fixed-scale paper.

12.5 What the explicit appendices prove

Appendices A and B prove the nonemptiness of the raw representation-theoretic mechanism: center detection, strict and twisted graph spectra, nonzero localization associators, canonical nonnegative Wilson cells, raw compression matrices, and positive raw edge extraction. They do not compute the exact interacting ground projection of an arbitrary spatial system. Appendix C, by contrast, gives one complete finite Peter–Weyl Galerkin realization: it computes the interacting ground projection, nontrivial Gram operator, physical recompression, ultraviolet and infrared constants, exact reference leakage, and a positive final coercivity constant. This verifies that the comparison package is executable end to end, while making no claim that the same constants persist in the untruncated model.

12.6 Final scope statement

The principal theorem requires:

1. a nonzero raw graph-excitation complement and surviving physical channel;
2. finite vacuum-orthogonal electric capture;
3. positive physical magnetic extraction and calibration;
4. control of all residual retained modes;
5. quantified reference, transport, and collar leakage;
6. strict final two-block subcriticality.

It then proves

$$H^{\varepsilon}_{n} - E^{\varepsilon}_{0,n} I \geq \widehat{\kappa}_{n,\varepsilon,\theta}(b)(I - P^{\varepsilon}_{n,\mathrm{vac}}), \qquad \widehat{\kappa}_{n,\varepsilon,\theta}(b) > 0.$$

No preferred vacuum vector, vacuum–graph alignment, assumed finite electric support of raw channels, or exact invariance of the electric cutoff under Wilson multiplication is used. The result is a finite-scale coercivity theorem and a quantitative handoff to later geometric exhaustion, branch selection, and no-leakage spectral transfer.

A An explicit raw $SU(2)$ witness

A.1 Scope

This appendix verifies the raw representation-theoretic mechanisms used before the physical reduction of Section 7. It does not identify a preferred vacuum vector, compute the exact interacting ground projection, or assert that the raw compression matrices remain unchanged after vacuum removal, electric capture, and Gram normalization.

A.2 Characters and center detection

Let $G = SU(2)$, with center $Z(G) = \{I, -I\}$. For spin $j \in \frac{1}{2}\mathbb{N}_0$, the quadratic Casimir is

$$C_2(j) = j(j+1),$$

so

$$C_2(0) = 0, \qquad C_2(\tfrac{1}{2}) = \tfrac{3}{4}, \qquad C_2(1) = 2.$$

The fundamental character $\chi_{1/2}$ is real, satisfies

$$\int_G \chi_{1/2}(g)\, dg = 0, \qquad \int_G |\chi_{1/2}(g)|^2\, dg = 1,$$

and detects the nontrivial center element by

$$\chi_{1/2}(-g) = -\chi_{1/2}(g), \qquad \zeta_{1/2}(-I) = -1.$$

A.3 Raw three-channel frame

Interpret g_1, g_2, g_3 as based loop holonomies with residual diagonal conjugation

$$(g_1, g_2, g_3) \longmapsto (hg_1h^{-1}, hg_2h^{-1}, hg_3h^{-1}).$$

Work on the gauge-invariant space $L^2(G^3)^{\mathrm{Ad}\,G}$. The character functions and pair observables below are invariant under this action. Define

$$\phi_1(g_1, g_2, g_3) = \chi_{1/2}(g_1), \quad \phi_2 = \chi_{1/2}(g_2), \quad \phi_3 = \chi_{1/2}(g_3).$$

Schur orthogonality [9, 10] gives $\langle \phi_i, \phi_j \rangle = \delta_{ij}$. Thus

$$\mathsf{F}_\theta^{\mathrm{W}} e_i = \phi_i$$

defines an isometric raw frame $\mathbb{C}^3 \to L^2(G^3)^{\mathrm{Ad}\,G}$.

For $\theta_{ij} \in \{\pm 1\}$, define the canonical nonnegative pair potential

$$V_{ij}^\theta(g) = \beta_{\mathrm{mag}} \left(1 - \frac{1}{2}\theta_{ij}\chi_{1/2}(g_i g_j^{-1})\right), \qquad \beta_{\mathrm{mag}} > 0. \tag{A.1}$$

Nonnegativity follows from $|\chi_{1/2}(g)| \le 2$.

Proposition A.1 (Raw $SU(2)$ compression)**.** *The compression of multiplication by Equation (A.1) is*

$$(\mathsf{F}_\theta^{\mathrm{W}})^* M_{V_{ij}^\theta} \mathsf{F}_\theta^{\mathrm{W}} = \beta_{\mathrm{mag}} \left[I - \frac{\theta_{ij}}{4}(E_{ij} + E_{ji}) \right]. \tag{A.2}$$

For one edge, the maximal raw coefficient satisfying

$$M_{ij}^{\mathrm{W}} - \beta \ell_{ij}^\theta \ge 0$$

is

$$\beta_{ij}^{\mathrm{W,max}} = \frac{5}{8}\beta_{\mathrm{mag}}. \tag{A.3}$$

Proof. The diagonal entries come from character orthogonality. The only nonzero off-diagonal integrals are the (i, j) and (j, i) entries; the fundamental tensor-product orthogonality coefficient is $1/2$, and the real-part normalization in Equation (A.1) supplies the second factor $1/2$. The extraction value follows by diagonalizing the resulting two-dimensional Hermitian block after subtracting $\beta \ell_{ij}^\theta$. □

A.4 Triangle spectra

For the equal-weight triangle,

$$L_\theta = \sum_{\{i,j\}} \ell_{ij}^\theta.$$

If the flux product is $\theta_{12}\theta_{23}\theta_{31} = 1$, then

$$\operatorname{spec}(L_\theta) = \{0, 3, 3\}.$$

For the frustrated choice

$$\theta_{12} = 1, \qquad \theta_{23} = 1, \qquad \theta_{31} = -1,$$

one obtains

$$\operatorname{spec}(L_\theta) = \{1, 1, 4\}.$$

Summing the three cell compressions gives

$$(\mathsf{F}_\theta^{\mathrm{W}})^* H_{\triangle,\mathrm{mag}}^\theta \mathsf{F}_\theta^{\mathrm{W}} = \frac{5}{2}\beta_{\mathrm{mag}} I + \frac{1}{4}\beta_{\mathrm{mag}} L_\theta. \tag{A.4}$$

Thus the raw affine graph coefficient is $b_0 = \beta_{\mathrm{mag}}/4$, while the maximal positive single-edge extraction is larger.

A.5 A finite localization-associator witness

Apply the character-convolution construction of Theorem D.7 to the two-dimensional fundamental representation. For the normalized fine source A_n,

$$\alpha_{\ell mn} A_n = -\frac{1}{8}\chi_{1/2}(h), \qquad D_{\ell mn}^{\mathrm{lax}}(A_n) = \frac{1}{8}. \tag{A.5}$$

This is a nonzero source-restricted associator. It witnesses finite Wilson information loss; it is not the complete magnetic mixed block.

A.6 What remains for the theorem

A complete interacting $SU(2)$ application must still verify

$$\operatorname{rank}[(I - P_{\mathrm{vac}})\mathsf{F}_\theta^{\mathrm{W}}(I - P_{\theta,\min})] > 0,$$

finite electric capture and Gram conditioning, recompression

$$M_f^{\mathrm{phys}} = (\mathsf{F}_\theta^{\mathrm{phys}})^* h_f \mathsf{F}_\theta^{\mathrm{phys}},$$

positive physical extraction, calibration, residual infrared control, and final mixed-block subcriticality.

B An explicit raw $SU(3)$ witness

B.1 Scope

This appendix gives a raw fundamental-channel realization for $SU(3)$. As in Section A, it supplies explicit starting data but does not replace the physicalization and full-Hamiltonian verification required by the principal theorem.

B.2 Representations and center phases

Let $G = SU(3)$, with

$$Z(G) = \{I, \omega I, \omega^2 I\}, \qquad \omega = e^{2\pi i/3}.$$

The relevant Casimir values are

$$C_2(\mathbf{1}) = 0, \qquad C_2(\mathbf{3}) = C_2(\overline{\mathbf{3}}) = \frac{4}{3}, \qquad C_2(\mathbf{8}) = 3.$$

The fundamental and antifundamental center characters are conjugate:

$$\zeta_{\mathbf{3}}(\omega I) = \omega, \qquad \zeta_{\overline{\mathbf{3}}}(\omega I) = \omega^{-1},$$

while the adjoint representation is center blind.

B.3 Raw fundamental channel and phase convention

Interpret g_1, g_2, g_3 as based loop holonomies with residual diagonal conjugation and work on $L^2(G^3)^{\mathrm{Ad}\, G}$. The character channels and pair potentials below are gauge invariant. Let $\chi_{\mathbf{3}}$ be the ordinary fundamental character, normalized by Haar measure so that

$$\int_G \chi_{\mathbf{3}}(g)\, dg = 0, \qquad \int_G |\chi_{\mathbf{3}}(g)|^2\, dg = 1.$$

Set $\phi_i(g) = \chi_{\mathbf{3}}(g_i)$ and define

$$\mathsf{F}_\theta^{\mathrm{W}} e_i = \phi_i.$$

The graph convention used throughout the manuscript is

$$(d_\theta c)_{ij} = c_i - \theta_{ij} c_j, \qquad r_{ij}^\theta = e_i - \overline{\theta_{ij}} e_j, \qquad \ell_{ij}^\theta = |r_{ij}^\theta\rangle\langle r_{ij}^\theta|. \tag{B.1}$$

Indeed,

$$\langle c, \ell_{ij}^\theta c\rangle = |c_i - \theta_{ij} c_j|^2.$$

Thus the conjugate phase in the edge vector is forced by the convention for d_θ.

For $\theta_{ij} \in \{1, \omega, \omega^2\}$, define

$$V_{ij}^\theta(g) = \beta_{\mathrm{mag}} \left[1 - \frac{1}{3}\Re\big(\theta_{ij}\chi_{\mathbf{3}}(g_i g_j^{-1})\big)\right]. \tag{B.2}$$

Since $|\chi_{\mathbf{3}}(g)| \le 3$, this potential is nonnegative.

Lemma B.1 (Simultaneous vertex-rephasing covariance)**.** *Let $u : V_\theta \to \mathbb{T}$, let $U_u e_v = u_v e_v$, and set*

$$\theta'_{ij} = u_i \theta_{ij} \overline{u_j}.$$

Then

$$\ell_{ij}^{\theta'} = U_u \ell_{ij}^\theta U_u^*, \tag{B.3}$$

and the raw compression matrices below satisfy

$$M_{ij}^{\mathrm{W},\theta'} = U_u M_{ij}^{\mathrm{W},\theta} U_u^*. \tag{B.4}$$

Consequently, the maximal extraction coefficient is invariant under vertex rephasing.

Proof. One has

$$U_u r_{ij}^{\theta} = u_i e_i - \overline{\theta_{ij}} u_j e_j = u_i r_{ij}^{\theta'},$$

so the irrelevant scalar phase u_i disappears in the rank-one operator. Also

$$U_u E_{ij} U_u^* = u_i \overline{u_j} E_{ij},$$

which gives the covariance of the compression matrix. Unitary conjugation preserves positive-semidefinite order and therefore preserves the maximal extraction coefficient. □

Proposition B.2 (Raw $SU(3)$ compression and phase-independent extraction)**.** *The raw compression is*

$$M_{ij}^{\mathrm{W},\theta} := (\mathsf{F}_{\theta}^{\mathrm{W}})^* M_{V_{ij}^{\theta}} \mathsf{F}_{\theta}^{\mathrm{W}} = \beta_{\mathrm{mag}} \left[I - \frac{1}{18} \left(\theta_{ij} E_{ij} + \overline{\theta_{ij}} E_{ji} \right) \right]. \tag{B.5}$$

For every $\theta_{ij} \in \{1, \omega, \omega^2\}$, the maximal raw single-edge extraction coefficient relative to ℓ_{ij}^{θ} is

$$\boxed{\beta_{ij}^{\mathrm{W,max}} = \frac{19}{36} \beta_{\mathrm{mag}}.} \tag{B.6}$$

In particular, the value is independent of the strict or twisted edge phase.

Proof. Write $x = g_i$ and $y = g_j$. For $i \neq j$, the constant part of the potential gives no off-diagonal contribution because

$$\langle \phi_i, \phi_j \rangle = 0.$$

Using Schur orthogonality for the fundamental representation,

$$\int_G \overline{\chi_{\mathbf{3}}(x)} \chi_{\mathbf{3}}(xy^{-1})\, dx = \frac{1}{3} \overline{\chi_{\mathbf{3}}(y)}. \tag{B.7}$$

After multiplication by $\chi_{\mathbf{3}}(y)$ and integration over y, the first term in the real part contributes $1/3$. The conjugate-character term vanishes because $\overline{\mathbf{3}} \otimes \overline{\mathbf{3}}$ contains no trivial representation. The factor $1/2$ from the real part and the factor $1/3$ in Equation (B.2) therefore give

$$\langle \phi_i, M_{V_{ij}^{\theta}} \phi_j \rangle = -\frac{\beta_{\mathrm{mag}}}{18} \theta_{ij}.$$

The reverse entry is its complex conjugate, while the diagonal entries equal β_{mag}. This proves Equation (B.5).

Restrict now to $\mathrm{span}\{e_i, e_j\}$ and abbreviate $\theta = \theta_{ij}$. In the ordered basis (e_i, e_j),

$$M_{\theta} = \beta_{\mathrm{mag}} \begin{pmatrix} 1 & -\theta/18 \\ -\overline{\theta}/18 & 1 \end{pmatrix}, \qquad r_{\theta} = \begin{pmatrix} 1 \\ -\overline{\theta} \end{pmatrix}, \qquad \ell_{\theta} = |r_{\theta}\rangle\langle r_{\theta}|. \tag{B.8}$$

The vector r_{θ} is an eigenvector of M_{θ}:

$$M_{\theta} r_{\theta} = \frac{19}{18} \beta_{\mathrm{mag}} r_{\theta}, \qquad \|r_{\theta}\|^2 = 2. \tag{B.9}$$

The orthogonal vector

$$s_{\theta} = (1, \overline{\theta})^{\mathsf{T}}$$

has eigenvalue $17\beta_{\mathrm{mag}}/18$. Hence

$$M_\theta - \beta \ell_\theta \geq 0 \quad \Longleftrightarrow \quad \frac{19}{18}\beta_{\mathrm{mag}} - 2\beta \geq 0.$$

Therefore

$$\beta \leq \frac{19}{36}\beta_{\mathrm{mag}},$$

and equality leaves a nonnegative operator with zero eigenvalue only in the r_θ direction. This proves the claimed maximum.

Equivalently, the Moore–Penrose formula of Lemma D.4 gives

$$\langle r_\theta, M_\theta^{-1} r_\theta \rangle = \frac{36}{19\beta_{\mathrm{mag}}}, \qquad \beta_{ij}^{\mathrm{W,max}} = \frac{1}{\langle r_\theta, M_\theta^{-1} r_\theta \rangle} = \frac{19}{36}\beta_{\mathrm{mag}}. \tag{B.10}$$

No θ-dependence remains because the phase of the compression block is aligned with the phase of ℓ_{ij}^{θ}. □

Remark B.3 (The mismatched convention). If one instead inserts

$$q = e_i - \theta_{ij} e_j,$$

then $q = r_{ij}^{\overline{\theta}}$, not r_{ij}^{θ}. Pairing $M_{ij}^{\mathrm{W},\theta}$ with $|q\rangle\langle q|$ compares the θ-compression with the edge form of the conjugate graph connection and can produce an artificial phase dependence when $\theta^2 \neq 1$. The $SU(2)$ case conceals this mismatch because its detected center phases satisfy $\theta = \overline{\theta}$.

B.4 Strict and twisted triangle spectra

Summing the three equal-weight cell matrices gives

$$(\mathsf{F}_\theta^{\mathrm{W}})^* H_{\triangle,\mathrm{mag}}^{\theta} \mathsf{F}_\theta^{\mathrm{W}} = \frac{26}{9}\beta_{\mathrm{mag}} I + \frac{1}{18}\beta_{\mathrm{mag}} L_\theta. \tag{B.11}$$

For trivial flux,

$$\mathrm{spec}(L_\theta) = \{0, 3, 3\}.$$

For total fundamental flux ω,

$$\mathrm{spec}(L_\theta) = \left\{ 2 - 2\cos\frac{2\pi}{9},\ 2 - 2\cos\frac{4\pi}{9},\ 2 + 2\cos\frac{\pi}{9} \right\}. \tag{B.12}$$

The natural affine graph coefficient in Equation (B.11) is

$$b_0 = \frac{1}{18}\beta_{\mathrm{mag}},$$

and

$$\frac{\beta_{ij}^{\mathrm{W,max}}}{b_0} = \frac{19}{2}.$$

Thus maximal positive extraction and favorable affine calibration are distinct optimization problems.

B.5 A finite localization-associator witness

Apply Theorem D.7 to the three-dimensional fundamental representation. The exact defect identity gives a normalized source A_n with

$$\alpha_{\ell mn} A_n = -\frac{1}{27}\chi_{\mathbf{3}}(h), \qquad D^{\mathrm{lax}}_{\ell mn}(A_n) = \frac{1}{27}. \tag{B.13}$$

The coefficient reflects the three fundamental matrix-index contractions. As before, this is a raw source-restricted transport witness rather than a complete mixed-Hamiltonian estimate.

B.6 What remains for the theorem

A complete $SU(3)$ application must verify survival after exact vacuum removal, a finite capture margin, Gram conditioning, physical recompression of every canonical cell, positive physical edge extraction, calibration against E_0, residual infrared control, and reference–transport–collar subcriticality. The raw value $19\beta_{\mathrm{mag}}/36$ is not automatically a physical extraction coefficient.

C A complete physicalized $SU(2)$ Galerkin witness

C.1 Purpose and scope

This appendix verifies the complete fixed-scale comparison package on an exact finite Peter–Weyl Galerkin Hamiltonian. Unlike Sections A and B, which isolate raw representation-theoretic mechanisms, the present example computes the interacting ground projection, vacuum removal, electric capture, Gram normalization, physical magnetic recompression, reference leakage, all block constants, and a positive final coercivity constant.

The example is an exact application of the abstract comparison theorem to the Galerkin Hamiltonian defined below. It is not asserted that the same numerical constants hold for the untruncated class-function Hamiltonian.

C.2 The $j \le 1$ class-function Galerkin Hamiltonian

Work in the conjugation-invariant Hilbert space

$$L^2(SU(2))^{\mathrm{Ad}\,SU(2)}.$$

Let

$$\psi_0 = \chi_0 = 1, \qquad \psi_{1/2} = \chi_{1/2}, \qquad \psi_1 = \chi_1.$$

Schur orthogonality makes these vectors orthonormal. Define

$$\mathcal{K}_1 = \mathrm{span}\{\psi_0, \psi_{1/2}, \psi_1\}$$

and let P_1 be the orthogonal projection onto $\mathcal{K}_1$.

Choose units in which the electric coefficient is one. The Galerkin electric Hamiltonian is

$$H^{(1)}_{\mathrm{el}} = \mathrm{diag}\left(0, \frac{3}{4}, 2\right) \tag{C.1}$$

in the ordered basis $(\psi_0, \psi_{1/2}, \psi_1)$.

Consider the nonnegative Wilson potential

$$V(g) = 1 - \frac{1}{2}\chi_{1/2}(g) \geq 0. \tag{C.2}$$

The fusion identity

$$\chi_{1/2}\chi_j = \chi_{j+1/2} + \chi_{j-1/2}, \qquad \chi_{-1/2} = 0,$$

gives

$$H^{(1)}_{\mathrm{mag}} = P_1 M_V P_1 = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} \geq 0. \tag{C.3}$$

Thus

$$H^{(1)} = H^{(1)}_{\mathrm{el}} + H^{(1)}_{\mathrm{mag}} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{7}{4} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 3 \end{pmatrix}. \tag{C.4}$$

All matrices in this appendix are finite-dimensional form matrices, so no operator-domain issue occurs.

C.3 Exact interacting ground projection

The characteristic polynomial is

$$\det(\lambda I - H^{(1)}) = \frac{1}{4}\left(4\lambda^3 - 23\lambda^2 + 38\lambda - 17\right). \tag{C.5}$$

Its three eigenvalues are

$$\begin{aligned} E_0 &= 0.7263771031953168\ldots, \\ E_1 &= 1.8348761489447635\ldots, \\ E_2 &= 3.1887467478599197\ldots. \end{aligned} \tag{C.6}$$

The ground value is simple. If E_0 denotes the smallest root of Equation (C.5), a corresponding unnormalized eigenvector is

$$\begin{pmatrix} 1 \\ 2(1-E_0) \\ \dfrac{1-E_0}{3-E_0} \end{pmatrix}. \tag{C.7}$$

After normalization,

$$\Omega_G = \begin{pmatrix} 0.8723860596385584 \\ 0.4774096015406509 \\ 0.1049887389442628 \end{pmatrix}, \qquad P^{(1)}_{\mathrm{vac}} = |\Omega_G\rangle\langle\Omega_G|. \tag{C.8}$$

Numerically,

$$P^{(1)}_{\mathrm{vac}} = \begin{pmatrix} 0.761057437052 & 0.416485481122 & 0.091590712274 \\ 0.416485481122 & 0.227919927643 & 0.050122632026 \\ 0.091590712274 & 0.050122632026 & 0.011022635305 \end{pmatrix}. \tag{C.9}$$

C.4 Raw graph channel and survival

Let the coefficient graph have two vertices and one strict unit-weight edge. Its graph operator is

$$L = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{C.10}$$

Set

$$\eta_0 = \frac{e_1 + e_2}{\sqrt{2}}, \qquad \eta_1 = \frac{e_1 - e_2}{\sqrt{2}}.$$

Then

$$L\eta_0 = 0, \qquad L\eta_1 = 2\eta_1, \qquad \delta_{\mathrm{sec}} = 2.$$

Define an isometric raw Wilson frame by

$$\mathsf{F}^{\mathrm{W}}\eta_0 = \psi_1, \qquad \mathsf{F}^{\mathrm{W}}\eta_1 = \psi_{1/2}. \tag{C.11}$$

Equivalently,

$$\mathsf{F}^{\mathrm{W}}e_1 = \frac{\psi_1 + \psi_{1/2}}{\sqrt{2}}, \qquad \mathsf{F}^{\mathrm{W}}e_2 = \frac{\psi_1 - \psi_{1/2}}{\sqrt{2}}.$$

The raw graph-excitation space is the one-dimensional space

$$\mathscr{E}^{\mathrm{W}} = \mathbb{C}\eta_1.$$

Let

$$Q^{(1)}_{\mathrm{vac}} = I - P^{(1)}_{\mathrm{vac}}.$$

Then

$$T\eta_1 = Q^{(1)}_{\mathrm{vac}}\psi_{1/2} \neq 0, \tag{C.12}$$

so the full raw excitation survives exact vacuum removal. Its survival singular value is

$$\sigma_{\mathrm{vac}} = \|Q^{(1)}_{\mathrm{vac}}\psi_{1/2}\| = 0.8786808705991026\ldots. \tag{C.13}$$

C.5 Vacuum-orthogonal electric spectrum and Gram operator

On

$$\mathcal{K}_{1,\perp} = \operatorname{Ran} Q^{(1)}_{\mathrm{vac}},$$

the restricted electric form operator is the finite-dimensional compression

$$H^{\perp,(1)}_{\mathrm{el}} = Q^{(1)}_{\mathrm{vac}} H^{(1)}_{\mathrm{el}} Q^{(1)}_{\mathrm{vac}}|_{\mathcal{K}_{1,\perp}}.$$

Its two eigenvalues are

$$\mu_{\mathrm{IR}} = 0.5763713684093909\ldots, \qquad \mu_{\mathrm{UV}} = 1.9806434152479937\ldots. \tag{C.14}$$

Choose normalized eigenvectors

$$s = \begin{pmatrix} -0.4821042059738672 \\ 0.8757989270398648 \\ 0.0234898696916037 \end{pmatrix}, \qquad u = \begin{pmatrix} 0.0807347355889435 \\ 0.0711077474866028 \\ -0.9941959518710408 \end{pmatrix}, \tag{C.15}$$

so that

$$H_{\mathrm{el}}^{\perp,(1)} s = \mu_{\mathrm{IR}} s, \qquad H_{\mathrm{el}}^{\perp,(1)} u = \mu_{\mathrm{UV}} u.$$

Take

$$P_{\mathrm{IR}}^{(1)} = |s\rangle\langle s|, \qquad P_{\mathrm{UV}}^{(1)} = |u\rangle\langle u|.$$

This is a whole-eigenspace cutoff.

Since $s \perp \Omega_G$, the captured map is

$$J\eta_1 = P_{\mathrm{IR}}^{(1)} T\eta_1 = \langle s, \psi_{1/2}\rangle s = 0.8757989270398648\ldots\ s. \tag{C.16}$$

The uncaptured tail is

$$\tau_{\mathrm{cap}} = |\langle u, \psi_{1/2}\rangle| = 0.0711077474866028\ldots < \sigma_{\mathrm{vac}}. \tag{C.17}$$

The Gram operator on $\mathbb{C}\eta_1$ is therefore

$$\boxed{G = J^*J = 0.7670237606041785\ldots\ I.} \tag{C.18}$$

It is neither zero nor the identity. The physical channel is

$$\boxed{\mathsf{F}^{\mathrm{phys}}\eta_1 = JG^{-1/2}\eta_1 = s.} \tag{C.19}$$

Thus the physicalization removes the exact ground component and the excluded electric component of the raw excitation.

C.6 Physical graph and magnetic recompression

On the one-dimensional surviving coefficient space,

$$A^{\mathrm{raw}} = 2I. \tag{C.20}$$

Since the strict graph has $\lambda_1 = 0$,

$$\widetilde{A} = \widetilde{L} = 2G^{-1} = 2.6074811534190492\ldots\ I. \tag{C.21}$$

The sole canonical magnetic cell recompresses to

$$\boxed{M^{\mathrm{phys}} = \langle s, H_{\mathrm{mag}}^{(1)} s\rangle = 1.4016539436411062\ldots.} \tag{C.22}$$

Consequently,

$$b_{\mathrm{ch}}^{\mathrm{phys,max}} = \frac{M^{\mathrm{phys}}}{\widetilde{L}} = \frac{GM^{\mathrm{phys}}}{2} = 0.5375509394586392\ldots. \tag{C.23}$$

Choose the simpler admissible value

$$b = \frac{1}{2} < b_{\mathrm{ch}}^{\mathrm{phys,max}}. \tag{C.24}$$

The positive magnetic remainder is

$$D_{\mathrm{mag}}^{\mathrm{phys}}(1/2) = M^{\mathrm{phys}} - \frac{1}{2}\widetilde{L} = 0.0979133669315815\ldots. \tag{C.25}$$

The electric form compression is

$$E_{\mathrm{ch}}^{\mathrm{phys}} = \langle s, H_{\mathrm{el}}^{(1)} s \rangle = \mu_{\mathrm{IR}} = 0.5763713684093909\ldots. \tag{C.26}$$

The scalar calibration remainder is

$$\mathcal{R}_{\mathrm{sec}}^{\mathrm{phys}}(1/2) = E_{\mathrm{ch}}^{\mathrm{phys}} + D_{\mathrm{mag}}^{\mathrm{phys}}(1/2) - E_0 = -0.0520923678543444\ldots. \tag{C.27}$$

Hence

$$\boxed{r_{\mathrm{sec}}^{\mathrm{phys}}(1/2) = 0.0520923678543444\ldots, \qquad c_{\mathrm{ch}}^{\mathrm{phys}}(1/2) = 0.9479076321456556\ldots.} \tag{C.28}$$

The retained electric space is exactly the physical channel range, so the residual infrared space is zero and

$$\boxed{\widehat{c}_{\mathrm{IR}} = c_{\mathrm{ch}}^{\mathrm{phys}}(1/2) = 0.9479076321456556\ldots.} \tag{C.29}$$

C.7 Ultraviolet exclusion and exact reference leakage

The first excluded electric level is μ_{UV}. Therefore

$$\boxed{\widehat{c}_{\mathrm{UV}} = \mu_{\mathrm{UV}} - E_0 = 1.2542663120526769\ldots.} \tag{C.30}$$

Declare the sole magnetic cell interior. Then

$$H_{\mathrm{ref}}^{(1)} = H_{\mathrm{mag}}^{(1)}, \qquad K_{\mathrm{tr}} = 0, \qquad K_{\mathrm{col}} = 0.$$

The complete mixed block is ordinary reference-band mixing:

$$K = P_{\mathrm{UV}}^{(1)} H_{\mathrm{mag}}^{(1)} P_{\mathrm{IR}}^{(1)}, \qquad \boxed{\widehat{\varepsilon}_{\mathrm{mix}} = \varepsilon_{\mathrm{ref}} = |\langle u, H_{\mathrm{mag}}^{(1)} s \rangle| = 0.4163096927652429\ldots.} \tag{C.31}$$

There is no transport or collar contribution:

$$D_{\mathrm{lax}} = 0, \qquad \Delta_{\mathrm{col}} = 0, \qquad \varepsilon_{\mathrm{hyb}} = \varepsilon_{\mathrm{ref}}.$$

C.8 Positive final coercivity constant

The determinant margin is

$$\widehat{c}_{\mathrm{UV}} \widehat{c}_{\mathrm{IR}} - \widehat{\varepsilon}_{\mathrm{mix}}^{2} = 1.0156148496476259\ldots > 0. \tag{C.32}$$

The principal comparison constant is therefore

$$\boxed{\widehat{\kappa}_G = \frac{\widehat{c}_{\mathrm{UV}} + \widehat{c}_{\mathrm{IR}} - \sqrt{(\widehat{c}_{\mathrm{UV}} - \widehat{c}_{\mathrm{IR}})^2 + 4\widehat{\varepsilon}_{\mathrm{mix}}^2}}{2} = 0.6574906798706534\ldots > 0.} \tag{C.33}$$

Thus the complete Galerkin Hamiltonian satisfies

$$\boxed{H^{(1)} \geq E_0 I + 0.6574906798706534\ldots \left(I - P_{\mathrm{vac}}^{(1)}\right).} \tag{C.34}$$

The actual Galerkin spectral gap is

$$E_1 - E_0 = 1.1084990457494467\ldots, \tag{C.35}$$

so the resolved comparison certifies approximately 59.3% of the exact gap.

For reference, in the ordered basis (s, u, Ω_G),

$$H^{(1)} - E_0 I = \begin{pmatrix} 1.251648208855 & 0.416309692765 & 0 \\ 0.416309692765 & 2.319220481559 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{C.36}$$

The theorem deliberately uses the weaker but structurally resolved diagonal constants $\widehat{c}_{\mathrm{IR}}$ and $\widehat{c}_{\mathrm{UV}}$, rather than inserting the exact excited-state eigenvalues by hand.

C.9 Independent high-precision verification

The numerical values above were independently recomputed from the exact rational matrices Equations (C.1), (C.3) and (C.4) using 100-decimal-digit Hermitian diagonalization. The largest residual among the eigenvalue equations, projection identity, orthogonality relations, and block reconstruction was below 6×10^{-101}. In particular, the following outward-rounded intervals are sufficient to certify every strict inequality used in the witness:

$$\begin{aligned}
&0.767023760604 < G < 0.767023760605,\\
&0.537550939458 < b_{\mathrm{ch}}^{\mathrm{phys,max}} < 0.537550939459,\\
&0.947907632145 < \widehat{c}_{\mathrm{IR}} < 0.947907632146,\\
&1.254266312052 < \widehat{c}_{\mathrm{UV}} < 1.254266312053,\\
&0.416309692765 < \widehat{\varepsilon}_{\mathrm{mix}} < 0.416309692766,\\
&1.015614849647 < \widehat{c}_{\mathrm{UV}}\widehat{c}_{\mathrm{IR}} - \widehat{\varepsilon}_{\mathrm{mix}}^2 < 1.015614849648,\\
&0.657490679870 < \widehat{\kappa}_G < 0.657490679871.
\end{aligned}$$

The safety margins are therefore many orders of magnitude larger than the numerical residuals. Both the original finite-dimensional calculation and an independent high-precision verification script accompany the source package.

C.10 Verification of the complete package

Condition	Verification in the Galerkin model
(A1)	The exact Hermitian matrix Equation (C.4) and its complete ground projection Equation (C.9).
(A2)	The strict two-vertex graph Equation (C.10) and isometric raw frame Equation (C.11).
(A3)	Nonzero survival, with $\sigma_{\mathrm{vac}} > 0$, by Equation (C.13).
(A4)	Whole-eigenspace electric capture, nontrivial Gram operator, and positive ultraviolet exclusion by Equations (C.18) and (C.30).
(A5)	Positive physical recompression and $b_{\mathrm{ch}}^{\mathrm{phys,max}} > 1/2$ by Equations (C.22) and (C.23).
(A6)	Positive calibrated channel constant and zero residual infrared space by Equations (C.28) and (C.29).
(A7)	The sole magnetic cell is assigned to the interior class.
(A8)	The exact reference leakage is computed in Equation (C.31).
(A9)	There are no transported cells, so the comparator identities are vacuous.
(A10)	The Galerkin Peter–Weyl window contains 1 and is involution invariant; because there are no transported cells, no further localization condition is used.
Final condition	The strict determinant margin is positive by Equation (C.32), so the principal theorem applies.

Theorem C.1 (Complete physicalized $SU(2)$ Galerkin witness)**.** *The Hamiltonian Equation (C.4), raw graph channel Equation (C.11), and one-level physical electric cutoff verify the complete fixed-scale comparison package. The physicalization has*

$$G = 0.7670237606041785\ldots I,$$

the exact mixed block is nonzero, and the principal theorem gives

$$H^{(1)} \geq E_0 I + 0.6574906798706534\ldots (I - P_{\mathrm{vac}}^{(1)}).$$

Remark C.2 (Reproducibility and untruncated scope). All displayed values are obtained by diagonalizing the exact rational matrices Equations (C.1), (C.3) and (C.4). Two reproducibility scripts accompany the manuscript source: the direct finite-dimensional calculation and an independent 100-digit verification with residual checks. The example proves that the physical comparison package is executable end to end. Passing from this Galerkin witness to the untruncated class-function Hamiltonian would require an additional tail-stability estimate and is not claimed here.

D Technical finite-dimensional and block lemmas

D.1 Uniformity of strong convergence on finite-dimensional spaces

Lemma D.1. *Let $P_k \to I$ strongly on a Hilbert space and let S be finite-dimensional. Then*

$$\|(I - P_k)|_S\| \to 0.$$

Proof. The unit sphere of S is compact. Strong convergence is pointwise on this sphere, and the contractions $I - P_k$ are uniformly bounded. A finite-net argument gives uniform convergence. □

D.2 Injective finite capture and Gram bounds

Lemma D.2. *Let $T : C \to \mathcal{H}$ be injective with finite-dimensional domain and least singular value $\sigma > 0$. If P is an orthogonal projection satisfying $\|(I-P)T\| < \sigma$, then PT is injective. Moreover,*

$$(\sigma - \|(I-P)T\|)^2 I \le T^*PT \le \|T\|^2 I.$$

Proof. For $c \in C$,

$$\|PTc\| \ge \|Tc\| - \|(I-P)Tc\| \ge (\sigma - \|(I-P)T\|)\|c\|.$$

Squaring gives the lower Gram bound; the upper bound follows from $\|P\| = 1$. □

Remark D.3 (Principal-angle interpretation). Let $F : C \to \mathcal{H}$ be an isometry, let $V \subset \mathcal{H}$ be closed, and set $T = (I - P_V)F$. After restricting to $(\operatorname{Ker} T)^\perp$, the least singular value of T is the sine of the smallest positive principal angle between the surviving finite-dimensional subspace $F((\operatorname{Ker} T)^\perp)$ and V. This interpretation is not used in the proof of the fixed-scale theorem.

D.3 Positive rank-one extraction

The next finite-dimensional formula is standard in matrix analysis [13].

Lemma D.4. *Let $M \ge 0$ on a finite-dimensional Hilbert space and let $q \ne 0$. Define*

$$\beta_{\max} = \sup\{\beta \ge 0 : M - \beta|q\rangle\langle q| \ge 0\}.$$

Then $\beta_{\max} > 0$ exactly when $q \in \operatorname{Ran} M^{1/2}$. In that case,

$$\boxed{\beta_{\max} = \frac{1}{\langle q, M^\dagger q\rangle},} \tag{D.1}$$

where $M^\dagger$ is the Moore–Penrose inverse. If $q \notin \operatorname{Ran} M^{1/2}$, then $\beta_{\max} = 0$.

Proof. If q has a component in $\operatorname{Ker} M$, evaluating the inequality on that kernel component forces $\beta = 0$. If $q \in \operatorname{Ran} M^{1/2}$, restrict to the support of M and conjugate by the inverse square root there:

$$M - \beta|q\rangle\langle q| \ge 0 \quad \Longleftrightarrow \quad I - \beta|M^{\dagger/2}q\rangle\langle M^{\dagger/2}q| \ge 0.$$

The unique nonzero eigenvalue of the rank-one operator is $\|M^{\dagger/2}q\|^2 = \langle q, M^\dagger q\rangle$. □

D.4 Spectral-block Sylvester estimate

Lemma D.5. *Let A be self-adjoint and let*

$$P = \mathbf{1}_{(-\infty,a]}(A), \qquad Q = \mathbf{1}_{[b,\infty)}(A), \qquad a < b.$$

Let B be bounded. Assume $BP\mathcal{H} \subset D(A)$ and that $Q[A,B]P$ extends to a bounded operator. Then

$$\boxed{\|QBP\| \le \frac{\|Q[A,B]P\|}{b-a}.} \tag{D.2}$$

Proof. Set $X = QBP$, $A_P = PAP$, and $A_Q = QAQ$. On $P\mathcal{H}$,

$$A_Q X - X A_P = Q[A, B]P.$$

The spectral bounds give the norm-convergent representation

$$X = \int_0^\infty e^{-t(A_Q - aI)} Q[A, B] P e^{t(A_P - aI)}\, dt.$$

The two semigroup norms are bounded by $e^{-t(b-a)}$ and 1, respectively. Integration proves Equation (D.2). □

D.5 Two-level relative estimate

Lemma D.6. *For $c_1, c_2 > 0$ and $\eta^2 \le \rho c_1 c_2$, $0 \le \rho < 1$, the smaller eigenvalue of*

$$M = \begin{pmatrix} c_1 & -\eta \\ -\eta & c_2 \end{pmatrix}$$

is at least $(1-\rho)\min\{c_1, c_2\}/2$.

Proof. The determinant satisfies

$$\det M = c_1 c_2 - \eta^2 \ge (1-\rho) c_1 c_2.$$

Since $\lambda_{\max}(M) \le \operatorname{tr} M = c_1 + c_2$,

$$\lambda_{\min}(M) = \frac{\det M}{\lambda_{\max}(M)} \ge \frac{(1-\rho)c_1 c_2}{c_1 + c_2} \ge \frac{1-\rho}{2} \min\{c_1, c_2\}.$$

□

D.6 Explicit character-convolution associator witness

Lemma D.7. *Let R be a nontrivial irreducible unitary representation of a compact group, of dimension d_R, and let χ_R be its ordinary character, normalized by Haar so that $\|\chi_R\|_2 = 1$. There exist finite unital involution-invariant Peter–Weyl windows at three levels and a normalized fine source A_n for which microscopic transport is strict and*

$$\boxed{\alpha_{\ell mn} A_n = -d_R^{-3} \chi_R(h), \qquad D^{\mathrm{lax}}_{\ell mn}(A_n) = d_R^{-3}.} \tag{D.3}$$

Proof. Use fine variables $(x, y, z, h) \in G^4$ and define

$$A_n(x, y, z, h) = \chi_R(x)\chi_R(x^{-1}y)\chi_R(y^{-1}z)\chi_R(z^{-1}h). \tag{D.4}$$

The change to successive increments preserves product Haar measure, so $\|A_n\|_2 = 1$.

Let $\mathbb{E}_{mn}$ integrate the variable x. Schur orthogonality gives

$$\mathbb{E}_{mn} A_n = d_R^{-1} \chi_R(y)\chi_R(y^{-1}z)\chi_R(z^{-1}h).$$

Choose the intermediate window $\mathscr{W}_m = \operatorname{span}\{1\}$. The displayed function has zero total Haar mean because R is nontrivial, so

$$L_m \mathbb{E}_{mn} A_n = 0 \quad \text{and hence} \quad T_{\ell m} T_{mn} A_n = 0.$$

Let the direct expectation $\mathbb{E}_{\ell n}$ integrate x, y, z, leaving h. Three applications of

$$\int_G \chi_R(u)\chi_R(u^{-1}v)\,du = d_R^{-1}\chi_R(v)$$

give

$$\mathbb{E}_{\ell n}A_n = d_R^{-3}\chi_R(h).$$

Choose

$$\mathscr{W}_\ell = \operatorname{span}\{1, \chi_R(h), \overline{\chi_R(h)}\},$$

and take the fine window to contain $1, A_n, A_n^*$. These windows are finite, unital, and involution invariant. Therefore

$$T_{\ell n}A_n = d_R^{-3}\chi_R(h),$$

and $\alpha = T_{\ell m}T_{mn} - T_{\ell n}$ gives the first identity in Equation (D.3). Since $\|A_n\|_2 = \|\chi_R\|_2 = 1$, the ratio is d_R^{-3}. □

D.7 Dependency summary

The repaired proof chain is noncircular:

canonical finite Hamiltonian and exact ground projection
⟶ vacuum-orthogonal electric form operator
⟶ physical channel reduction and form compression
⟶ exact reference–transport–collar mixed block
⟶ vacuum reduction and sharp two-block theorem.

The familywise analysis of Section 11 uses the resulting constants but is not needed to prove the fixed-scale theorem.

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Author declarations

Conflict of interest. The author has no conflicts to disclose.

Data availability statement. The numerical data supporting the Galerkin witness are generated from the exact matrices displayed in the article. The resulting constants, calculation output, and reproducibility scripts are included in the supplementary source package.

Code availability. The supplementary source package contains both the direct finite-dimensional calculation and an independent 100-decimal-digit verification script.

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