

Scale-Bordism Criteria for Uniform Coercivity and Center-Banded Vacuum Descent in Yang–Mills Wilson–Tube Systems

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We separate the finite-to-continuum organization problem for Yang–Mills Wilson–tube Hamiltonians into an abstract full-scale criterion and an explicit fixed-stratum realization. For the full multiparameter scale category, directed finite-shape amalgamation, compact coefficient-line diagram spaces, coherent continuous restriction, and universal finite extension yield a nonempty inverse limit; any resulting higher pseudosection carrying one quantitative profile inherits a uniform positive off-ground coercive constant. In the five-paper sequence, the principal upstream instance is the selected triality-sensitive $SU(3)$ packet constructed in Paper I. Independently, a four-character $SU(2)$ seed is tensorized here to an end-to-end depth exhaustion at one fixed stratum, with complete physicalization, vacuum-line scale bordism, center-background covariance, and depth-independent coercive constants verified explicitly. The same finite analytic family supports strict and twisted coefficient-line reassociation laws. The nontrivial degree-three class for the classifying space of the two-element center with circle coefficients admits ordinary Hilbert realizations on every finite-depth chart but no global band-local stationary trivialization under the declared higher equivalences. Paper II therefore supplies a refinement-compatible coercive branch criterion together with explicit strict/twisted higher-vacuum witnesses, while spacetime reconstruction, thin descent, and invariant-mass spectral transfer remain assigned to Papers III–V.

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I. INTRODUCTION

A finite-dimensional gauge-invariant Hamiltonian has a discrete spectrum and a positive separation above its complete ground eigenspace. That elementary fact does not solve the finite-to-continuum organization problem. A constructive Yang–Mills program must select, across many simultaneous refinements, the entire finite coercive mechanism: the Hamiltonian, complete ground projection, physicalized Wilson channel, Gram operator, magnetic recompression, ultraviolet and residual blocks, mixed terms, and the admissible comparison data relating different scales.

A scale argument needs more than a family of positive numbers. It needs typed arrows, endpoint control, and a proof that the package survives the passage.

At scale γ , the finite estimate is

$$H_\gamma \geq E_{0,\gamma}I + \hat{\kappa}_\gamma(I - P_{\text{vac},\gamma}), \quad P_{\text{vac},\gamma} = \mathbf{1}_{\{E_{0,\gamma}\}}(H_\gamma). \quad (1)$$

The objective is not merely to know that every $\hat{\kappa}_\gamma$ is positive, but to organize compatible finite packages with one lower bound

$$H_\gamma \geq E_{0,\gamma}I + \kappa_*(I - P_{\text{vac},\gamma}), \quad \kappa_* > 0. \quad (2)$$

The companion finite-scale analysis¹ supplies the coercive package without selecting a preferred vacuum vector, without vacuum–graph alignment, and without an infrared block-closure assumption. The vacuum datum is always the complete ground spectral projection.

The selected triality-sensitive SU(3) packet of Paper I is the principal upstream finite-scale instance used by the five-paper sequence. Paper II imports that packet and its exact selected floor κ_{cof} ; it does not recompute the Paper-I SU(3) triality calculation. The explicit SU(2) family constructed below is a separate fixed-stratum validation of the recursive analytic, scale-bordism, and coefficient-line mechanism. Its numerical constants are not the selected SU(3) constants.

A. Scale bordism and coefficient-line descent

A strict inductive system is generally too rigid for the refinement problem. A refinement arrow $a : \gamma \rightarrow \gamma'$ carries an analytic bordism and a vacuum-line correspondence. Different refinement paths may agree only through admissible higher comparison, and the coefficient

lines of composable bordisms may possess a nontrivial reassociator. We therefore use a scale-bordism bicategory in the standard coherence sense^{2,3}. Three layers are kept distinct:

- (i) raw geometric refinement;
- (ii) a relative refinement-to-admissible filling class $[\Delta_\zeta, \tau]_{\text{ref/adm}}$;
- (iii) a coefficient-line reassociation class $[\Omega]_{\text{band}}$.

No transgression between the relative and reassociation classes is assumed.

The coefficient lines are essential. In the twisted case the vacuum transport cannot be represented by one globally chosen family of ordinary unitary lifts with scalar compositors: ordinary operator associativity would force the scalar associator to be trivial. The invariant arrows are instead correspondences

$$\mathbb{U}_a^{\text{vac}} : \mathcal{V}_{s(a)} \hat{\otimes} L_a \longrightarrow \mathcal{V}_{t(a)},$$

with multiplication and reassociation retained in the coefficient category. Ordinary Hilbert realizations arise only after choosing local tensorators on finite charts.

B. Principal results

The first result is conditional and full scale.

Theorem I.1 (Abstract full-scale criterion, informal). *Let the finite demand shapes be directed under common finite extension. Suppose one good seed exists; every fixed-profile space of coefficient-line physical diagrams is compact Hausdorff; restriction maps are coherent and continuous; and every finite good diagram extends surjectively while preserving the analytic package, canonical bordisms, coefficient-line law, and lifting two-presheaf. Then the full directed inverse limit is nonempty. Every selected higher pseudosection carrying the fixed profile satisfies (2).*

The proof uses Tychonoff compactness and the finite-intersection property over the entire directed family of finite shapes. It does not assume a countable cofinal chain. The hypotheses remain a criterion: the explicit model below does not verify universal extension in the full multiparameter category.

The second result is constructive but fixed stratum.

Theorem I.2 (Explicit fixed-stratum realization, informal). *Fix $\sigma_0 = (\varepsilon_0, \theta_0, \Lambda_0, b_0)$. A four-character $\mathrm{SU}(2)$ Galerkin seed tensorizes to a depth-cofinal family in $\mathfrak{R}_{\mathrm{depth}}(\sigma_0)$ carrying the complete physicalized package, one canonical vacuum-line scale-bordism correspondence, center-background covariance, and depth-independent coercive constants. In particular,*

$$\widehat{H}_n \geq 0.24975(I - P_{\mathrm{vac},n}), \quad \widehat{H}_n \geq 3.093219423836 \dots (I - P_{\mathrm{vac},n}).$$

The construction does not vary $\varepsilon_0, \theta_0, \Lambda_0$, or b_0 , and hence is not a full continuum-directed branch.

The same analytic family supports two higher globalizations.

Theorem I.3 (Center-banded reconstruction dichotomy, informal). *The fundamental Wilson-shell marking is center odd. Within the center-natural, shell-local, translation-stationary, coefficient-monoidal higher-equivalence category, admissible scalar higher gauges are exactly pullbacks of normalized cochains on the fixed $\mathbb{Z}_2$ band. The trivial reassociation law gives strict Hilbert descent. The nontrivial generator $[\Omega_1] \in H^3(B\mathbb{Z}_2; U(1))$, in the standard finite-group cohomology convention^{4,5}, admits local Hilbert realizations on every finite depth simplex but no global band-local stationary tensorator. The local realizations and their rooted one- and two-cochain transitions form a contractible lifting two-presheaf.*

The two globalizations use the same Hamiltonians, ground projections, physicalized packages, canonical projective comparisons, overlap singular values, analytic refinement defects, and coercive constants. Consequently:

the complete finite analytic shadow does not determine the higher vacuum descent type.

The finite Hamiltonian also does not select the strict or twisted coefficient reassociation law. The distinction lies in global higher gluing relative to the declared physical equivalence category.

C. Scope and organization

No continuum Hamiltonian, continuum ground projection, local observable net, Poincaré representation, or continuum mass gap is constructed here. The abstract theorem gives a conditional route to full multiparameter cofinality; the explicit model verifies the complete

recursive mechanism only along tensor depth at fixed σ_0 . A later reconstruction must simultaneously realize $n \rightarrow \infty$, $\varepsilon \rightarrow 0$, $\Lambda \rightarrow \infty$, and the remaining geometric and boundary refinements while preserving the bordism, quantitative profile, coefficient law, and higher gluing.

Sections II–IV define the finite package, scale geometry, coefficient-line diagrams, and the directed inverse-limit criterion. Section V separates full-scale and fixed-stratum coercive consequences. Section VI defines the banded vacuum type and admissible higher equivalence. Section VII constructs the explicit strict/twisted pair and proves finite-shadow indistinguishability and the higher-descent obstruction. Section VIII states the exact reconstruction handoff. Extended seed arithmetic, perturbative variants, and the optional comparison functional are placed in the Supplementary Material.

II. FINITE COERCIVE PACKAGES AND SCALE GEOMETRY

A. Geometric scales and marked packages

Following the Hamiltonian lattice-gauge viewpoint^{6–8}, a geometric scale is denoted

$$\beta = (n, \varepsilon, \theta, \Lambda, b),$$

where n records the finite region or lattice depth, ε the tube thickness, θ geometric regularity data, Λ a representation cutoff, and b boundary information. The geometric scales form a directed category $\mathfrak{R}_{\text{geom}}$. A marked analytic package over β is denoted $\tilde{\gamma}$; its underlying analytic scale is γ .

A refinement arrow may enlarge the region, thin the tubes, increase the representation cutoff, refine the physical channel, or alter the admissible boundary description. Write

$$\sigma = (\varepsilon, \theta, \Lambda, b), \quad \beta = (n, \sigma).$$

n :	discrete region or tensor depth.
$\sigma = (\varepsilon, \theta, \Lambda, b)$:	fixed finite stratum.
$\beta = (n, \sigma)$:	geometric demand scale.
$\tilde{\gamma}$:	marked package over a geometric scale.

γ :	underlying analytic package index.
$P_{\text{vac},\gamma} = P_{\text{cg},\gamma}$:	complete exact ground spectral projection.
$\kappa_*(\mathbf{q}) = \kappa_{\text{II},\text{prof}}$:	generic profilewise sufficient bound.
κ_{cof} :	Paper-I exact selected SU(3) finite-packet floor.
$\mathfrak{R}_{\text{geom}}$:	full multiparameter geometric scale category.
$\mathfrak{R}_{\text{depth}}(\sigma_0)$:	fixed-stratum depth subcategory.

The continuum-directed category orders larger regions and cutoffs, thinner tubes, and finer regularity and boundary demands as more refined.

Definition II.1 (Full-scale cofinality and fixed-stratum depth exhaustion). A sequence $\{\beta_m\} \subset \mathfrak{R}_{\text{geom}}$ is *full-scale cofinal* if for every geometric demand $\beta \in \mathfrak{R}_{\text{geom}}$ there is an m with $\beta \preceq \beta_m$. In particular, for the usual continuum demand system this requires region depth tending to infinity, tube thickness tending to zero, representation cutoff tending to infinity, and the declared regularity and boundary data becoming cofinal.

For a fixed finite stratum $\sigma_0 = (\varepsilon_0, \theta_0, \Lambda_0, b_0)$, let

$$\mathfrak{R}_{\text{depth}}(\sigma_0) = \{\beta_n^{\sigma_0} = (n, \sigma_0) : n \geq 1\}$$

be the full depth subcategory. A compatible family indexed by these objects is a *fixed-stratum depth exhaustion*; it is depth-cofinal in $\mathfrak{R}_{\text{depth}}(\sigma_0)$, not automatically in $\mathfrak{R}_{\text{geom}}$.

Proposition II.2 (A fixed-stratum depth path is not a full continuum cofinal path). *Assume $\mathfrak{R}_{\text{geom}}$ contains either a demand with thickness $\varepsilon' < \varepsilon_0$ or a demand with cutoff $\Lambda' > \Lambda_0$. Then the path $\beta_n^{\sigma_0} = (n, \sigma_0)$ is not cofinal in $\mathfrak{R}_{\text{geom}}$, although it is cofinal in $\mathfrak{R}_{\text{depth}}(\sigma_0)$.*

Proof. Changing only n never dominates a demand requiring a thinner tube or a larger representation cutoff than the fixed values in σ_0 . Cofinality in the depth subcategory is immediate because every finite depth is eventually exceeded. $\square$

Throughout the paper, unqualified “cofinal” refers to the full multiparameter category. The explicit tensor family of Section VII is called depth-exhaustive or depth-cofinal only after its fixed stratum is displayed.

B. The Paper-I package

For each admissible γ , the inherited finite data are

$$H_\gamma = H_{\text{el},\gamma} + H_{\text{mag},\gamma}, \quad E_{0,\gamma} = \inf \text{spec}(H_\gamma), \quad P_{\text{vac},\gamma} = \mathbf{1}_{\{E_{0,\gamma}\}}(H_\gamma),$$

and

$$\widehat{H}_\gamma = H_\gamma - E_{0,\gamma}I.$$

Let $\mathbf{F}_\gamma^W$ be the finite Wilson-channel map. Vacuum removal gives

$$T_\gamma = (I - P_{\text{vac},\gamma})\mathbf{F}_\gamma^W, \quad \mathbf{G}_\gamma = T_\gamma^* T_\gamma.$$

a. Complete-ground, Gram, and packet conventions. The symbol $P_{\text{vac},\gamma}$ denotes the complete exact spectral projection of H_γ at its ground energy. It neither selects a preferred vector nor imposes rank one in the abstract theory. At the sequence interface the same datum is denoted $P_{\text{cg},\gamma}$, so

$$P_{\text{cg},\gamma} = P_{\text{vac},\gamma}.$$

The active Paper-II symbol is retained to preserve the theorem and equation graph. Also, $\mathbf{G}_\gamma = T_\gamma^* T_\gamma$ is the Gram operator of the vacuum-removed Wilson channel; it is not the gauge group G .

The finite packet consumed from Paper I is

$$(H_\gamma, E_{0,\gamma}, P_{\text{cg},\gamma}, \mathbf{F}_\gamma^W, T_\gamma, \mathbf{G}_\gamma, \mathbf{F}_\gamma^{\text{phys}}, P_{\text{ch},\gamma}, P_{\text{res},\gamma}, P_{\text{UV},\gamma}, K_\gamma),$$

with $P_{\text{cg},\gamma} = P_{\text{vac},\gamma}$ in the active notation. The selected triality-sensitive $\text{SU}(3)$ packet of Paper I is the principal upstream realization of this interface. No incision, regional W^* -system, Friedrichs operator, Mosco limit, thin observable system, relativistic spectrum, or mass conclusion is imported into Paper II. On the good locus $\mathbf{G}_\gamma$ is invertible, and the normalized physical channel is

$$\mathbf{F}_\gamma^{\text{phys}} = T_\gamma \mathbf{G}_\gamma^{-1/2}.$$

Its range projection is

$$P_{\text{ch},\gamma} = \mathbf{F}_\gamma^{\text{phys}} (\mathbf{F}_\gamma^{\text{phys}})^*.$$

The physical magnetic recompression is

$$\mathbf{B}_{\text{mag},\gamma}^{\text{phys}} = (\mathbf{F}_\gamma^{\text{phys}})^* H_{\text{mag},\gamma} \mathbf{F}_\gamma^{\text{phys}}.$$

The remaining finite package includes a residual infrared projection $P_{\text{res},\gamma}$, an ultraviolet projection $P_{\text{UV},\gamma}$, and the exact combined mixed operator K_γ . No assumption is made that these blocks reduce the Hamiltonian separately; the mixed term is retained and estimated.

Definition II.3 (Finite good package). A marked package is good for a threshold profile

$$\mathbf{q} = (g_*, G^*, c_{\text{ch},*}, c_{\text{res},*}, c_{\text{UV},*}, \varepsilon_{\text{sr},*}, \varepsilon_{\text{mix},*}, \rho_*)$$

if it satisfies:

- (i) exact ground-sector isolation and the prescribed rank stratum;
- (ii) $g_* I \leq G_\gamma \leq G^* I$;
- (iii) $\mathbf{B}_{\text{mag},\gamma}^{\text{phys}} \geq c_{\text{ch},*} I$;
- (iv) residual and ultraviolet lower bounds $c_{\text{res},*}$ and $c_{\text{UV},*}$;
- (v) the declared channel–residual and ultraviolet–infrared mixed-block inequalities with subcriticality constants below one;
- (vi) positive vacuum overlap for all declared refinement arrows;
- (vii) the prescribed relative and higher-coherence data.

In a refinement system built from vacuum-line correspondences, the phrase “positive vacuum overlap” refers to a local Hilbert-space representative of $\mathbb{J}_a$. Since the vacuum restriction of $\mathbb{J}_a$ is unitary, the invariant overlap is one; a smaller number obtained from a raw cylindrical chart records only the transversality of that chart before polar physicalization.

The finite coercivity theorem of Paper I then gives

$$\widehat{H}_\gamma \geq \widehat{\kappa}(\mathbf{q})(I - P_{\text{vac},\gamma}), \quad (3)$$

where $\widehat{\kappa}(\mathbf{q}) > 0$ is an explicit function of the threshold profile.

For the selected Paper-I $\text{SU}(3)$ packet, the exact retained common floor is denoted κ_{cof} . That source constant is not identified with a generic Paper-II profile bound merely because both are positive.

C. Topological strata and continuity

All finite spaces are stratified by combinatorial type, representation content, cutoff rank, vacuum rank, and physical-channel rank. On a fixed stratum the Hamiltonian and transport matrices carry the operator-norm topology, while finite central bands carry the discrete topology. Spectral separation implies norm continuity of the Riesz ground projection; the Gram lower bound implies continuity of $\mathbf{G}^{-1/2}$; and positive overlap implies continuity of the polar vacuum comparison^{9–12}. These observations are the finite-dimensional basis for compact good loci. The Supplementary Material records the corresponding quantitative perturbation estimates and the necessary distinction between an isolated low-energy island and an exact higher-rank ground sector.

III. RELATIVE SCALE BORDISM AND ADMISSIBLE COHERENCE

Reader’s guide. Three kinds of information enter the refinement problem, and they must not be conflated. A refinement arrow changes the finite geometry; a relative filling class records whether two refinement histories are admissibly comparable; and a coefficient-line reassociation class records the residual higher gluing of vacuum transport. The first two control compatibility of finite packets. The third determines the global vacuum descent type. No transgression identifying these layers is assumed.

A. The refinement-to-admissible mapping fiber

Let $(A^\bullet, d_A)$ encode raw refinement defects, let $(B^\bullet, d_B)$ encode admissible correction data, and let

$$Q : A^\bullet \longrightarrow B^\bullet$$

be the comparison cochain map. Using the standard shifted mapping-cone convention¹³, we define the mapping fiber

$$\mathrm{Fib}(Q)^k = A^k \oplus B^{k-1}, \quad D_Q(\alpha, \beta) = (d_A \alpha, Q\alpha - d_B \beta).$$

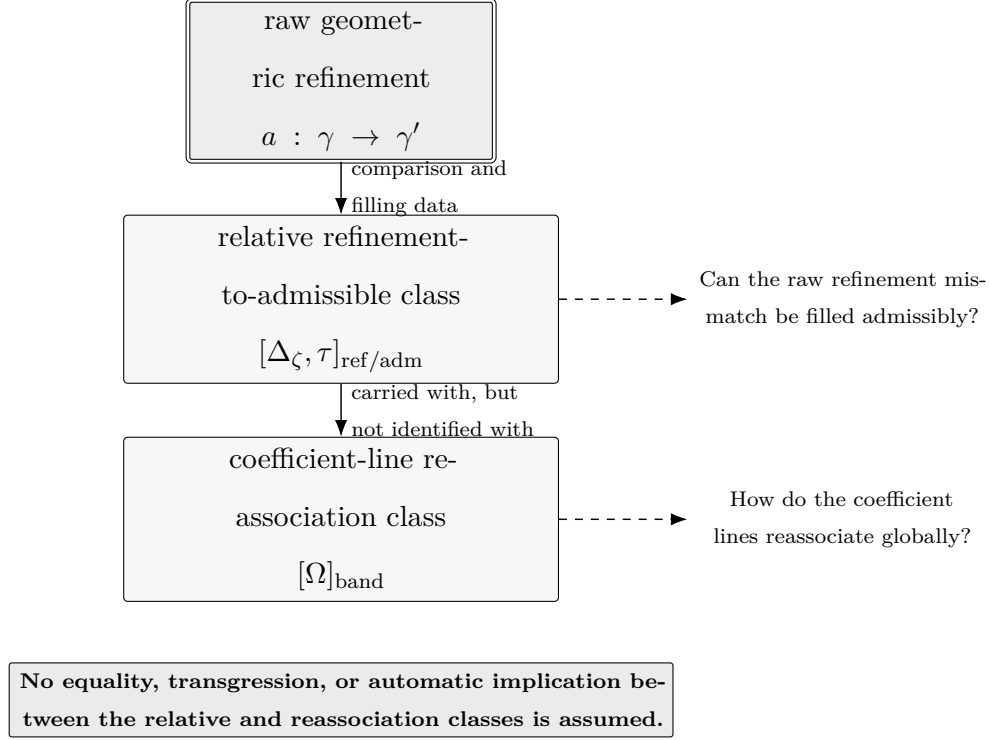


FIG. 1. The three-layer refinement datum. The relative class and the coefficient-line reassociation class belong to different complexes and answer different structural questions.

Since $Qd_A = d_BQ$, one has $D_Q^2 = 0$. A raw refinement mismatch $\Delta_\zeta \in A^k$ together with an admissible filling $\tau \in B^{k-1}$ defines the relative class

$$[\Delta_\zeta, \tau]_{\text{ref/adm}} \in H^k(\text{Fib}(Q)).$$

The cocycle equations are

$$d_A\Delta_\zeta = 0, \quad Q\Delta_\zeta = d_B\tau.$$

A representative change has the form

$$\Delta'_\zeta = \Delta_\zeta + d_A\lambda, \quad \tau' = \tau + Q\lambda - d_B\mu.$$

The class records whether raw refinement mismatch can be filled admissibly. It is not the associator class introduced below.

a. Cohomology-class firewall. The relative class $[\Delta_\zeta, \tau]_{\text{ref/adm}}$ and the coefficient-line band class $[\Omega]_{\text{band}}$ live in different complexes and answer different questions. The first measures admissible filling of refinement mismatch; the second measures coefficient-line reassociation. No equality, transgression, or automatic implication between them is assumed.

B. Scale bordisms and analytic comparison cells

A scale bordism $\mathfrak{B}_a : \tilde{\gamma} \rightarrow \tilde{\gamma}'$ over a refinement arrow a contains the geometric refinement, analytic comparison maps $\mathbb{T}_a^{(\nu)}$, the vacuum compression $\mathbb{C}_a^{\text{vac}}$, and the relative filling data.

For the recursive realization below, these data are induced from one phase-free vacuum-line correspondence

$$\mathbb{J}_a : \mathcal{H}_{s(a)} \hat{\otimes} L_a \longrightarrow \mathcal{H}_{t(a)}, \quad (4)$$

where L_a is the added vacuum refinement line, equipped with its center marking. A local unit section $e_a \in L_a$ produces the Hilbert-space representative

$$J_a^{e_a}(\psi) = \mathbb{J}_a(\psi \otimes e_a),$$

but $\mathbb{J}_a$, not the section, is the invariant bordism datum. The coefficient correspondence, vacuum comparison, package restriction, and higher comparison cells are all induced from $\mathbb{J}_a$. Thus different coordinate representatives used in a raw cylindrical chart are not distinct physical refinement arrows.

a. Physical-arrow firewall. The line-assisted correspondence $\mathbb{J}_a$ is the physical scale arrow together with its coefficient line. It is not assumed invertible. Reassociation of coefficient lines may repair higher coherence, but it does not turn one-sided physical transport into an equivalence.

For composable arrows

$$\tilde{\gamma}_0 \xrightarrow{a} \tilde{\gamma}_1 \xrightarrow{b} \tilde{\gamma}_2,$$

the actual pairwise composition of $\mathbb{J}_a$ and $\mathbb{J}_b$ induces a compositor

$$\chi_{b,a} : \mathfrak{B}_b \circ \mathfrak{B}_a \Longrightarrow \mathfrak{B}_{ba}.$$

In the separated-shell realization this compositor is the exact tensor-concatenation identity on the full finite Hilbert spaces. Local unit sections of the coefficient lines turn it into a unitary comparison defined up to one central phase.

The analytic non-flatness in channel ν is

$$\mathfrak{D}_{b,a}^{(\nu)} = U_{\chi_{b,a}}^{(\nu)} \mathbb{T}_b^{(\nu)} \mathbb{T}_a^{(\nu)} - \mathbb{T}_{ba}^{(\nu)}. \quad (5)$$

For three composable arrows, the two pastings of these *induced* pairwise compositors differ only through reassociation of the coefficient lines. If the fixed banded coefficient reassociator

is

$$\alpha_{c,b,a}^\Omega = \Omega(|c|, |b|, |a|)I,$$

then the higher comparison between the two parenthesized triple composites is the coefficient-line two-cell

$$\mathbf{a}_{c,b,a} = \mathbb{U}_{cba}^{\text{vac}} \circ (I \hat{\otimes} \alpha_{c,b,a}^\Omega) = \Omega(|c|, |b|, |a|) \mathbb{U}_{cba}^{\text{vac}}. \quad (6)$$

This is a two-cell between parenthesized coefficient-line correspondences, not a conjugation by an inverse of the full isometry and not an equality among globally evaluated ordinary lifts. Thus the associator is not an independent scalar assigned after the bordisms are constructed. It is the coefficient-line reassociation carried by their actual triple composition. The pentagon is the cocycle identity

$$\delta_{\text{cat}} \omega = 1. \quad (7)$$

The class $[\omega]_{\text{adm}}$ still depends on the specified coefficient band and arrow action. With ordinary Hilbert reassociation, $\Omega = 1$ and the phase-free correspondence is strict; a nontrivial class requires the nontrivial banded coefficient law and is not generated by endpoint matrix multiplication alone.

C. Vacuum compression and the coefficient band

The complete ground spaces are

$$\mathcal{V}_\gamma = \text{Ran } P_{\text{vac}, \gamma}.$$

When $\mathbf{C}_a^{\text{vac}}$ has positive lower singular value, its polar factor gives an isometric or, on a constant-rank branch, unitary comparison $\mathbf{U}_a^{\text{vac}}$. Vacuum compression sends admissible compositors and associators to vacuum-sector cells. On a fixed coefficient band this is a cochain map, hence induces maps on relative and associator cohomology. The ambient class may become trivial after vacuum compression; the vacuum type must therefore be computed from the compressed diagram, not inferred from the ambient marking.

For the principal twisted example the band is the $\mathbb{Z}_2$ center grade of the fundamental $\text{SU}(2)$ Wilson character. The normalized cocycle

$$\Omega_1(g, h, k) = (-1)^{\widetilde{g} \epsilon_2(h, k)} \quad (8)$$

represents the generator of $H^3(B\mathbb{Z}_2; U(1))^{4,5}$, with

$$\Omega_1(1, 1, 1) = -1.$$

The band is essential: the nerve of a bare directed depth category is contractible, so the pullback of Ω_1 can be trivialized by arbitrary arrow-dependent phases. Appendix F constructs the two center-twisted background fibers and proves that the marked elementary Wilson-shell operator is center odd. Center-background-natural marked equivalences preserve the grade. The higher-equivalence class is then restricted separately by shell locality, translation stationarity, and coefficient monoidality; Theorem F.7 proves that these gauges are exactly the pullbacks of band cochains. The reconstruction-relevant obstruction is the failure of strictification within that declared local higher-equivalence category.

IV. COEFFICIENT-LINE GOOD LOCI AND DIRECTED COFINAL SELECTION

A. Finite coefficient-line physical diagrams

Let D be a finite demand-diagram shape in the geometric scale category. A *finite coefficient-line physical diagram* over D is a pair

$$\mathfrak{D}_D^{\text{cl}} = (\mathfrak{A}_D, \mathfrak{C}_D^\Omega), \quad (9)$$

where the analytic part

$$\mathfrak{A}_D = \left(\{\mathfrak{P}_x\}_{x \in D_0}, \{\mathfrak{B}_a, \mathbb{J}_a\}_{a \in D_1}, \{\mathfrak{D}_{b,a}^{(\nu)}\}_{(b,a) \in D_2, \nu}, (\Delta_\zeta, \tau)_D \right)$$

contains the complete physicalized packages, canonical scale-bordism correspondences, analytic comparison defects, and relative mapping-fiber data. The coefficient-line part is

$$\mathfrak{C}_D^\Omega = \left(\{L_a\}_{a \in D_1}, \{m_{b,a} : L_b \hat{\otimes} L_a \rightarrow L_{ba}\}_{(b,a) \in D_2}, \alpha_D^\Omega, \{\mathbb{U}_a^{\text{vac}}\}_{a \in D_1}, | \cdot |_D, \mathcal{B}_D^\Omega, \mathbf{Triv}_{\Omega_D} \right). \quad (10)$$

Here α_D^Ω is the pulled-back coefficient reassociator, $\mathbb{U}_a^{\text{vac}} : \mathcal{V}_{s(a)} \hat{\otimes} L_a \rightarrow \mathcal{V}_{t(a)}$ is the vacuum restriction of $\mathbb{J}_a$, and $\mathbf{Triv}_{\Omega_D}$ is the lifting two-presheaf assigning to each rooted finite simplicial chart $E \subset N(D)$ the two-groupoid $\text{Triv}_{\Omega_D|E}(E)$ of coefficient-line trivializations. A point of the finite-diagram space contains this whole coefficient-line system; it does not contain a globally chosen ordinary unitary lift or a preferred local tensorator.

Two such diagrams represent the same point when they are related by objectwise package equivalences, center-background-natural bordism intertwiners, and a shell-local, translation-stationary, coefficient-monoidal equivalence of coefficient systems. Local κ - and h -transitions inside $\mathbf{Triv}_{\Omega_D}$ are morphisms of the lifting two-presheaf, not additional scalar compositor coordinates of the finite diagram.

B. Invariant finite-diagram curvature

The bounded good locus must be defined from invariant data, not from a chosen Hilbert lifting chart. We therefore set

$$\text{Curv}^{\text{raw}}(D) = \sum_{(b,a),\nu} w_\nu \|\mathfrak{D}_{b,a}^{(\nu)}\|^2 + w_{\text{rel}} \|(\Delta_\zeta, \tau)_D\|^2 + w_{\text{band}} \sum_{(c,b,a) \in D_3} \ell_Z(\Omega_D(|c|, |b|, |a|)). \quad (11)$$

No term involving a scalar $v_{b,a}$ or a chosen local tensorator appears in (11); those are chart data rather than invariants of the coefficient-line descent object. The band term is constant on a fixed marked type and may be omitted when only analytic compactness is being tested.

Let $\text{Vol}_{\text{WT}}(D) > 0$ be the geometric Wilson-tube volume and set

$$\mathcal{J}_D^{\text{raw}} = \frac{\text{Curv}^{\text{raw}}(D)}{\text{Vol}_{\text{WT}}(D)}.$$

Fix a quantitative profile $\mathbf{q}_*$, a curvature bound M , and a finite diagram shape D . Denote by

$$\mathfrak{X}_D^{\text{good}}(\mathbf{q}_*, M) \quad (12)$$

the set of band-local equivalence classes of finite coefficient-line physical diagrams (9) whose objects are good packages, whose canonical bordisms satisfy the declared overlap and relative filling conditions, whose coefficient-line data satisfy the module pentagon, and whose normalized invariant curvature is at most M . The topology in the selection theorem is a topology on these equivalence classes together with their functorially attached lifting two-presheaves; compactness is an explicit hypothesis, not a consequence of choosing local trivializations.

C. Restriction of the full descent datum

For an inclusion $i : D \hookrightarrow D'$, restriction acts simultaneously on all levels:

$$r_{D',D}(\mathfrak{A}_{D'}, \mathfrak{C}_{D'}^\Omega) = (\mathfrak{A}_{D'}|_D, \mathfrak{C}_{D'}^\Omega|_D). \quad (13)$$

In particular it restricts packages, bordisms, analytic defects, relative fillings, coefficient lines, multiplication maps, the grade, the pulled-back reassociator, and the vacuum correspondences. On lifting charts it is the cochain-restriction two-functor

$$\mathrm{Triv}_{\Omega_{D'}}(E') \longrightarrow \mathrm{Triv}_{\Omega_D}(E)$$

for every rooted chart $E \subset D$ and compatible root-preserving chart $E' \subset D'$. Thus the bonding maps used below are maps of the repaired coefficient-line objects, not restrictions of an obsolete scalar-cell pseudosection.

D. The directed compact inverse system

Let $\mathfrak{J}_{\mathrm{fin}}$ denote the partially ordered set of finite demand-diagram shapes, ordered by inclusion. It is understood to test the full scale category: every finite collection of objects and arrows of $\mathfrak{R}_{\mathrm{geom}}$ is contained in some member of $\mathfrak{J}_{\mathrm{fin}}$.

Hypothesis IV.1 (Directed finite-shape amalgamation and a seed). The index set $\mathfrak{J}_{\mathrm{fin}}$ is directed under common finite extension: every finite family $D_1, \dots, D_m$ is contained in some finite shape E . There are a shape $D_\bullet \in \mathfrak{J}_{\mathrm{fin}}$ and a point

$$x_\bullet \in \mathfrak{X}_{D_\bullet}^{\mathrm{good}}(\mathbf{q}_*, M).$$

Hypothesis IV.2 (Compact Hausdorff coefficient-line good loci). For every $D \in \mathfrak{J}_{\mathrm{fin}}$, the space $\mathfrak{X}_D^{\mathrm{good}}(\mathbf{q}_*, M)$ of equivalence classes defined in (12) is compact Hausdorff.

Hypothesis IV.3 (Coherent continuous restriction). For every inclusion $D \subset D'$, the full restriction (13) induces a continuous map

$$r_{D',D} : \mathfrak{X}_{D'}^{\mathrm{good}}(\mathbf{q}_*, M) \longrightarrow \mathfrak{X}_D^{\mathrm{good}}(\mathbf{q}_*, M),$$

and these maps satisfy

$$r_{D,D} = \mathrm{id}, \quad r_{D'',D} = r_{D',D} \circ r_{D'',D'} \quad (D \subset D' \subset D'').$$

The identities include the restriction two-functors of the lifting presheaves.

Hypothesis IV.4 (Universal finite extension feasibility). For every inclusion $D \subset D'$, the full coefficient-line restriction map $r_{D',D}$ is surjective. Equivalently, every good finite analytic

and higher-descent datum over D extends to D' without weakening $\mathbf{q}_*$, changing the declared rank stratum or marked band class, or replacing the coefficient-line law by an unrelated scalar-cell model.

The final hypothesis contains the model-specific analytic and higher-descent difficulty. Compactness alone does not prove extension feasibility. Directed amalgamation is also essential: one seed can populate every finite shape only because the seed shape and any prescribed finite demand admit a common finite extension.

Interpretive lead-in. The finite good loci and restriction maps now form a directed system indexed by finite demand shapes. The next theorem addresses the genuinely global selection question in the abstract part of the paper: whether all finite compatibility requirements can be satisfied simultaneously. Its conclusion is conditional on universal finite extension and compact coherent restriction; it is not inferred from the explicit depth family constructed later.

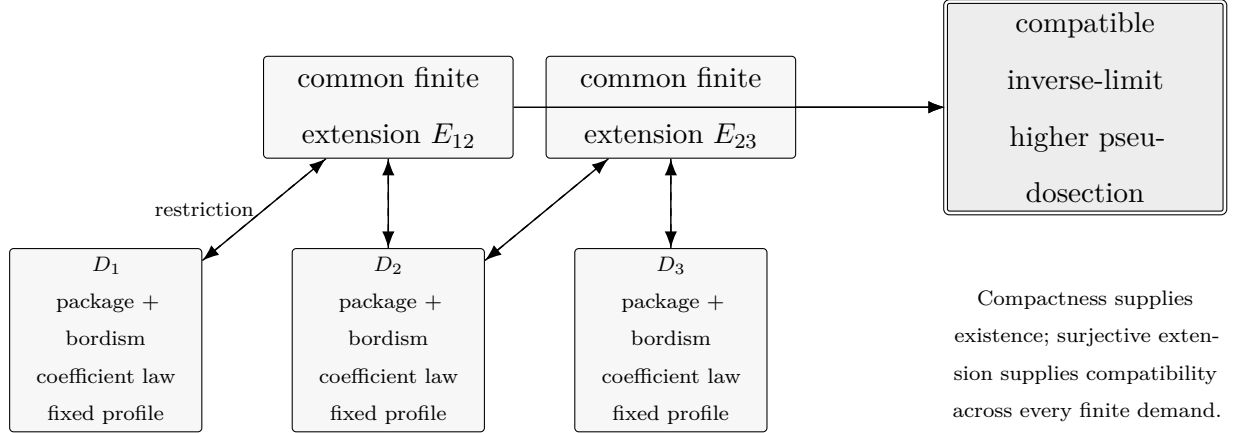


FIG. 2. Directed finite-shape selection. Dashed arrows indicate extension or amalgamation; solid arrows indicate restriction. The inverse-limit point retains the complete finite analytic and higher-descent datum, not only a sequence of coercive constants.

Theorem IV.5 (Directed compact inverse-limit selection criterion). *Assume Hypotheses IV.1–IV.4. Then every coefficient-line good-diagram space is nonempty and*

$$\varprojlim_{D \in \mathcal{J}_{\text{fin}}} \mathfrak{X}_D^{\text{good}}(\mathbf{q}_*, M) \neq \emptyset. \quad (14)$$

Every point $(x_D)_{D \in \mathcal{J}_{\text{fin}}}$ of this inverse limit is a compatible family of finite physical packages, canonical bordisms, coefficient-line systems, and lifting two-presheaves. Since $\mathcal{J}_{\text{fin}}$ tests all

finite parts of $\mathfrak{R}_{\text{geom}}$, such a family determines a full multiparameter cofinal physical higher pseudosection with the fixed profile $\mathbf{q}_*$.

Proof. First fix $D \in \mathfrak{J}_{\text{fin}}$. Directed amalgamation gives a finite shape E containing both $D_\bullet$ and D . Surjectivity of $r_{E,D_\bullet}$ provides x_E with $r_{E,D_\bullet}(x_E) = x_\bullet$, and then $r_{E,D}(x_E) \in \mathfrak{X}_D^{\text{good}}(\mathbf{q}_*, M)$. Thus every factor is nonempty.

Consider the compact product

$$K = \prod_{D \in \mathfrak{J}_{\text{fin}}} \mathfrak{X}_D^{\text{good}}(\mathbf{q}_*, M),$$

which is compact by Tychonoff's theorem. For each inclusion $D \subset D'$, let

$$C_{D',D} = \{(x_F)_F \in K : r_{D',D}(x_{D'}) = x_D\}.$$

The set $C_{D',D}$ is closed because it is the inverse image of the diagonal in the Hausdorff space $\mathfrak{X}_D^{\text{good}} \times \mathfrak{X}_D^{\text{good}}$ under the continuous map $(x_F)_F \mapsto (r_{D',D}(x_{D'}), x_D)$.

The family $\{C_{D',D}\}$ has the finite-intersection property. For a finite collection of bonding equations choose one finite shape E containing every shape that occurs. Choose x_E and set $x_D = r_{E,D}(x_E)$ for every occurring D . The full restriction identities, including those of the coefficient-line and lifting data, make all selected equations hold. The remaining coordinates may be chosen arbitrarily because every factor is nonempty. Compactness of K gives

$$\bigcap_{D \subset D'} C_{D',D} \neq \emptyset,$$

which is precisely the inverse limit in (14). □

Remark IV.6 (No hidden countability or scalar-cell hypothesis). The proof does not choose a countable cofinal chain, and it does not select scalar compositors or globally evaluated ordinary vacuum lifts. Its factors are the repaired coefficient-line good loci and its bonding maps restrict the entire higher-descent datum.

Remark IV.7. The theorem is a criterion, not a microscopic Yang–Mills extension theorem. Its universal extension hypothesis concerns every equivalence class in each full coefficient-line good-diagram space over the full multiparameter scale category. Section VII instead constructs one depth-cofinal family inside a fixed finite stratum. It neither proves full-scale cofinality nor surjectivity of the restriction maps on the full good-diagram spaces. Fixed-stratum depth exhaustion is a theorem in its own habitat. It becomes something else only after the remaining scale variables have been controlled.

E. Conditional common refinement

Two full-scale higher pseudosections are physically cofinally equivalent if every pair of finite restrictions admits, after enlargement of the finite demand shape, a common good extension whose analytic bordisms and coefficient-line descent systems are related by a band-local equivalence with positive vacuum overlap. This requires a separate good-amalgamation hypothesis. Even when such an amalgam exists, it need not preserve an optional asymptotic comparison cost: physical cofinal equivalence and cost neutrality are different conditions.

V. PROFILEWISE UNIFORM COERCIVITY AND THE COFINAL CONSEQUENCE

Let $c_{\text{ch},*}, c_{\text{res},*}, c_{\text{UV},*} > 0$ be uniform channel, residual, and ultraviolet lower bounds on a compatible family of finite packages. Let $\rho_{\text{sr}} < 1$ control the channel–residual mixed term and let $\rho < 1$ control the exact ultraviolet–infrared mixed block. Define

$$c_{\text{IR},*} = \frac{1 - \rho_{\text{sr}}}{2} \min\{c_{\text{ch},*}, c_{\text{res},*}\}, \quad (15)$$

and

$$\kappa_* = \frac{1 - \rho}{2} \min\{c_{\text{UV},*}, c_{\text{IR},*}\}. \quad (16)$$

a. Coercive-constant firewall. The displayed κ_* is profile dependent; when that dependence must be visible we write $\kappa_*(\mathbf{q})$ and set

$$\kappa_{\text{II},\text{prof}} := \kappa_*(\mathbf{q}).$$

It is not a synonym for the exact selected Paper-I SU(3) constant κ_{cof} . The values 0.24975, 3.093219423836..., and 3.092840380512857... belong only to the explicit fixed-stratum SU(2) realization.

Theorem V.1 (Uniform coercivity from one fixed profile). *Let $\mathfrak{S}_{\text{phys}}$ be any compatible family of finite physicalized packages carrying one fixed threshold profile, whether or not its index set is full-scale cofinal. If that profile satisfies the strict inequalities entering (15) and (16), then $\kappa_* > 0$ and every package in the family obeys*

$$\boxed{H_\gamma \geq E_{0,\gamma} I + \kappa_*(I - P_{\text{vac},\gamma}).} \quad (17)$$

Proof. The channel and residual diagonal bounds, together with the channel–residual mixed estimate, give

$$\widehat{H}_\gamma \geq c_{\text{IR},*} P_{\text{IR},\gamma}$$

on the complete infrared complement of the ground sector. Combining this with the ultraviolet lower bound and the exact ultraviolet–infrared mixed estimate yields (16) by the elementary two-block inequality

$$2|\langle x, Ky \rangle| \leq \rho(c_{\text{UV},*}\|x\|^2 + c_{\text{IR},*}\|y\|^2).$$

No block-reducing hypothesis is used. The full mixed operator is estimated directly. $\square$

What has been established. The inverse-limit theorem and the profile theorem perform different jobs. Compactness selects an entire compatible diagram, whereas the retained profile transfers one quantitative coercive floor to every object of any compatible family carrying it. The explicit $SU(2)$ construction below verifies this quantitative mechanism only along tensor depth at one fixed stratum.

Corollary V.2 (Full-scale and fixed-stratum consequences). *Theorem V.1 has two distinct applications:*

- (i) *a full-scale cofinal pseudosection obtained from Theorem IV.5 has one uniform cofinal coercive constant;*
- (ii) *the explicit family of Section VII has one uniform constant along its fixed-stratum tensor-depth exhaustion.*

The second statement is not a continuum-cofinal conclusion and requires no universal finite-extension hypothesis.

A. Vacuum overlap and exact ground sectors

For every selected refinement arrow a , assume

$$(C_a^{\text{vac}})^* C_a^{\text{vac}} \geq \nu_*^2 P_{\text{vac},s(a)}, \quad \nu_* > 0. \tag{18}$$

Then the polar vacuum comparison is well defined and varies continuously on the fixed rank stratum. When higher-rank ground sectors are allowed, the selected projection must be

the complete exact ground projection. A stable low-energy island of the same rank is not sufficient unless a no-splitting theorem identifies the island Hamiltonian with a scalar on that subspace. In the integrated recursive realization below, the ground sector is rank one, so this issue does not arise.

B. What the theorem does and does not say

Equation (17) is a profilewise statement. When the compatible family is full-scale cofinal, it becomes uniform cofinal coercivity. On the explicit fixed-stratum family, it is only uniform depth coercivity. Neither version constructs a limit operator, proves convergence of quadratic forms, or supplies the no-leakage theorem required for a continuum mass gap.

VI. COEFFICIENT-LINE VACUUM TYPE AND FIXED-TYPE SELECTION

Reader's guide. The analytic packet alone does not determine how vacuum transport reassociates globally. We therefore fix the coefficient band and treat the reassociation law as part of the selected datum. This creates two logically separate questions: whether a branch exists with the prescribed finite analytic profile, and whether its coefficient-line descent is strict or carries a nontrivial higher class. The fixed-type inverse system keeps both questions visible.

A. Coherent vacuum descent diagrams

Vacuum type is not attached to an isolated Hamiltonian package and is not encoded by a tuple of globally chosen ordinary lifts. It is attached to the coefficient-line vacuum part of a finite physical diagram:

$$\mathfrak{D}_D^{\text{vac}} = \left(\{\mathcal{V}_x\}, \{L_a\}, \{m_{b,a}\}, \alpha_D^\Omega, \{\mathbb{U}_a^{\text{vac}}\}, \mathbf{Triv}_{\Omega_D}, |\cdot|_D, \mathcal{B}_D^\Omega \right). \quad (19)$$

The lifting two-presheaf $\mathbf{Triv}_{\Omega_D}$ records the local Hilbert realizations and their one- and two-transitions; no object Λ of a local lifting chart is part of the invariant vacuum type. The induced type is

$$\text{Type}_{\text{vac}}(\mathfrak{D}_D) = \left(\mathfrak{t}_{\text{sc}}^{\text{vac}}(\mathfrak{D}_D), [\Omega_D]_{\text{band}} \right). \quad (20)$$

Restriction of the full coefficient-line diagram preserves this type.

B. Strict and twisted cyclic bands

For a fixed cyclic coefficient band, the strict branch has trivial coefficient reassociator. A twisted branch has the same canonical coefficient-line arrows but a nonzero banded reassociator class. In the $\mathbb{Z}_2$ case the generator Ω_1 in (8) cannot be removed by a normalized band tensorator. Indeed, every normalized 2-cochain λ on $\mathbb{Z}_2$ satisfies

$$(\delta\lambda)(1, 1, 1) = 1,$$

whereas $\Omega_1(1, 1, 1) = -1$.

Theorem VI.1 (Complete banded strict/twisted alternative). *Let $\mathfrak{D}_D^{\text{vac}}$ be a canonical constant-rank coefficient-line vacuum diagram obtained from the scale-bordism correspondences of Appendix O, with fixed normalized $\mathbb{Z}_2$ coefficient band and vanishing relative scale class. Then the following are equivalent:*

- (i) $[\Omega_D]_{\text{band}} = 0$;
- (ii) *the coefficient-line system admits a center-background-natural, shell-local strong monoidal trivialization and hence an ordinary Hilbert descent;*
- (iii) *it is band-preservingly equivalent to the explicit strict control generated by the maps $J_{n;k}^e$ of Appendix S.*

If instead $[\Omega_D]_{\text{band}} = [\Omega_1]_{\text{band}} \neq 0$, none of these conditions holds. The invariant vacuum datum is then the coefficient-line-valued higher descent object of Appendix I: it is locally Hilbert realizable on finite rooted depth charts but has no band-local global section.

Proof. The strictification argument is Theorem H.5. A vanishing band class removes the coefficient reassociator by a normalized band tensorator. The remaining chartwise central comparison is an ordinary normalized 2-cocycle on the rooted batch-refinement category and is contracted by an arrow 1-cochain; the resulting strict realization differs from $J_{n;k}^e$ by an objectwise scalar gauge. Conversely, a band-local strong monoidal trivialization has trivial reassociator and therefore zero band class. The nonzero case is Theorem I.11. $\square$

Remark VI.2 (Why the band is part of the theorem). The unmarked depth category has contractible nerve. Arbitrary arrow-dependent tensorators may trivialize the pulled-back cocycle on a local finite chart. The nontrivial statement is the obstruction to one center-natural, shell-local, stationary, coefficient-monoidal trivialization of the repeated-shell coefficient law. These strict and twisted alternatives classify Paper-II coefficient-band globalizations; they are not the Sector-I/Sector-II/Sector-III terminology of the later spacetime realization. Coefficient-line equivalence is not physical scale equivalence.

C. Center-background-natural admissibility

Let $Z\chi_j = (-1)^{2j}\chi_j$ on the retained character space and let X be the compressed fundamental Wilson multiplier. Then

$$ZXZ^{-1} = -X.$$

The seed therefore has two center-twisted background fibers

$$H^{[\epsilon]} = H_{\text{el}} + I - \frac{(-1)^\epsilon}{2}X, \quad ZH^{[\epsilon]}Z^{-1} = H^{[\epsilon+1]}.$$

Appendix F proves that the complete physicalized package and the canonical coefficient-line scale bordism extend naturally over these fibers.

Theorem VI.3 (Physical invariance of the Wilson-shell band). *Within the category of center-background-natural marked Wilson-shell systems, an elementary refinement has invariant grade one and a batch refinement has grade $|a_{n;k}| = \bar{k}$. Center-natural marked equivalences preserve this grade. If coefficient-system tensorators are additionally required to be shell local, translation stationary, and coefficient monoidal, they factor uniquely through normalized cochains on the fixed band. The rooted arrow-dependent contraction is background natural but fails shell stationarity and is therefore a local lifting-chart object, not an admissible global equivalence.*

The theorem separates two inputs: center covariance fixes the physical grade, while shell locality and stationarity fix which monoidal transformations count as local equivalences of the repeated-shell coefficient law. It does not assert that the finite Hamiltonian selects Ω_1 over Ω_0 . The two remain distinct coefficient-line globalizations in the declared category.

D. Fixed-type cofinal higher pseudosections

For a finite diagram shape D and a banded type $\mathfrak{v}$, let

$$\mathfrak{X}_D^{\text{good}, \mathfrak{v}} = \{[\mathfrak{D}_D^{\text{cl}}] \in \mathfrak{X}_D^{\text{good}} : \text{Type}_{\text{vac}}(\mathfrak{D}_D^{\text{vac}}) = \mathfrak{v}|_D\}. \quad (21)$$

The brackets emphasize that the point is a band-local equivalence class of the full coefficient-line diagram; it is not a selected local tensorator.

Theorem VI.4 (Fixed-type cofinal selection). *Assume that the finite demand shapes form the directed system of Hypothesis IV.1; that one coefficient-line fixed-type seed exists; that every $\mathfrak{X}_D^{\text{good}, \mathfrak{v}}$ is compact Hausdorff; and that the full restriction maps are coherent, continuous, and surjective. Then*

$$\varprojlim_{D \in \mathcal{I}_{\text{fin}}} \mathfrak{X}_D^{\text{good}, \mathfrak{v}} \neq \emptyset.$$

Every point of this inverse limit is a full-scale cofinal physical higher pseudosection of type $\mathfrak{v}$, and the uniform coercive estimate (17) holds along its analytic packages.

Proof. Apply Theorem IV.5 to the fixed-type inverse subsystem. Directed amalgamation and the fixed-type seed imply nonemptiness of every factor by common extension; compact Hausdorffness and coherent continuous surjective restriction give the finite-intersection argument. The restriction theorem of Appendix Q acts on the coefficient lines, reassociator, vacuum correspondences, and lifting two-presheaf, so type jumping is impossible. $\square$

The theorem does not preselect a type objectwise and does not introduce a second scalar-cell model. The type is the compatible marked class of the coefficient-line higher descent system itself.

VII. AN END-TO-END FIXED-STRATUM STRICT/TWISTED DEPTH PAIR

Fix a finite scale stratum

$$\sigma_0 = (\varepsilon_0, \theta_0, \Lambda_0, b_0),$$

where Λ_0 is the four-character retained cutoff and the tube, regularity, and boundary data are held fixed. This section constructs one explicit calibrated $\text{SU}(2)$ family that is depth-cofinal

in $\mathfrak{R}_{\text{depth}}(\sigma_0)$. By Proposition II.2, it is not a full multiparameter cofinal branch. The construction is independent of the universal extension hypothesis in Theorem IV.5. Exact matrices and human-readable interval certificates are supplied in the Supplementary Material.

a. Role of the explicit realization. The abstract selection criterion is group agnostic at the level of a finite good package. The selected triality-sensitive SU(3) packet imported from Paper I is the principal upstream branch for the later sequence. The construction in this section is instead an explicit SU(2) fixed-stratum validation of the recursive analytic, scale-bordism, and coefficient-line mechanisms. It neither replaces the selected SU(3) packet nor transfers its numerical constants to that packet.

A. One-shell seed

Let $\mathcal{K}_2 \cong \mathbb{C}^4$ be the four-character gauge-invariant seed space. Its finite Hamiltonian H_2 has a rank-one complete ground projection $P_{\text{vac}}^{(2)}$. The physicalized Wilson channel satisfies

$$G_2 = (\mathbf{F}_2^W)^*(I - P_{\text{vac}}^{(2)})\mathbf{F}_2^W = 0.99999999715420\dots,$$

and the physical magnetic recompression is

$$(\mathbf{F}_2^{\text{phys}})^* H_{\text{mag}}^{(2)} \mathbf{F}_2^{\text{phys}} = 1.059584273164\dots$$

The exact mixed block obeys

$$\|K_2\| = 0.051998558862\dots,$$

and a refinement-ready finite coercive certificate is

$$\widehat{H}_2 \geq 3.093219423836\dots (I - P_{\text{vac}}^{(2)}). \quad (22)$$

B. Recursive tensor family and global physicalization

At depth n , set

$$\mathcal{H}_n = \mathcal{K}_2^{\otimes n}, \quad H_n = \sum_{m=1}^n I^{\otimes(m-1)} \otimes H_2 \otimes I^{\otimes(n-m)},$$

with ground energy $E_{0,n} = nE_{0,2}$ and complete ground projection

$$P_{\text{vac},n} = (P_{\text{vac}}^{(2)})^{\otimes n}. \quad (23)$$

The coefficient space is the orthogonal direct sum of the n vacuum-anchored local Wilson lines. The global Wilson map inserts the local channel in one shell and anchors every other shell in its vacuum line. Vacuum removal makes the shell channels orthogonal. Consequently,

$$\mathbf{G}_n = G_2 I \quad (24)$$

exactly for every n . The physical magnetic recompression is diagonal on the coefficient sum:

$$\mathbf{B}_{\text{mag},n}^{\text{phys}} = (1.059584273164 \dots + (n-1)0.838794220733 \dots) I. \quad (25)$$

The complete mixed block is the orthogonal sum of the local mixed blocks, and hence

$$\|K_n\| = \|K_2\|. \quad (26)$$

Multi-excitation sectors contribute only positive ultraviolet energy.

C. One canonical scale-bordism correspondence

Let $\mathcal{V} = \text{Ran } P_{\text{vac}}^{(2)}$. The declared elementary refinement is not a chosen map $\mathcal{H}_n \rightarrow \mathcal{H}_{n+1}$, but the phase-free correspondence

$$\mathbb{J}_{n+1,n} : \mathcal{H}_n \hat{\otimes} \mathcal{V} \longrightarrow \mathcal{H}_{n+1}, \quad (27)$$

given by canonical tensor reassociation. It intertwines the shifted Hamiltonian, complete vacuum projection, Wilson map, Gram operator, physicalized channel, magnetic recompression, and mixed block as proved in Theorem C.6. Its vacuum restriction is unitary, so the physical overlap constant is $\nu_* = 1$.

A local unit section $e \in \mathcal{V}$ gives

$$J_{n+1,n}^e(\psi) = \mathbb{J}_{n+1,n}(\psi \otimes e),$$

which is the representative used in both the strict and twisted lifts. The older map $\psi \mapsto \psi \otimes \chi_0$ is only a raw cylindrical chart; the polar factor of its vacuum compression is $J_{n+1,n}^{e_0}$, where $e_0 = P\chi_0/\|P\chi_0\|$. Hence package restriction, vacuum transport, overlap, and higher descent are induced by one scale-bordism datum.

Theorem VII.1 (End-to-end physicalized fixed-stratum depth family). *The family $\{\mathcal{H}_n, H_n\}_{n \geq 1}$ carries the complete physicalized package at every depth with one common profile*

$$\begin{aligned} g_* &= 0.999999999715, & G^* &= 1, & c_{\text{ch},*} &= 1, & c_{\text{res},*} &= 1, \\ c_{\text{UV},*} &= 6, & \varepsilon_{\text{mix},*} &= 0.052, & \rho_* &= 0.001, & \nu_* &= 1. \end{aligned}$$

The Paper-I package formula gives

$$\widehat{H}_n \geq 0.24975(I - P_{\text{vac},n}), \quad (28)$$

while the direct tensor-sum estimate gives

$$\widehat{H}_n \geq 3.093219423836 \dots (I - P_{\text{vac},n}). \quad (29)$$

The displayed depth maps give a specified compatible extension from the canonical depth- n package to the canonical depth- $n + 1$ package without weakening the profile. This proves existence of the explicit depth exhaustion in $\mathfrak{R}_{\text{depth}}(\sigma_0)$. It does not vary $\varepsilon_0, \theta_0, \Lambda_0$, or b_0 , does not give a full-scale cofinal sequence, and does not assert that every arbitrary element of a full good-diagram space extends.

Proof. Equations (23)–(26) show that every physicalized block is either a direct sum of the one-shell block or receives an additional positive tensor-sum contribution. Thus the displayed thresholds are independent of depth. The single correspondence (27) transports the complete package and has exact vacuum overlap one. The package estimate follows directly from Theorem V.1 and Corollary V.2, applied to this explicitly constructed fixed-stratum family. The stronger bound follows because every vector orthogonal to the product ground line excites at least one shell and costs at least the one-shell certificate (22). $\square$

D. Exact cofinality status and relation to the abstract criterion

Let $\mathfrak{D}_n^{\text{rec}}$ denote the canonical coherent diagram through depth n constructed above and let $\beta_n^{\sigma_0} = (n, \sigma_0)$. The sequence

$$\mathfrak{D}_1^{\text{rec}} \subset \mathfrak{D}_2^{\text{rec}} \subset \dots$$

is a compatible physical pseudosection over $\mathfrak{R}_{\text{depth}}(\sigma_0)$ and is cofinal only in that depth subcategory. It may be encoded as the singleton inverse system

$$\mathfrak{X}_n^{\text{rec}} = \{\mathfrak{D}_n^{\text{rec}}\},$$

whose bonding maps are bijective. This bookkeeping observation neither verifies Hypothesis IV.4 nor supplies the missing motions $\varepsilon \rightarrow 0$, $\Lambda \rightarrow \infty$, and cofinal refinement of θ and b . A full-scale cofinal branch must be obtained separately from the abstract criterion or by a new multiparameter construction that extends this depth model while preserving its profile and higher coefficient law.

E. Strict control

Choose temporarily a unit representative e of the one-shell vacuum line. The strict comparison is the local representative of the canonical bordism,

$$J_{n;k}^e(\xi) = \mathbb{J}_{n;k}(\xi \otimes e^{\otimes k}).$$

Then

$$J_{n+k;\ell}^e J_{n;k}^e = J_{n;k+\ell}^e. \quad (30)$$

If $e' = ze$, the corresponding comparisons satisfy

$$J_{n;k}^{e'} = W_{n+k}(z) J_{n;k}^e W_n(z)^{-1}, \quad W_n(z) = z^n I,$$

so the choice of representative is only gauge. The strict diagram has type $(0, 0)$.

F. Twisted center-banded composition induced from the same bordisms

For an arrow $a_{n;k}$, the coefficient line of the canonical correspondence is $L_a = \mathcal{V}^{\otimes k}$. Its center grade $|a| = \bar{k}$ is the additive parity of the marked fundamental Wilson multipliers on the added shells. The exact identity $ZXZ^{-1} = -X$, together with center-background naturality of the complete package and of $\mathbb{J}$, proves that this marking is invariant under center-natural marked equivalence. Theorem VI.3 separately characterizes physically local higher gauges by shell stationarity and coefficient monoidality. Pairwise composition of the actual correspondences is exact:

$$\mathbb{J}_b(\mathbb{J}_a \hat{\otimes} I_{L_b}) = \mathbb{J}_{ba}(I \hat{\otimes} m_{b,a}).$$

Hence the pairwise compositor is induced by $\mathbb{J}$, not separately postulated.

Equip the coefficient lines with the center-banded reassociator

$$\alpha_{c,b,a}^{\Omega_1} = \Omega_1(|c|, |b|, |a|)I.$$

The two parenthesized triple composites are related by the coefficient-line two-cell

$$\mathbf{a}_{c,b,a}^{\text{vac}} = \mathbb{U}_{cba}^{\text{vac}} \circ (I \hat{\otimes} \alpha_{c,b,a}^{\Omega_1}) = \Omega_1(|c|, |b|, |a|) \mathbb{U}_{cba}^{\text{vac}}. \quad (31)$$

For three elementary insertions this two-cell is $-I$. It is not an equality among globally chosen ordinary unitary lifts. On every finite depth simplex the rooted arrow-dependent tensorator Λ_N with $\delta\Lambda_N = \hat{\Omega}_1$ gives an ordinary local Hilbert realization; changes of Λ_N give the higher gluing transitions. The rooted tensorator is background natural but fails shell stationarity. By Theorem F.7, no shell-local stationary tensorator exists because every such tensorator would depend only on the fixed grades. The twisted diagram has type $(0, [\Omega_1]_{\text{band}})$.

With the ordinary coefficient reassociator $\Omega_0 = 1$, the same correspondence gives the strict branch. The center-background completion fixes the physical grade, while shell locality and stationarity exclude the depth-dependent rooted contraction as a global higher equivalence. The finite matrices nevertheless do not by themselves select Ω_1 over Ω_0 ; the two are distinct global reassociation laws within the same physically marked center-equivariant refinement category.

Interpretive lead-in. The strict and twisted constructions now share the same finite Hamiltonians, physical channels, comparison maps, defect norms, and coercive constants. The next theorem isolates the remaining difference: not a finite spectral difference, but the global reassociation law of the coefficient lines. This is the precise sense in which the finite analytic shadow underdetermines the higher vacuum descent type.

Theorem VII.2 (Paired fixed-stratum depth globalizations). *The same recursively physicalized family admits two globalizations that are inequivalent under center-natural, shell-local stationary higher equivalence:*

$$\mathfrak{S}^{\text{str}} \text{ of type } (0, 0), \quad \mathfrak{S}^{\text{tw}} \text{ of type } (0, [\Omega_1]_{\text{band}}).$$

They have identical finite Hamiltonians, complete ground projections, physicalized packages, canonical projective comparisons, vacuum-overlap singular values, analytic refinement-defect norms, and the same uniform coercive bounds (28) and (29).

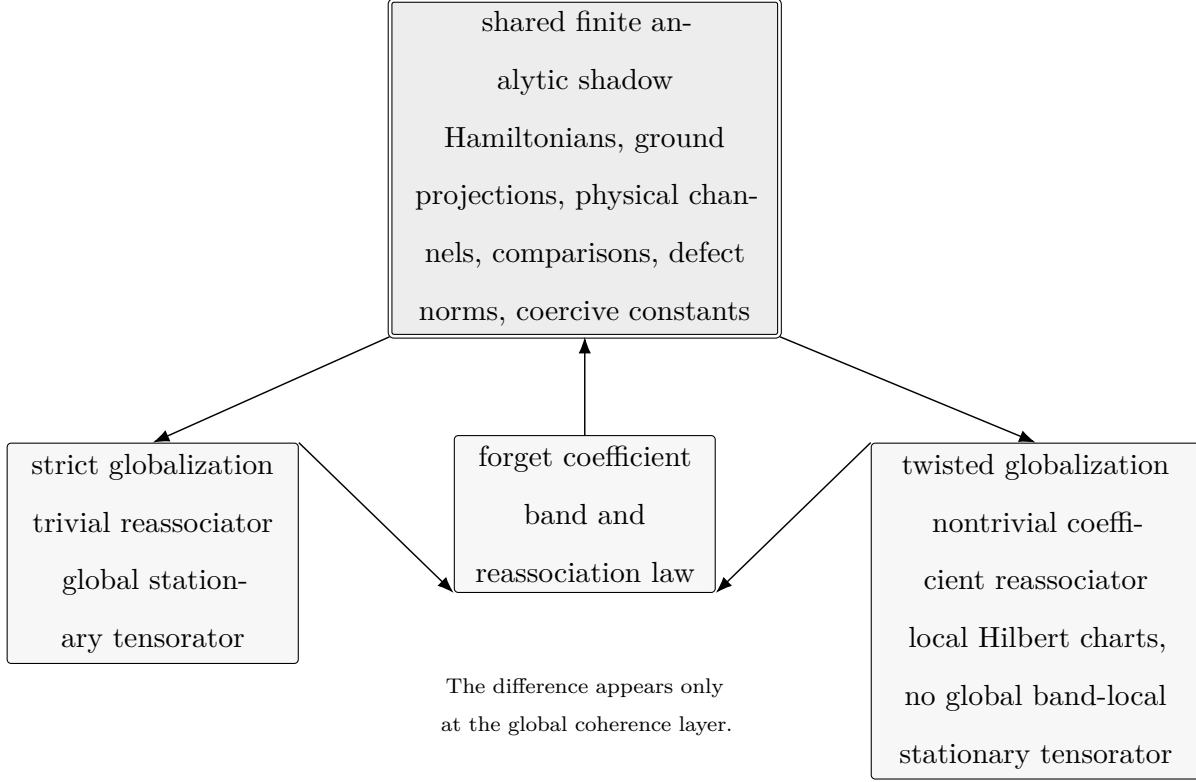


FIG. 3. The same finite analytic shadow supports two inequivalent global descent types. Any discriminator factoring only through the shared finite data necessarily assigns the same value to both branches.

Proof. The two branches use the same center-background-natural canonical pairwise bordisms and differ only in the reassociation law of their physically marked center-graded coefficient lines; the corresponding vacuum associators are induced from that law by the module pentagon. Multiplication of a lift by a central phase leaves every objectwise operator unchanged. Coefficient-space phase changes conjugate the Gram operator and recompressed magnetic form unitarily, leave the channel projection fixed, and multiply analytic defects only by unit-modulus scalars. Hence all singular values, spectra, norms, and coercive constants agree. The banded types differ, so a center-band-preserving equivalence cannot exist. $\square$

G. Coefficient-line higher descent and reconstruction obstruction

The twisted vacuum cannot be encoded by one global family of ordinary unitary lifts with scalar compositors: ordinary operator associativity would force the scalar associator to be

trivial. The invariant arrows remain the coefficient-line correspondences

$$\mathbb{U}_a^{\text{vac}} : \mathcal{V}_{s(a)} \hat{\otimes} L_a \longrightarrow \mathcal{V}_{t(a)}.$$

Theorem VII.3 (Twisted higher vacuum reconstruction). *For the recursive twisted depth family, every finite depth simplex has a nonempty two-groupoid of arrow-dependent tensorators Λ satisfying $\delta\Lambda = \hat{\Omega}_1$. Each such tensorator gives an ordinary Hilbert realization of the coefficient-line vacuum transport, and one- and zero-cochains give the coherent transitions and modifications between realizations. These data form the fibred higher descent object*

$$\mathbf{Vac}_{\text{Hilb}}^{\Omega_1}.$$

There is no shell-local stationary global tensorator. By Theorem F.7, any such tensorator would factor through the fixed center grades and require a normalized $\lambda : \mathbb{Z}_2^2 \rightarrow U(1)$ with $\delta\lambda = \Omega_1$. The strict control $\Omega_0 = 1$ has such a global section.

Proof. The explicit rooted tensorator is

$$\Lambda_N(i, j, k) = 1 \quad (i = 0), \quad \Lambda_N(i, j, k) = \hat{\Omega}_N(0, i, j, k) \quad (i > 0),$$

and the cocycle identity gives $\delta\Lambda_N = \hat{\Omega}_N$. The lifting two-groupoid and its Hilbert realization are constructed in Appendix I. A grade-only section would satisfy $\delta\lambda = \Omega_1$, impossible because $(\delta\lambda)(1, 1, 1) = 1$ while $\Omega_1(1, 1, 1) = -1$. $\square$

What has been established. Every finite depth chart admits an ordinary Hilbert realization, and the local lifting two-presheaf is contractible. The obstruction appears only when one asks for a single band-local, translation-stationary, coefficient-monoidal trivialization over the entire branch. Local realizability and global strictifiability are therefore different properties, even though all finite analytic certificates agree.

This is a statement about the fixed-stratum finite-depth higher descent object, not a continuum vacuum. A continuum construction must take a limit of its local Hilbert realizations and higher transition data and separately prove analytic convergence and no leakage of the uniform threshold.

Ordinary Hilbert realizations on every finite depth chart do not supply one global shell-local, translation-stationary, coefficient-monoidal tensorator. Local realizability and global strictification are different assertions; the obstruction lives precisely in that difference.

H. Finite-shadow indistinguishability

Define the complete finite analytic shadow

$$\text{Sh}_{\text{fin}}(\mathfrak{S}) = \left(\{\mathfrak{P}_n\}, \{\overline{U}_a^{\text{vac}}\}, \{\text{sv}(\mathbb{C}_a^{\text{vac}})\}, \{\|\mathfrak{D}_{b,a}^{(\nu)}\|\} \right),$$

where $\mathfrak{P}_n$ denotes the complete physicalized package.

Theorem VII.4 (No finite-shadow discriminator). *For the paired branches,*

$$\text{Sh}_{\text{fin}}(\mathfrak{S}^{\text{str}}) = \text{Sh}_{\text{fin}}(\mathfrak{S}^{\text{tw}}).$$

Consequently, every invariant, estimator, or algorithm whose full input factors through this shadow assigns the same value to the two branches.

Proof. The equality of all components was established in Theorem VII.2. Factorization through the same input gives equal output. $\square$

Corollary VII.5. *Any strict-twisted discriminator must use band-sensitive or reconstruction-sensitive data omitted by the finite analytic shadow.*

VIII. CONCLUSION AND RECONSTRUCTION HANDOFF

The paper establishes two distinct finite-to-continuum statements. The abstract statement is a genuine directed inverse-limit criterion: compact coefficient-line diagram spaces, coherent continuous restriction, directed finite-shape amalgamation, and universal finite extension produce a full multiparameter physical higher pseudosection. Any such pseudosection carrying one quantitative profile inherits the uniform estimate

$$H_\gamma \geq E_{0,\gamma} I + \kappa_*(I - P_{\text{vac},\gamma}).$$

The theorem is conditional on its extension hypotheses and does not rely on countable cofinality.

The principal upstream instance is the selected triality-sensitive $\text{SU}(3)$ finite packet imported from Paper I, with exact retained floor κ_{cof} . The constructive statement internal to Paper II is an end-to-end $\text{SU}(2)$ depth exhaustion at a fixed finite stratum σ_0 . One canonical vacuum-line scale-bordism correspondence transports the complete physicalized package, has

exact vacuum overlap, and is natural over the two center-background fibers. The family has depth-independent coercive constants but does not implement tube thinning, cutoff removal, or the remaining geometric and boundary refinements.

The same fixed-stratum analytic family admits strict and nontrivially center-banded coefficient-line globalizations. Their finite analytic shadows are identical. In the twisted case, every finite depth chart admits an ordinary Hilbert realization, and the choices form a locally contractible lifting two-presheaf with explicit rooted transitions and modifications. What fails is a shell-local, translation-stationary, coefficient-monoidal global tensorator. Thus the obstruction concerns admissible global higher gluing, not local Hilbert realizability.

The explicit datum passed forward is therefore

$$\mathfrak{D}_{\text{II},\sigma_0}^{\text{tw}} = \left(\mathfrak{R}_{\text{depth}}^{\Omega_1}(\sigma_0), \{\mathcal{H}_n, H_n, P_{\text{vac},n}, \mathfrak{P}_n\}, \{\mathbb{J}_a\}, \mathbf{Vac}_{\text{Hilb}}^{\Omega_1}, [\Delta_\zeta, \tau]_{\text{ref/adm}}, [\Omega_1]_{\text{band}}, \mathbf{q}_*, \kappa_{\text{II},\text{prof}} \right).$$

Here $\mathfrak{P}_n$ is the complete finite physicalized package, each $\mathbb{J}_a$ is a line-assisted physical correspondence rather than an assumed equivalence, and the two displayed cohomology classes retain their separate meanings. This handoff contains no incision, regional W^* -system, Friedrichs or Mosco operator, thin limit, relativistic spectrum, or mass conclusion. Paper II exports a typed finite-depth obstruction-and-coercivity packet; it does not export a spacetime theory by anticipation.

Scope at this boundary. The full multiparameter theorem remains a criterion. The explicit model is cofinal only in tensor depth at fixed stratum and does not by itself perform the simultaneous geometric, ultraviolet, collar, and boundary refinements required for spacetime reconstruction. Paper III begins from a selected packet and addresses that realization problem.

A continuum reconstruction must extend this structure to a genuinely cofinal multiparameter subsystem with

$$n \rightarrow \infty, \quad \varepsilon \rightarrow 0, \quad \Lambda \rightarrow \infty,$$

and cofinal refinement of θ and b . It must then prove analytic tightness, convergence of local observable systems and quadratic forms, effective higher descent, and no leakage of the uniform threshold. The scale bordism is the indispensable bridge: it is the common structure that transports both the finite coercive package and the category of vacuum gluing across continuum-directed scales.

SUPPLEMENTARY MATERIAL

See the supplementary material for the extended $SU(2)$ seed calculations, exact matrices and interval certificates, cyclic associator and center-twist checks, paired strict/twisted realization data, and the supporting certificate and reproducibility documentation for the finite calculations used in this article.

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USE OF GENERATIVE ARTIFICIAL INTELLIGENCE

The author used OpenAI’s ChatGPT, including GPT-4.5 through GPT-5.6, as an interactive research and adversarial editorial tool during the development of this manuscript and the broader five-paper series. Its use included assistance with organizing mathematical material, testing formulations and proof strategies, improving exposition, checking notation and cross-references, preparing and reviewing $\text{\LaTeX}$ source, and supporting computational and document-audit workflows.

The conceptual framework, ontologies, definitions, conjectures, identification of the principal obstructions, load-bearing components of the proofs, and overall proof strategies originated with and remain solely those of the author.

All mathematical claims, definitions, theorems, proofs, computations, interpretations, citations, and final wording were reviewed and approved by the author, who assumes full responsibility for the content of the manuscript. No generative artificial-intelligence system is credited as an author.

AUTHOR DECLARATIONS

Conflict of Interest. The author has no conflicts to disclose.

Author Contributions. Jeffrey S Little: Conceptualization; Formal analysis; Investigation; Methodology; Software; Validation; Writing–original draft; Writing–review and editing.

DATA AVAILABILITY

The data and code that support the findings of this study are available within the article, its supplementary material, and the accompanying computational reproducibility package.

Appendix A: Compressed certificate for the physicalized SU(2) seed

This appendix records the data needed by the journal argument. The complete algebraic and interval calculation is given in the Supplementary Material.

Work in the ordered character basis

$$\mathcal{K}_2 = \text{span}\{\chi_0, \chi_{1/2}, \chi_1, \chi_{3/2}\}.$$

With electric normalization $4j(j+1)$ and magnetic multiplier $1 - \frac{1}{2}\chi_{1/2}$, the compressed Hamiltonian is

$$H^{(2)} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 4 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 9 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 16 \end{pmatrix}. \quad (\text{A1})$$

Its lowest eigenvalue is

$$E_0^{(2)} = 0.918058177772217\dots,$$

and the complete ground projection is $P_{\text{vac}}^{(2)} = \mathbf{1}_{\{E_0^{(2)}\}}(H^{(2)})$. A positive representative of the ground line is

$$\Omega_2 = (0.986786015025, 0.161718088440, 0.010025465323, 0.000332366530)^T + O(10^{-12}),$$

but only the projection is part of the package.

On the one-dimensional coefficient excitation line, take the normalized raw Wilson vector

$$\phi_W = \frac{-16\chi_0 + 97\chi_{1/2} + 10\chi_1}{\sqrt{9765}}. \quad (\text{A2})$$

After complete ground-sector removal,

$$T_2 = (I - P_{\text{vac}}^{(2)})\phi_W, \quad G_2 = T_2^* T_2 = G_2 I,$$

with

$$0.999999999715 < G_2 < 0.999999999716.$$

The Gram-normalized physical vector $s_2 = G_2^{-1/2} T_2$ satisfies

$$\|s_2\| = 1, \quad P_{\text{vac}}^{(2)} s_2 = 0.$$

The actual magnetic recompression obeys

$$1.059584273164 < \langle s_2, H_{\text{mag}}^{(2)} s_2 \rangle < 1.059584273165.$$

Let $P_{\text{ch}}^{(2)} = |s_2\rangle\langle s_2|$ and $P_{\text{UV}}^{(2)} = I - P_{\text{vac}}^{(2)} - P_{\text{ch}}^{(2)}$. There is no residual block. With $\widehat{H}^{(2)} = H^{(2)} - E_0^{(2)} I$,

$$P_{\text{UV}}^{(2)} \widehat{H}^{(2)} P_{\text{UV}}^{(2)} \geq 8.096308674897 P_{\text{UV}}^{(2)},$$

and the exact mixed block

$$K_2 = P_{\text{UV}}^{(2)} \widehat{H}^{(2)} P_{\text{ch}}^{(2)}$$

is nonzero, with

$$0.051998558862 < \|K_2\| < 0.051998558863.$$

Thus the witness exercises genuine mixed-block control rather than block closure.

For the robust profile used throughout the paper, choose

$$c_{\text{ch},2} = 3.093770498, \quad c_{\text{UV},2} = 8, \quad \varepsilon_{\text{mix},2} = 0.052.$$

The determinant margin is

$$c_{\text{ch},2} c_{\text{UV},2} - \varepsilon_{\text{mix},2}^2 = 24.747459984 > 0.$$

The smaller eigenvalue of the associated scalar block matrix is

$$\widehat{\kappa}_{\text{SU}(2)} = 3.093219423836803 \dots$$

Consequently,

$$H^{(2)} \geq E_0^{(2)} I + 3.093219423836 \dots (I - P_{\text{vac}}^{(2)}). \quad (\text{A3})$$

Theorem A.1 (Complete physicalized $SU(2)$ seed). *The data above form a finite physicalized good package with complete ground projection, positive Gram operator, positive magnetic recompression, positive channel and ultraviolet diagonal constants, zero residual sector, a nonzero ultraviolet–infrared mixed block, and the certified coercive estimate (A3).*

Remark A.2. The certificate proves finite good-locus nonemptiness for this four-character compression. It does not prove persistence under cutoff enlargement, tube thinning, convergence of the ground projections, or survival of the bound in a continuum limit.

Appendix B: Explicit cyclic associators and nontriviality audit

1. Purpose and coefficient convention

This appendix supplies the exact cyclic associator systems used in Section III. The group-cohomological background is standard^{4,5}. The construction has four purposes:

- (i) to define normalized cocycles on the additive refinement-label group $\mathbb{Z}_N$;
- (ii) to verify the pentagon in the compositor ordering fixed in the compositor convention of Section III;
- (iii) to prove directly that the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ examples are not coboundaries;
- (iv) to keep the additive center labels, the embedded central unitaries, and the resulting $U(1)$ -valued cohomology class logically distinct.

Let

$$z_N = \exp\left(\frac{2\pi i}{N}\right), \quad \mu_N = \{z_N^{\tilde{k}} : k \in \mathbb{Z}_N\} \subset U(1). \quad (\text{B1})$$

For each label $k \in \mathbb{Z}_N$, let $\tilde{k} \in \{0, 1, \dots, N-1\}$ denote its preferred lift. The coefficient embedding is the homomorphism

$$\iota_N : \mathbb{Z}_N \longrightarrow U(1), \quad \iota_N(k) = z_N^{\tilde{k}}. \quad (\text{B2})$$

For $SU(N)$, the same scalar is interpreted as the central unitary $z_N^{\tilde{k}} I_N$. The notation does not identify $H^3(\mathbb{Z}_N; \mathbb{Z}_N)$ with $H^3(\mathbb{Z}_N; U(1))$. Rather, (B2) induces a coefficient map between the two cochain systems.

2. The cyclic carry function

For $a \in \mathbb{Z}_N$, write

$$\tilde{a} \in \{0, 1, \dots, N-1\}$$

for its preferred representative. Define

$$\epsilon_N(a, b) = \frac{\tilde{a} + \tilde{b} - \widetilde{a+b}}{N} \in \{0, 1\}. \quad (\text{B3})$$

Thus $\epsilon_N(a, b) = 1$ precisely when addition of the preferred lifts produces one carry across N .

Lemma B.1 (Carry identity). *For all $a, b, c \in \mathbb{Z}_N$,*

$$\epsilon_N(a, b) + \epsilon_N(a+b, c) = \epsilon_N(b, c) + \epsilon_N(a, b+c). \quad (\text{B4})$$

Moreover,

$$\widetilde{a+b} = \tilde{a} + \tilde{b} - N\epsilon_N(a, b). \quad (\text{B5})$$

Proof. Equation (B5) is the definition of the carry. Applying it to the two parenthesizations of $\tilde{a} + \tilde{b} + \tilde{c}$ gives

$$\tilde{a} + \tilde{b} + \tilde{c} - \widetilde{a+b+c} = N(\epsilon_N(a, b) + \epsilon_N(a+b, c))$$

and also

$$\tilde{a} + \tilde{b} + \tilde{c} - \widetilde{a+b+c} = N(\epsilon_N(b, c) + \epsilon_N(a, b+c)).$$

Equality of the right-hand sides proves (B4). □

The function ϵ_N is normalized:

$$\epsilon_N(0, a) = \epsilon_N(a, 0) = 0. \quad (\text{B6})$$

3. The normalized cyclic associator

For $m \in \mathbb{Z}_N$, define the exponent label

$$\kappa_m(a, b, c) = m\tilde{a}\epsilon_N(b, c) \pmod{N} \quad (\text{B7})$$

and the embedded central phase

$$\boxed{\Omega_m(a, b, c) = \iota_N(\kappa_m(a, b, c)) = z_N^{m\tilde{a}\epsilon_N(b, c)}}. \quad (\text{B8})$$

Equivalently,

$$\Omega_m(a, b, c) = \exp\left(\frac{2\pi i m}{N} \tilde{a}\epsilon_N(b, c)\right).$$

The sign in this formula is fixed so that

$$\Omega_1(1, 2, 1) = e^{2\pi i/3}$$

for the $\mathbb{Z}_3$ example below.

The cocycle is normalized:

$$\Omega_m(0, b, c) = \Omega_m(a, 0, c) = \Omega_m(a, b, 0) = 1. \quad (\text{B9})$$

Under the arrow ordering of the compositor convention of Section III, an additive triple of refinement labels (c, b, a) is assigned

$$\omega_{c,b,a} = \Omega_m(c, b, a). \quad (\text{B10})$$

Thus the ordinary multiplicative group coboundary below is exactly the central specialization of the categorical differential used in the categorical coboundary convention of Section III.

4. Pentagon verification

For a normalized $U(1)$ -valued 3-cochain on the additive group $\mathbb{Z}_N$, use the convention

$$\begin{aligned} (\delta\Omega)(d, c, b, a) &= \Omega(c, b, a)\Omega(d + c, b, a)^{-1}\Omega(d, c + b, a) \\ &\quad \times \Omega(d, c, b + a)^{-1}\Omega(d, c, b). \end{aligned} \quad (\text{B11})$$

All sums in the arguments are taken modulo N .

Theorem B.2 (Cyclic pentagon). *For every $N \geq 2$ and every $m \in \mathbb{Z}_N$,*

$$\boxed{\delta\Omega_m = 1.} \quad (\text{B12})$$

Consequently, (B10) defines a coherent central associator satisfying the five-term pentagon of the pentagon ordering fixed in Section III.

Proof. Write

$$F_m(a, b, c) = m\tilde{a}\epsilon_N(b, c) \in \mathbb{Z}.$$

The exponent of $(\delta\Omega_m)(d, c, b, a)$ is, modulo N ,

$$\begin{aligned} D_m(d, c, b, a) &= F_m(c, b, a) - F_m(d + c, b, a) + F_m(d, c + b, a) \\ &\quad - F_m(d, c, b + a) + F_m(d, c, b). \end{aligned}$$

By (B5),

$$\widetilde{d + c} = \tilde{d} + \tilde{c} - N\epsilon_N(d, c).$$

Hence the first two terms contribute

$$m(-\tilde{d} + N\epsilon_N(d, c))\epsilon_N(b, a).$$

The remaining three terms contribute

$$m\tilde{d}(\epsilon_N(c + b, a) - \epsilon_N(c, b + a) + \epsilon_N(c, b)).$$

The carry identity (B4), applied to (c, b, a) , gives

$$\epsilon_N(c + b, a) - \epsilon_N(c, b + a) + \epsilon_N(c, b) = \epsilon_N(b, a).$$

The terms proportional to $\tilde{d}$ cancel, leaving

$$D_m(d, c, b, a) = mN\epsilon_N(d, c)\epsilon_N(b, a). \tag{B13}$$

This is divisible by N , so

$$z_N^{D_m(d, c, b, a)} = 1.$$

That is precisely (B12). □

Remark B.3 (Strict operator coboundaries). Theorem B.2 produces admissible higher gluing data. This does not contradict the fact that a globally strict operator coboundary has trivial associator. For $m \neq 0$, the cocycle below is not the coboundary of one globally defined normalized 2-cochain, and it is not obtained solely by multiplying a single family of global finite-dimensional transport matrices.

5. A direct nontriviality detector

For a normalized 3-cocycle Ω , define the cyclic detector

$$\mathfrak{I}_N(\Omega) = \prod_{b=0}^{N-1} \Omega(1, b, 1). \tag{B14}$$

Lemma B.4 (Coboundary invariance of the detector). *Let $\lambda : \mathbb{Z}_N^2 \rightarrow U(1)$ be a normalized 2-cochain, with*

$$(\delta\lambda)(a, b, c) = \lambda(b, c)\lambda(a + b, c)^{-1}\lambda(a, b + c)\lambda(a, b)^{-1}. \quad (\text{B15})$$

Then

$$\mathfrak{I}_N(\delta\lambda) = 1. \quad (\text{B16})$$

Consequently,

$$\mathfrak{I}_N(\Omega \delta\lambda) = \mathfrak{I}_N(\Omega).$$

Proof. Using (B15),

$$\mathfrak{I}_N(\delta\lambda) = \prod_{b=0}^{N-1} \frac{\lambda(b, 1)}{\lambda(b + 1, 1)} \prod_{b=0}^{N-1} \frac{\lambda(1, b + 1)}{\lambda(1, b)}.$$

Both products telescope cyclically on $\mathbb{Z}_N$, so their product is one. $\square$

Proposition B.5 (Nontriviality and pairwise inequivalence). *For the cocycle Ω_m ,*

$$\boxed{\mathfrak{I}_N(\Omega_m) = z_N^m}. \quad (\text{B17})$$

Therefore Ω_m is not a coboundary whenever $m \not\equiv 0 \pmod{N}$, and the N cocycles $\Omega_0, \dots, \Omega_{N-1}$ are pairwise inequivalent.

Proof. For $b \in \{0, \dots, N - 1\}$, the carry $\epsilon_N(b, 1)$ vanishes except at $b = N - 1$, where it equals one. Hence

$$\mathfrak{I}_N(\Omega_m) = \prod_{b=0}^{N-1} z_N^{m\epsilon_N(b, 1)} = z_N^m.$$

Coboundary invariance from Theorem B.4 proves both claims. $\square$

Remark B.6 (Scope of the detector). The detector proves exactly what is required here: the displayed cocycles satisfy the pentagon, are not coboundaries, and represent pairwise distinct classes within the explicit family. No exhaustiveness claim for all $U(1)$ -valued cyclic 3-cocycles is needed in this article.

6. The $\mathbb{Z}_2$ associator

Let $N = 2$ and $m = 1$. The only normalized triple with nontrivial value is $(1, 1, 1)$, and

$$\boxed{\Omega_1(1, 1, 1) = -1.} \quad (\text{B18})$$

The detector gives

$$\mathfrak{I}_2(\Omega_1) = -1,$$

so the class is nontrivial. With the central-length convention

$$\ell_N(k) = |1 - z_N^k|^2 = 2 - 2 \cos\left(\frac{2\pi k}{N}\right) = 4 \sin^2\left(\frac{\pi k}{N}\right), \quad (\text{B19})$$

one obtains

$$\boxed{\ell_2(1) = 4.} \quad (\text{B20})$$

This is the topological curvature contribution used for one extensive nontrivial $\mathbb{Z}_2$ associator cell per refinement shell.

7. The $\mathbb{Z}_3$ associator

Let $N = 3$ and $m = 1$. Since

$$\epsilon_3(2, 1) = 1,$$

the marked triple used in the main text satisfies

$$\boxed{\Omega_1(1, 2, 1) = z_3 = e^{2\pi i/3}.} \quad (\text{B21})$$

The detector gives

$$\mathfrak{I}_3(\Omega_1) = z_3,$$

so the class is nontrivial. Its central length is

$$\boxed{|1 - e^{2\pi i/3}|^2 = 3.} \quad (\text{B22})$$

The conjugate cocycle Ω_2 is also nontrivial and satisfies

$$\mathfrak{I}_3(\Omega_2) = z_3^2.$$

Nevertheless,

$$|1 - z_3|^2 = |1 - z_3^2|^2 = 3.$$

Thus the scalar central-length functional does not distinguish the two oppositely oriented nontrivial $\mathbb{Z}_3$ classes. The marked cocycle class must be retained in addition to its numerical curvature cost.

8. Vacuum-sector realization and possible annihilation

Let a vacuum coefficient band carry the central character

$$\rho_q(z_N^k) = z_N^{qk} I_{\mathcal{V}}, \quad q \in \mathbb{Z}_N. \quad (\text{B23})$$

The induced vacuum associator is

$$\Omega_m^{\text{vac}}(a, b, c) = \rho_q(\Omega_m(a, b, c)) = \Omega_{qm}(a, b, c) I_{\mathcal{V}}. \quad (\text{B24})$$

Therefore

$$\boxed{[\Omega_m^{\text{vac}}] = 0 \iff qm \equiv 0 \pmod{N}} \quad (\text{B25})$$

within this cyclic family. In particular, a trivial central character annihilates every ambient cyclic associator after vacuum realization. This is a concrete instance of the warning that an ambient nonzero class need not act nontrivially on the complete ground sector.

For the prime cyclic groups used here, every nontrivial character preserves the nontriviality of a nonzero class. The statement is about the coefficient band through which the associator acts; it is not an assertion that the finite Hamiltonian ground space itself carries a distinguished global center eigenvector.

9. Compatibility with the analytic curvature disk

The cyclic associators and the analytic two-shell defect of the analytic two-shell defects described in the Supplementary Material occupy different components of the total scale-coherence data. Their product gives a finite compatibility witness with

$$\mathfrak{D}_{B,A} \neq 0 \quad \text{and} \quad [\Omega_m] \neq 0.$$

No implication is claimed in either direction:

$$\mathfrak{D}_{B,A} \neq 0 \not\Rightarrow [\Omega_m] \neq 0,$$

$$[\Omega_m] \neq 0 \not\Rightarrow \mathfrak{D}_{B,A} \neq 0.$$

The analytic defect measures non-flat operator transport, whereas Ω_m records coherent higher gluing in the central coefficient band.

Warning B.7 (Compatibility is not microscopic derivation). Taking the direct product of the Wilson-character analytic seed and a cyclic associator system proves simultaneous nonemptiness of the analytic and cohomological mechanisms. It does not derive the associator from the finite two-plaquette Hamiltonian, nor does it identify the cyclic label group with the complete microscopic refinement category.

10. Machine-checkable finite audit

For $N = 2$ and $N = 3$, all cocycle and nontriviality statements reduce to a finite enumeration. The supplementary verification script checks:

- (i) the carry identity for every triple;
- (ii) the pentagon exponent for every quadruple and every m ;
- (iii) normalization of Ω_m ;
- (iv) the detector value $\mathfrak{I}_N(\Omega_m) = z_N^m$;
- (v) the exact $\mathbb{Z}_2$ and $\mathbb{Z}_3$ marked values;
- (vi) the central lengths 4 and 3.

The script uses integer exponents modulo N , avoiding floating-point phase comparisons in the cocycle and pentagon tests.

Appendix C: End-to-end recursive physicalization of the $\mathrm{SU}(2)$ seed

1. Purpose

This appendix closes the finite-to-recursive verification gap left by the separate one-scale and shell-model constructions. Starting from the complete physicalized four-character seed

of Section A, we construct, at every depth, a single separated-shell Wilson-character package carrying all of the Article-I data:

$$H_n, \quad E_{0,n}, \quad P_{\text{vac},n}, \quad \mathbf{F}_n^W, \quad \mathbf{G}_n, \quad \mathbf{F}_n^{\text{phys}}, \quad \mathbf{B}_{\text{mag},n}^{\text{phys}},$$

$$P_{\text{ch},n}, \quad P_{\text{res},n}, \quad P_{\text{UV},n}, \quad K_n.$$

The Gram, physical magnetic, diagonal, mixed-block, vacuum-overlap, and coercive constants are independent of depth. The construction therefore produces one complete compatible tensor-depth exhaustion directly inside a fixed finite scale stratum. It does not claim finite extension feasibility for every arbitrary point of the full good-package spaces used in the abstract inverse-limit criterion.

The family remains a calibrated separated-shell Galerkin realization at a fixed stratum $\sigma_0 = (\varepsilon_0, \theta_0, \Lambda_0, b_0)$. It is not identified with the unmodified constant-coupling Kogut–Susskind Hamiltonian. The purpose of the construction is narrower and exact: to show that the complete finite physicalization mechanism survives arbitrary tensor depth with one common quantitative profile. It does not thin the tubes, remove the representation cutoff, or refine the remaining geometric and boundary coordinates.

2. The local physicalized package

Let

$$\mathcal{K} = \text{span} \left\{ \chi_0, \chi_{\frac{1}{2}}, \chi_1, \chi_{\frac{3}{2}} \right\} \cong \mathbb{C}^4$$

and let

$$H = H_{\text{el}} + H_{\text{mag}}$$

be the exact matrix of (A1). Write

$$E_0 = E_0^{(2)}, \quad P = P_{\text{vac}}^{(2)}, \quad Q = I - P, \quad h = H - E_0 I.$$

The complete vacuum datum is the rank-one spectral projection P . Let

$$\mathcal{V} = \text{Ran } P$$

be the corresponding vacuum line. No phase choice on that line is part of the package.

Let $\mathfrak{e} = \mathbb{C}g_+$ be the one-dimensional local coefficient space, and let

$$F^W : \mathfrak{e} \longrightarrow \mathcal{K}$$

be the raw Wilson map of (A2). Define

$$T = (I - P)F^W, \quad G = T^*T = G_2 I_{\mathfrak{e}}, \quad F^{\text{phys}} = TG^{-1/2}.$$

Let

$$p_{\text{ch}} = F^{\text{phys}}(F^{\text{phys}})^*, \quad p_{\text{uv}} = Q - p_{\text{ch}}.$$

The local residual projection is zero.

The estimates of Section A imply

$$\begin{aligned} G_2 &> 0.999999999715, & (F^{\text{phys}})^* H_{\text{mag}} F^{\text{phys}} &> 1.059584273164, \\ p_{\text{ch}} h p_{\text{ch}} &> 3.114097481076 p_{\text{ch}}, & p_{\text{uv}} h p_{\text{uv}} &> 8.096308674897 p_{\text{uv}}, \\ \|p_{\text{uv}} h p_{\text{ch}}\| &< 0.051998558863, & h &\geq \hat{\kappa}_{\text{SU}(2)} Q, \end{aligned} \tag{C1}$$

where

$$\hat{\kappa}_{\text{SU}(2)} = 3.093219423836803087 \dots$$

3. The separated-shell exhaustion

For $n \geq 1$, define the depth- n Hilbert space

$$\mathcal{H}_n = \mathcal{K}^{\otimes n}. \tag{C2}$$

This is the finite retained class-function sector of n separated $\text{SU}(2)$ Wilson shells. For $1 \leq m \leq n$, let a superscript (m) denote amplification to the m th tensor factor. Set

$$\begin{aligned} H_{\text{el},n} &= \sum_{m=1}^n H_{\text{el}}^{(m)}, \\ H_{\text{mag},n} &= \sum_{m=1}^n H_{\text{mag}}^{(m)}, \\ H_n &= H_{\text{el},n} + H_{\text{mag},n}, \\ \widehat{H}_n &= H_n - nE_0 I = \sum_{m=1}^n h^{(m)}. \end{aligned} \tag{C3}$$

Because P is the complete local ground projection,

$$E_{0,n} = nE_0, \quad P_{\text{vac},n} = P^{\otimes n}. \tag{C4}$$

In particular, the complete vacuum rank is one at every depth.

Proposition C.1 (Exact tensor-sum spectral separation). *For every $n \geq 1$,*

$$\widehat{H}_n \geq \widehat{\kappa}_{\text{SU}(2)}(I - P_{\text{vac},n}). \quad (\text{C5})$$

Proof. Since $h \geq \widehat{\kappa}_{\text{SU}(2)}Q$ and P, Q reduce h , the tensor product decomposes into excitation-pattern sectors

$$\Pi_S = \bigotimes_{m \in S} Q^{(m)} \bigotimes_{m \notin S} P^{(m)}, \quad S \subseteq \{1, \dots, n\}.$$

On $\Pi_S \mathcal{H}_n$ one has

$$\widehat{H}_n \geq |S| \widehat{\kappa}_{\text{SU}(2)}.$$

The sector $S = \emptyset$ is exactly $\text{Ran } P_{\text{vac},n}$, and every other sector has $|S| \geq 1$. $\square$

4. A phase-free anchored coefficient space

To avoid choosing a vacuum vector, define the depth- n coefficient space by

$$\mathfrak{E}_n = \bigoplus_{m=1}^n \mathcal{V}^{\otimes(m-1)} \otimes \mathfrak{e} \otimes \mathcal{V}^{\otimes(n-m)}. \quad (\text{C6})$$

Every summand is one-dimensional, but the definition uses only the vacuum line $\mathcal{V}$, not a preferred vector in it.

On the m th summand, define the raw Wilson map by

$$\mathbb{F}_n^W = I_{\mathcal{V}}^{\otimes(m-1)} \otimes F^W \otimes I_{\mathcal{V}}^{\otimes(n-m)}. \quad (\text{C7})$$

Equivalently, after choosing a temporary unit representative $\Omega \in \mathcal{V}$, the image of the m th unit coefficient is

$$\Omega^{\otimes(m-1)} \otimes F^W g_+ \otimes \Omega^{\otimes(n-m)}.$$

Changing the phase of Ω only conjugates the coefficient description by a scalar unitary and does not change the package.

Define

$$T_n = (I - P_{\text{vac},n}) \mathbb{F}_n^W, \quad \mathbb{G}_n = T_n^* T_n. \quad (\text{C8})$$

Theorem C.2 (Exact global Gram operator). *For every depth,*

$$\boxed{\mathbb{G}_n = G_2 I_{\mathfrak{E}_n}}. \quad (\text{C9})$$

Consequently,

$$0.999999999715 I_{\mathfrak{E}_n} < \mathbf{G}_n < I_{\mathfrak{E}_n} \quad (\text{C10})$$

for every n .

Proof. Write

$$F^W g_+ = a\Omega + t, \quad t \perp \Omega, \quad \|t\|^2 = G_2.$$

After removal of the joint vacuum, the m th anchored vector is

$$\Omega^{\otimes(m-1)} \otimes t \otimes \Omega^{\otimes(n-m)}.$$

Two such vectors with distinct excited shell indices are orthogonal because one tensor factor contains $t \perp \Omega$. Every diagonal norm equals G_2 . $\square$

The physicalized map is therefore

$$\mathbf{F}_n^{\text{phys}} = T_n \mathbf{G}_n^{-1/2}. \quad (\text{C11})$$

It is an isometry. Set

$$P_{\text{ch},n} = \mathbf{F}_n^{\text{phys}} (\mathbf{F}_n^{\text{phys}})^*. \quad (\text{C12})$$

5. Exact physical magnetic recompression

Let

$$m_{\text{vac}} = \text{Tr}(PH_{\text{mag}}) = \langle \Omega, H_{\text{mag}} \Omega \rangle = 0.838794220733 \dots$$

This scalar is phase independent. Let

$$b_{\text{mag}} = (F^{\text{phys}})^* H_{\text{mag}} F^{\text{phys}} = 1.059584273164 \dots$$

Proposition C.3 (Global physical magnetic recompression). *For every n ,*

$$\boxed{\mathbf{B}_{\text{mag},n}^{\text{phys}} = (\mathbf{F}_n^{\text{phys}})^* H_{\text{mag},n} \mathbf{F}_n^{\text{phys}} = [b_{\text{mag}} + (n-1)m_{\text{vac}}] I_{\mathfrak{E}_n}.} \quad (\text{C13})$$

In particular,

$$\mathbf{B}_{\text{mag},n}^{\text{phys}} > 1.059584273164 I_{\mathfrak{E}_n}. \quad (\text{C14})$$

Proof. A normalized physical coefficient at shell m is the tensor product of the local physical vector on the m th factor and vacuum lines on all other factors. The m th magnetic term contributes b_{mag} , while each other factor contributes m_{vac} . Off-diagonal matrix elements vanish because distinct anchored physical vectors contain one factor orthogonal to the vacuum line. $\square$

6. Global channel, residual, and ultraviolet projections

For a subset $S \subseteq \{1, \dots, n\}$, retain the excitation projection Π_S from the proof of Theorem C.1. The global channel projection is the sum of the one-excitation local channel projections:

$$P_{\text{ch},n} = \sum_{m=1}^n P^{\otimes(m-1)} \otimes p_{\text{ch}} \otimes P^{\otimes(n-m)}. \quad (\text{C15})$$

Set

$$P_{\text{res},n} = 0, \quad P_{\text{UV},n} = I - P_{\text{vac},n} - P_{\text{ch},n}. \quad (\text{C16})$$

The ultraviolet sector splits orthogonally as

$$P_{\text{UV},n} = P_{\text{one-uv},n} + P_{\text{multi},n}, \quad (\text{C17})$$

where

$$P_{\text{one-uv},n} = \sum_{m=1}^n P^{\otimes(m-1)} \otimes p_{\text{uv}} \otimes P^{\otimes(n-m)}$$

and

$$P_{\text{multi},n} = \sum_{|S| \geq 2} \Pi_S.$$

Both summands reduce the ultraviolet compression of $\widehat{H}_n$.

Proposition C.4 (Uniform diagonal bounds). *For every n ,*

$$P_{\text{ch},n} \widehat{H}_n P_{\text{ch},n} > 3.114097481076 P_{\text{ch},n}, \quad (\text{C18})$$

$$P_{\text{UV},n} \widehat{H}_n P_{\text{UV},n} > 6.186438847673 P_{\text{UV},n}. \quad (\text{C19})$$

Proof. The channel compression is a direct sum of copies of the local scalar $p_{\text{ch}} h p_{\text{ch}}$. On the one-excitation ultraviolet sector, the compression is a direct sum of copies of $p_{\text{uv}} h p_{\text{uv}}$ and is bounded below by 8.096308674897. On every multi-excitation sector Π_S ,

$$\widehat{H}_n \geq |S| \widehat{\kappa}_{\text{SU}(2)} \geq 2 \widehat{\kappa}_{\text{SU}(2)} = 6.186438847673606 \dots$$

The smaller of the two ultraviolet lower bounds is therefore the displayed multi-excitation value. $\square$

7. The exact global mixed block

Define

$$K_n = P_{\text{UV},n} \widehat{H}_n P_{\text{ch},n}. \quad (\text{C20})$$

Since $P_{\text{res},n} = 0$, this is the complete ultraviolet–infrared mixed block.

Theorem C.5 (Depth-independent exact mixed norm). *For every n ,*

$$\boxed{\|K_n\| = \|K_2\| < 0.051998558863.} \quad (\text{C21})$$

Moreover, K_n has no component from the channel sector into the multi-excitation ultraviolet sector.

Proof. The tensor-sum Hamiltonian preserves every excitation pattern Π_S . Acting on the channel vector anchored at shell m , only the local term $h^{(m)}$ is nonzero. Its channel-to-ultraviolet component is the local block $p_{\text{uv}} h p_{\text{ch}}$. The resulting ranges for different m are orthogonal. Thus K_n is the orthogonal direct sum of n copies of the local mixed block, and the norm of a direct sum is the largest summand norm. $\square$

8. One common Article-I threshold profile

The empty residual sector is treated by the standard zero-dimensional convention:

$$c_{\text{res},*} = 1, \quad \varepsilon_{\text{sr},*} = 0, \quad \rho_{\text{sr},*} = 0.$$

The corresponding inequalities hold vacuously on $\text{Ran } P_{\text{res},n} = \{0\}$. Choose the depth-independent rational profile

$$\boxed{\begin{aligned} g_* &= 0.999999999715, & G^* &= 1, \\ c_{\text{ch},*} &= 1, & c_{\text{res},*} &= 1, \\ c_{\text{UV},*} &= 6, & \varepsilon_{\text{sr},*} &= 0, \\ \varepsilon_{\text{mix},*} &= 0.052, & \rho_{\text{sr},*} &= 0, \\ \rho_* &= 0.001. \end{aligned}} \quad (\text{C22})$$

The common channel constant $c_{\text{ch},*} = 1$ is below both the physical magnetic lower bound and the shifted Hamiltonian channel lower bound.

The inherited infrared constant is

$$c_{\text{IR},*} = \frac{1 - \rho_{\text{sr},*}}{2} \min\{c_{\text{ch},*}, c_{\text{res},*}\} = \frac{1}{2}. \quad (\text{C23})$$

The mixed inequality is strict because

$$(0.052)^2 = 0.002704 < 0.003 = \rho_* c_{\text{UV},*} c_{\text{IR},*}. \quad (\text{C24})$$

Therefore the Article-I package formula gives

$$\boxed{\widehat{H}_n \geq \kappa_{\text{pkg},*}(I - P_{\text{vac},n}), \quad \kappa_{\text{pkg},*} = \frac{1 - 0.001}{2} \cdot \frac{1}{2} = 0.24975.} \quad (\text{C25})$$

This conservative constant is derived from one common package profile. The stronger direct tensor-sum estimate (C5) remains valid simultaneously.

If one uses the sharper two-block constants

$$c_{\text{ch}}^{\text{cert}} = 3.093770498, \quad c_{\text{UV}}^{\text{rec}} = 6, \quad \varepsilon_{\text{mix}}^{\text{cert}} = 0.052,$$

the depth-independent resolved two-block value is

$$\kappa_{\text{block},*} = 3.092840380512857 \dots \quad (\text{C26})$$

9. One canonical vacuum-line scale-bordism correspondence

Let

$$\mathcal{V} = \text{Ran } P$$

be the one-shell vacuum line. For a batch insertion of k shells define

$$\mathbb{J}_{n;k} : \mathcal{H}_n \widehat{\otimes} \mathcal{V}^{\otimes k} \longrightarrow \mathcal{H}_{n+k} \quad (\text{C27})$$

to be the canonical tensor reassociation map. Its range is the closed subspace in which every added shell lies in its complete ground line. No unit vector in $\mathcal{V}$ is chosen.

The corresponding old-coefficient subspace is

$$\mathfrak{E}_{n;k}^{\text{old}} = \mathfrak{E}_n \widehat{\otimes} \mathcal{V}^{\otimes k} \subseteq \mathfrak{E}_{n+k},$$

and the canonical coefficient correspondence is denoted

$$\mathbb{E}_{n;k} : \mathfrak{E}_n \widehat{\otimes} \mathcal{V}^{\otimes k} \longrightarrow \mathfrak{E}_{n+k}. \quad (\text{C28})$$

It identifies the source with the first n anchored coefficient summands at depth $n + k$.

Theorem C.6 (Exact package intertwining by the canonical bordism). *For every $n \geq 1$ and $k \geq 0$, the correspondence $\mathbb{J}_{n;k}$, together with $\mathbb{E}_{n;k}$, satisfies*

$$\mathbb{J}_{n;k}^* \widehat{H}_{n+k} \mathbb{J}_{n;k} = \widehat{H}_n \otimes I_{\mathcal{V}^{\otimes k}}, \quad (\text{C29})$$

$$\mathbb{J}_{n;k}(P_{\text{vac},n} \otimes I_{\mathcal{V}^{\otimes k}}) \mathbb{J}_{n;k}^* = P_{\text{vac},n+k}, \quad (\text{C30})$$

$$\mathbb{J}_{n;k}(\mathbf{F}_n^W \otimes I_{\mathcal{V}^{\otimes k}}) = \mathbf{F}_{n+k}^W \mathbb{E}_{n;k}, \quad (\text{C31})$$

$$\mathbb{E}_{n;k}^* \mathbf{G}_{n+k} \mathbb{E}_{n;k} = \mathbf{G}_n \otimes I_{\mathcal{V}^{\otimes k}}, \quad (\text{C32})$$

$$\mathbb{J}_{n;k}(\mathbf{F}_n^{\text{phys}} \otimes I_{\mathcal{V}^{\otimes k}}) = \mathbf{F}_{n+k}^{\text{phys}} \mathbb{E}_{n;k}, \quad (\text{C33})$$

$$\mathbb{E}_{n;k}^* \mathbf{B}_{\text{mag},n+k}^{\text{phys}} \mathbb{E}_{n;k} = (\mathbf{B}_{\text{mag},n}^{\text{phys}} + km_{\text{vac}} I) \otimes I_{\mathcal{V}^{\otimes k}}, \quad (\text{C34})$$

$$\mathbb{J}_{n;k}^* K_{n+k} \mathbb{J}_{n;k} = K_n \otimes I_{\mathcal{V}^{\otimes k}} \quad (\text{C35})$$

on the old channel and ultraviolet summands. The newly added channel and ultraviolet summands are orthogonal positive extensions. Hence one and the same bordism correspondence carries Hamiltonian extension, exact package restriction, vacuum transport, and the later higher-descent data.

Proof. The added local shifted Hamiltonians vanish on $\mathcal{V}$, which proves (C29) and (C30). Every old anchored Wilson vector acquires only vacuum tensor factors, giving (C31). Vacuum removal and Gram normalization commute with this tensor identification, proving (C32) and (C33). Each added magnetic term contributes its vacuum expectation m_{vac} , which gives (C34). The tensor-sum Hamiltonian preserves excitation patterns, so the old mixed block is transported exactly and has no component into a newly created multi-excitation sector; this proves (C35). $\square$

10. Vacuum comparison, local representatives, and the raw chart

Restricting $\mathbb{J}_{n;k}$ to vacuum lines gives the canonical unitary correspondence

$$\mathbb{U}_{n;k}^{\text{vac}} : \mathcal{V}_n \widehat{\otimes} \mathcal{V}^{\otimes k} \xrightarrow{\cong} \mathcal{V}_{n+k}. \quad (\text{C36})$$

Consequently the declared physical overlap is exact:

$$(\mathbb{U}_{n;k}^{\text{vac}})^* \mathbb{U}_{n;k}^{\text{vac}} = I, \quad \boxed{\nu_* = 1}. \quad (\text{C37})$$

Choosing temporarily a unit vector $e \in \mathcal{V}$ gives the representative

$$J_{n;k}^e(\psi) = \mathbb{J}_{n;k}(\psi \otimes e^{\otimes k}). \quad (\text{C38})$$

Changing e by a phase changes this map by the corresponding central phase and leaves its projective class fixed.

The traditional cylindrical chart obtained from the normalized trivial character χ_0 is retained only as a raw geometric coordinate map:

$$j_{n+1,n}^{\chi_0}(\psi) = \psi \otimes \chi_0. \quad (\text{C39})$$

Let

$$c_0 = \|P\chi_0\| = 0.986786015024\dots, \quad e_0 = \frac{P\chi_0}{\|P\chi_0\|} \in \mathcal{V}.$$

Then its vacuum compression is

$$P_{\text{vac},n+1} j_{n+1,n}^{\chi_0} P_{\text{vac},n} = c_0 J_{n;1}^{e_0} P_{\text{vac},n}. \quad (\text{C40})$$

Thus the polar factor of the raw cylindrical compression is precisely the unit-vector representative of the canonical correspondence $\mathbb{J}_{n;1}$. The number c_0 measures transversality of one raw coordinate chart; it is not the overlap constant of a second physical refinement arrow.

Corollary C.7 (The three former maps are one bordism datum). *Package restriction is the intertwining action of $\mathbb{J}_{n;k}$, vacuum transport is its restriction $\mathbb{U}_{n;k}^{\text{vac}}$, and every strict or twisted Hilbert-space comparison is a local lift $J_{n;k}^e$ of that same correspondence. The χ_0 -cylindrical map is merely a raw chart whose physicalized polar factor is such a lift.*

11. Canonical recursive extension of the complete package

The depth- n package extends to depth $n+1$ by:

- (i) tensoring the Hilbert space with one new retained Wilson-character shell $\mathcal{K}$;
- (ii) adding one copy of H_{el} and H_{mag} ;
- (iii) tensoring the complete vacuum projection with P ;
- (iv) adjoining one new anchored coefficient summand to $\mathfrak{E}_n$;
- (v) extending F_n^W by the new anchored local Wilson map;
- (vi) extending the channel projection by one orthogonal local channel line;

- (vii) extending the ultraviolet sector by the new one-shell ultraviolet and all new multi-excitation sectors.

Every old package component is recovered by the canonical correspondence of Section C 9, and the numerical profile (C22) is unchanged.

Theorem C.8 (End-to-end recursive physicalized realization). *The separated-shell $SU(2)$ Wilson-character family*

$$\left\{ \mathcal{H}_n, H_n, E_{0,n}, P_{\text{vac},n}, \mathbf{F}_n^W, \mathbf{G}_n, \mathbf{F}_n^{\text{phys}}, \mathbf{B}_{\text{mag},n}^{\text{phys}}, P_{\text{ch},n}, P_{\text{res},n}, P_{\text{UV},n}, K_n \right\}_{n \geq 1}$$

forms a recursively extendable complete physicalized package. At every depth:

- (i) *the complete vacuum projection is $P^{\otimes n}$;*
- (ii) *the Gram operator is exactly $G_2 I_{\mathfrak{E}_n}$;*
- (iii) *the physical magnetic recompression is strictly positive;*
- (iv) *the channel and ultraviolet diagonal bounds are uniform;*
- (v) *the residual sector is empty and the channel-residual block vanishes;*
- (vi) *the complete ultraviolet-infrared mixed norm equals the one-shell norm;*
- (vii) *the canonical vacuum-line bordism has exact overlap $\nu_* = 1$;*
- (viii) *the common Article-I threshold profile (C22) gives*

$$\widehat{H}_n \geq 0.24975(I - P_{\text{vac},n});$$

- (ix) *the stronger direct tensor-sum estimate*

$$\widehat{H}_n \geq 3.093219423836 \dots (I - P_{\text{vac},n})$$

also holds.

Consequently, the canonical recursive extension is verified end to end for the complete physicalized package in this calibrated family. This theorem constructs a specific fixed-stratum depth branch and makes no full-scale cofinality or universal surjectivity claim for restriction maps on arbitrary good diagrams.

Remark C.9 (Twisted scale-bordism handoff). The construction above fixes the complete analytic package and the single canonical vacuum-line bordism correspondence whose vacuum restriction determines the projective comparisons. Appendix P pulls those comparisons through the nontrivial categorical central extension determined by the generator of $H^3(B\mathbb{Z}_2; U(1))$ in the fixed center band. The resulting branch has the same finite package and coercive profile, but no globally coherent center-band-preserving strict unitary lift.

Appendix D: Center-graded coefficient lines and higher vacuum transport

1. Purpose and corrected categorical level

Appendix O constructs the phase-free scale-bordism correspondence

$$\mathbb{J}_{n;k} : \mathcal{H}_n \hat{\otimes} \mathcal{V}^{\otimes k} \longrightarrow \mathcal{H}_{n+k}$$

that transports the complete recursively physicalized package. The twisted vacuum structure is not a second family of refinement maps and is not a globally chosen family of ordinary unitary lifts with scalar multiplication laws. It is the coefficient-line-valued higher composition carried by the same bordisms before their one-dimensional coefficient lines are globally trivialized.

The distinction is essential. If ordinary maps U_a were globally chosen and satisfied $U_b U_a = v_{b,a} U_{ba}$, strict associativity of operator multiplication would force $\delta v = 1$. A nontrivial associator can therefore appear only while the coefficient lines, their parenthesizations, and their local tensorators remain explicit.

Definition D.1 (Ambient coefficient-line correspondence bicategory). The higher transport is interpreted in the bicategory **HilbLineCorr** $_{\Omega}$. Its objects are finite Hilbert spaces. A one-morphism $\mathcal{H} \rightarrow \mathcal{K}$ is a pair (L, U) , where L is an invertible Hilbert line and $U : \mathcal{H} \hat{\otimes} L \rightarrow \mathcal{K}$ is unitary. A two-morphism $(L, U) \Rightarrow (L', U')$ is a unitary line map $q : L \rightarrow L'$ satisfying

$$U' (I_{\mathcal{H}} \hat{\otimes} q) = U.$$

Horizontal composition tensors the coefficient lines and composes the correspondences. Its associator is the Hilbert tensor associator whiskered by the coefficient reassociator α^{Ω} . Thus a central scalar on a coefficient line is a two-endomorphism of a composite correspondence; it is not an equality between globally evaluated ordinary operators.

2. Batch refinements and the canonical coefficient-line correspondence

Let $\mathfrak{R}_{\text{bat}}$ have objects $n \in \mathbb{N}$ and arrows

$$a_{n;k} : n \longrightarrow n + k, \quad k \in \mathbb{N}_0,$$

with the evident batch composition. The coefficient line of $a_{n;k}$ is

$$L_{n;k} = \mathcal{V}^{\otimes k}. \quad (\text{D1})$$

The invariant vacuum arrow is the restriction

$$\mathbb{U}_{n;k}^{\text{vac}} : \mathcal{V}_n \hat{\otimes} L_{n;k} \longrightarrow \mathcal{V}_{n+k} \quad (\text{D2})$$

of $\mathbb{J}_{n;k}$. It is unitary because both source and target are the complete tensor-product ground sectors.

For composable arrows $a = a_{n;k}$ and $b = a_{n+k;\ell}$, ordinary tensor concatenation gives the phase-free identity

$$\mathbb{U}_b^{\text{vac}} \left(\mathbb{U}_a^{\text{vac}} \hat{\otimes} I_{L_b} \right) = \mathbb{U}_{ba}^{\text{vac}} \left(I_{\mathcal{V}_n} \hat{\otimes} m_{b,a}^0 \right), \quad (\text{D3})$$

where $m_{b,a}^0 : L_a \hat{\otimes} L_b \rightarrow L_{ba}$ is the strict tensor-concatenation unitary. Equation (D3) is the underlying unmarked Hilbert identity. It does not by itself define the twisted higher compositor.

Proposition D.2 (Exact phase-free pair composition). *Equation (D3) holds on the complete vacuum sectors and is the vacuum restriction of the full-Hilbert-space identity in Appendix O.*

Proof. Both sides concatenate the same old-shell tensor with the same ordered list of new-shell vacuum tensors. $\square$

3. The center band and its pointed coefficient category

The fundamental Wilson character is center odd:

$$\chi_{1/2}(-U) = -\chi_{1/2}(U).$$

Hence

$$|a_{n;k}| = \bar{k} \in \mathbb{Z}_2. \quad (\text{D4})$$

Let $\mathcal{B}^\Omega$ be the pointed $U(1)$ -banded coefficient category with simple grades $g \in \mathbb{Z}_2$, strict unit, and reassociator

$$\alpha_{g,h,k}^\Omega = \Omega(g, h, k)I. \quad (\text{D5})$$

The strict branch uses $\Omega_0 = 1$. The twisted branch uses

$$\Omega_1(g, h, k) = (-1)^{\tilde{g} \epsilon_2(h,k)}, \quad \Omega_1(1, 1, 1) = -1. \quad (\text{D6})$$

Appendix H proves $\delta\Omega_1 = 1$.

For a finite depth simplex D_N , pull $\mathcal{B}^\Omega$ back along $|\cdot|$. Its associator on $i < j < k < \ell$ is

$$\hat{\Omega}_N(i, j, k, \ell) = \Omega(|a_{k\ell}|, |a_{jk}|, |a_{ij}|). \quad (\text{D7})$$

A local coefficient-line lifting chart is a normalized two-cochain Λ_N satisfying

$$\Lambda_N(i, k, \ell)\Lambda_N(i, j, k) = \hat{\Omega}_N(i, j, k, \ell)\Lambda_N(i, j, \ell)\Lambda_N(j, k, \ell). \quad (\text{D8})$$

Equivalently, $\delta\Lambda_N = \hat{\Omega}_N$. The rooted formula

$$\Lambda_N(i, j, k) = \begin{cases} 1, & i = 0, \\ \hat{\Omega}_N(0, i, j, k), & i > 0 \end{cases} \quad (\text{D9})$$

solves (D8). Thus every finite depth chart admits a strong monoidal trivialization of the pulled-back coefficient lines, while no grade-only tensorator exists for Ω_1 .

4. The induced associator as a coefficient-line two-cell

For three composable refinements a, b, c , the two parenthesized coefficient-line domains are

$$\mathcal{V}_{s(a)} \hat{\otimes} ((L_a \hat{\otimes} L_b) \hat{\otimes} L_c)$$

and

$$\mathcal{V}_{s(a)} \hat{\otimes} (L_a \hat{\otimes} (L_b \hat{\otimes} L_c)).$$

The comparison between them is the formal coefficient reassociator $I \hat{\otimes} \alpha_{|c|, |b|, |a|}^\Omega$. Applying the canonical vacuum correspondence to the two parenthesized domains produces a central unitary two-cell between the two triple-composition pastings. It is not an equality of globally evaluated ordinary maps.

Theorem D.3 (Vacuum associator induced from the coefficient lines). *The coefficient-line-valued triple composition of the canonical vacuum bordisms has associator two-cell*

$$\boxed{\mathbf{a}_{c,b,a}^{\text{vac}} = \mathbb{U}_{cba}^{\text{vac}} \circ \left(I \hat{\otimes} \alpha_{|c|,|b|,|a|}^{\Omega} \right) = \Omega(|c|, |b|, |a|) \mathbb{U}_{cba}^{\text{vac}}} \quad (\text{D10})$$

as a two-cell between the two parenthesized coefficient-line correspondences. No inverse of the full isometry $\mathbb{J}_{cba}$ is used. After a local tensorator Λ_N is chosen, equation (D8) is exactly the coherence law that turns this higher correspondence into an ordinary chartwise Hilbert realization.

Proof. The phase-free tensor concatenations agree after the two source parenthesizations are related by the standard Hilbert associator. The only additional two-cell in the banded coefficient system is α^{Ω} , which is the central scalar $\Omega(|c|, |b|, |a|)$. Postcomposition with the vacuum restriction of $\mathbb{J}_{cba}$ therefore gives (D10). On a finite lifting chart, the strong monoidal functor with tensorator Λ_N converts the twisted coefficient parenthesization into the strict one precisely when $\delta\Lambda_N = \hat{\Omega}_N$. $\square$

Proposition D.4 (Odd triple reassociation). *For three elementary shell insertions,*

$$\mathbf{a}_{c,b,a}^{\text{vac}} = -I \quad (\text{D11})$$

on the coefficient-line triple composite, because $\Omega_1(1, 1, 1) = -1$.

5. Projective quotient and local Hilbert charts

Projectivizing the vacuum restriction of $\mathbb{J}_{n;k}$ gives

$$\overline{U}_{n;k}^{\text{vac}} \in \text{PU}(\mathcal{V}_n, \mathcal{V}_{n+k}).$$

The phase-free identity implies

$$\overline{U}_{n+k;\ell}^{\text{vac}} \overline{U}_{n;k}^{\text{vac}} = \overline{U}_{n;k+\ell}^{\text{vac}}. \quad (\text{D12})$$

Theorem D.5 (Coherence and local Hilbert realizability). *The coefficient-line associators satisfy the pentagon. On every finite depth simplex, the rooted tensorator Λ_N produces an ordinary Hilbert realization of the higher vacuum transport. Different local tensorators are related by one- and two-coherent coefficient-line gauges. The projective quotient remains the strict batch-refinement category.*

Proof. The pentagon is $\delta\Omega_1 = 1$. Local realizability follows from (D9) and (D8). The transition hierarchy is the lifting two-groupoid of Appendix I; its object connectedness and transition connectedness are proved explicitly in Theorem I.5. Projectivization removes all central coefficient-line two-cells. $\square$

6. The band-local obstruction

A band-local global trivialization is, by Theorem F.7, a shell-local stationary coefficient-monoidal tensorator and hence the pullback of a normalized $\lambda : \mathbb{Z}_2^2 \rightarrow U(1)$. It would satisfy

$$\delta\lambda = \Omega_1.$$

But $(\delta\lambda)(1, 1, 1) = 1$ for every normalized λ , whereas $\Omega_1(1, 1, 1) = -1$.

Theorem D.6 (Center-banded coefficient-line obstruction). *The twisted coefficient-line vacuum transport has no global trivialization whose tensorator factors through the fixed $\mathbb{Z}_2$ band. Its banded class is*

$$[\Omega_1]_{\text{band}} \neq 0 \quad \text{in} \quad H^3(B\mathbb{Z}_2; U(1)).$$

Nevertheless, every finite depth chart has an ordinary Hilbert realization through the arrow-dependent tensorator Λ_N , which is background natural but not shell stationary.

Warning D.7 (Banded, not absolute, nontriviality). The depth nerve is contractible, so the pullback of Ω_1 admits arrow-dependent trivializations. The rooted trivialization depends on the absolute base depth and fails shell stationarity. The obstruction is to one shell-local stationary tensorator on the fixed center band. Repair IX.1 therefore produces a two-presheaf of local Hilbert realizations rather than denying unbanded strictification.

7. Compatibility with the complete physicalized package

Theorem D.8 (End-to-end coefficient-line-valued twisted branch). *The recursively physicalized $SU(2)$ family of Theorem C.8 admits a twisted higher vacuum descent such that:*

- (i) *every depth carries the complete physicalized package;*
- (ii) *every invariant arrow is the vacuum restriction $\mathbb{U}_a^{\text{vac}} : \mathcal{V}_{s(a)} \otimes L_a \rightarrow \mathcal{V}_{t(a)}$;*

- (iii) *the coefficient-line associator is induced by the fixed reassociator Ω_1 and gives $-I$ on three odd insertions;*
- (iv) *every finite depth chart has an ordinary Hilbert realization;*
- (v) *no grade-only global tensorator trivializes the twisted coefficient law;*
- (vi) *the projective transport, finite analytic profile, and both coercive bounds are unchanged.*

Proof. Appendix O gives the finite package, the canonical correspondence, and the bounds. The induced two-cell is Theorem D.3; local realizability and pentagon coherence are Theorem D.5; the band-local obstruction is Theorem D.6; analytic invariance is Appendix R. $\square$

With $\Omega_0 = 1$, the same correspondence has a global strict coefficient-line trivialization and gives the control branch. With Ω_1 , the invariant finite arrows remain the same, but their higher coefficient-line realization is the fibred descent object of Appendix V. The finite Hamiltonian does not by itself choose between these two global coefficient laws.

Appendix E: Coefficient-line diagram-level vacuum type and inverse-system stability

1. Purpose and corrected logical level

Vacuum reassociators are defined on composable triples of coefficient-line scale comparisons. They are therefore invariants of coherent finite higher diagrams and compatible higher pseudosections, not of isolated Hamiltonian packages. This appendix uses one formalism throughout: the invariant arrow is

$$\mathbb{U}_a^{\text{vac}} : \mathcal{V}_{s(a)} \hat{\otimes} L_a \longrightarrow \mathcal{V}_{t(a)},$$

the coefficient lines and their multiplication remain explicit, and local ordinary Hilbert realizations are organized by a lifting two-presheaf. No globally chosen ordinary unitary lifts, scalar compositors, or scalar associator cells are included as independent finite-diagram coordinates.

The underlying directed depth category has contractible nerve. Consequently, the pullback of the cyclic cocycle to a finite rooted depth chart can be trivialized by an arrow-dependent

tensorator. The nontrivial obstruction in this article is relative to the center-natural, shell-local, translation-stationary, coefficient-monoidal equivalence category of Appendix F. The characterization theorem there identifies those global higher gauges exactly with normalized cochains on the fixed band $\mathbb{Z}_2$. This restriction is additional physical locality structure; it is not inferred from center naturality alone.

2. Finite coefficient-line vacuum descent diagrams

Let D be a finite demand subcategory. A *coefficient-line coherent vacuum diagram* over D consists of

$$\mathfrak{D}_D^{\text{vac}} = \left(\{\mathcal{V}_x\}_{x \in D_0}, \{L_a\}_{a \in D_1}, \{m_{b,a}\}_{(b,a) \in D_2}, \alpha_D^\Omega, \{\mathbb{U}_a^{\text{vac}}\}_{a \in D_1}, \mathbf{Triv}_{\Omega_D}, |\cdot|_D, \mathcal{B}_D^\Omega \right), \quad (\text{E1})$$

where:

- (i) $\mathcal{V}_x = \text{Ran } P_{\text{vac},x}$ is the complete finite vacuum sector;
- (ii) L_a is the invertible coefficient line of the refinement arrow a , and

$$m_{b,a} : L_b \hat{\otimes} L_a \longrightarrow L_{ba}$$

is its coefficient multiplication;

- (iii) α_D^Ω is the pulled-back reassociator, with component

$$\alpha_{c,b,a}^\Omega = \Omega_D(|c|_D, |b|_D, |a|_D)I, \quad \delta\Omega_D = 1; \quad (\text{E2})$$

- (iv) $\mathbb{U}_a^{\text{vac}} : \mathcal{V}_{s(a)} \hat{\otimes} L_a \rightarrow \mathcal{V}_{t(a)}$ is the vacuum restriction of the canonical scale-bordism correspondence;
- (v) $|\cdot|_D : D \rightarrow B\mathbb{Z}_2$ is the shell-parity grade and $\mathcal{B}_D^\Omega$ is the fixed pointed coefficient category carrying that grade and reassociator;
- (vi) $\mathbf{Triv}_{\Omega_D}$ is the two-presheaf on root-preserving finite simplicial charts $E \subset N(D)$ whose value is the lifting two-groupoid $\text{Triv}_{\Omega_D|E}(E)$. Its objects are local tensorators Λ with $\delta\Lambda = \hat{\Omega}_D|_E$, its one-morphisms are κ -transitions, and its two-morphisms are h -modifications.

The projective comparisons $\overline{U}_a^{\text{vac}}$ are derived shadows of $\mathbb{U}_a^{\text{vac}}$; they are not substitutes for the coefficient-line arrow data. Likewise, a chosen Λ is a local Hilbert chart, not an invariant coordinate of $\mathfrak{D}_D^{\text{vac}}$.

Definition E.1 (Band-local equivalence of coefficient-line diagrams). Two diagrams $\mathfrak{D}_D^{\text{vac}}$ and $\mathfrak{D}_D^{\text{vac}'}$ are *band-locally equivalent* when they are related by:

- (i) objectwise vacuum unitaries extending naturally over the center-background groupoid;
- (ii) a strong monoidal equivalence of coefficient systems whose tensorator is shell local, translation stationary, and coefficient monoidal, hence is the pullback of a unique normalized $\lambda : \mathbb{Z}_2^2 \rightarrow U(1)$;
- (iii) the induced two-natural equivalence between $\mathbf{Triv}_{\Omega_D}$ and $\mathbf{Triv}_{\Omega'_D}$, together with compatible intertwining of the coefficient-line vacuum correspondences.

The reassociator representative changes by

$$\Omega'_D = (\delta\lambda)\Omega_D.$$

Local κ - and h -transitions inside one lifting two-presheaf are internal changes of Hilbert chart. They do not change the coefficient law or its band class. Arbitrary depth-dependent tensorators may be objects of a local lifting chart, but they are not global equivalences unless they satisfy the shell-local conditions.

This is the equivalence relation relevant to center-graded reconstruction. Theorem F.9 derives the shell grade from the covariance of the actual center-twisted seed fibers and the odd fundamental Wilson marking. Theorem F.7 then characterizes the permitted monoidal transformations of the repeated-shell coefficient law.

3. Diagram-level induced vacuum type

Let $\mathfrak{t}_{\text{sc}}^{\text{vac}}(\mathfrak{D}_D)$ be the induced relative refinement-to-admissible class of the analytic bordism diagram, and let $[\Omega_D]_{\text{band}}$ be the band class of the coefficient reassociator. Define

$$\boxed{\text{Type}_{\text{vac}}(\mathfrak{D}_D) = (\mathfrak{t}_{\text{sc}}^{\text{vac}}(\mathfrak{D}_D), [\Omega_D]_{\text{band}})}. \quad (\text{E3})$$

No vacuum type is assigned to an isolated package, a projective shadow, or a chosen local tensorator.

Proposition E.2 (Representative invariance). *The type (E3) is invariant under band-local equivalence.*

Proof. The relative component changes by a mapping-fiber coboundary. The monoidal coefficient equivalence changes the reassociator by $\delta\lambda$ for a normalized band cochain λ . The induced two-natural equivalence of lifting presheaves changes only the choice of local Hilbert charts. Hence both cohomology classes in (E3) are unchanged. $\square$

4. Restriction stability of the complete coefficient-line datum

Let $i : D \hookrightarrow D'$ be an inclusion of finite demand diagrams. Restriction forgets data outside D and acts on every surviving level:

$$\text{res}_{D',D} : \mathfrak{D}_{D'}^{\text{vac}} \longmapsto \mathfrak{D}_D^{\text{vac}}. \quad (\text{E4})$$

It restricts the vacuum sectors, coefficient lines, multiplication maps, reassociator, grade, coefficient category, and vacuum correspondences. On local lifting data it is the root-preserving cochain-restriction two-functor

$$\mathbf{Triv}_{\Omega_{D'}}|_D \longrightarrow \mathbf{Triv}_{\Omega_D}. \quad (\text{E5})$$

Theorem E.3 (Restriction stability of coefficient-line vacuum type). *Suppose $\mathfrak{D}_{D'}^{\text{vac}}$ has marked type $\mathfrak{v} = (\mathfrak{t}, [\Omega]_{\text{band}})$. Then*

$$\text{Type}_{\text{vac}}(\text{res}_{D',D} \mathfrak{D}_{D'}^{\text{vac}}) = \mathfrak{v}|_D. \quad (\text{E6})$$

For a fixed global band and relative marking, $\mathfrak{v}|_D$ denotes the same marked class represented on the smaller diagram.

Proof. Restriction commutes with the mapping-fiber differential and with pullback of the coefficient cocycle. It preserves the fixed grade functor and restricts the coefficient multiplication and vacuum correspondences. Cochain restriction sends objects, one-morphisms, and two-morphisms of each lifting two-groupoid to the corresponding data on the smaller chart. Thus both type components and the attached higher-descent object restrict functorially. $\square$

Corollary E.4 (No diagram-level type jumping). *A compatible inverse system of coefficient-line coherent diagrams cannot change from the trivial band class to $[\Omega_1]_{\text{band}}$, or between distinct marked cyclic classes, under restriction.*

5. Fixed-type inverse systems

For a finite demand diagram D , let

$$\mathfrak{X}_D^{\text{good}, \mathfrak{v}} = \left\{ [\mathfrak{D}_D^{\text{cl}}] : \mathfrak{D}_D^{\text{cl}} \text{ is a good coefficient-line physical diagram and } \text{Type}_{\text{vac}}(\mathfrak{D}_D^{\text{vac}}) = \mathfrak{v}|_D \right\}. \quad (\text{E7})$$

The brackets denote band-local equivalence classes of the full analytic and higher-descent datum. Restriction is the map induced by (E4) together with restriction of the analytic packages and bordisms.

Theorem E.5 (Directed fixed-type inverse-limit selection). *Assume:*

- (i) *the finite demand shapes form a directed poset under common finite extension and test every finite part of the full scale category;*
- (ii) *one finite good coefficient-line diagram of marked type $\mathfrak{v}$ exists;*
- (iii) *every $\mathfrak{X}_D^{\text{good}, \mathfrak{v}}$ is compact Hausdorff;*
- (iv) *for every $D \subset D'$, the full coefficient-line restriction map is continuous and surjective, with the inverse-system identities.*

Then

$$\varprojlim_{D \in \mathcal{J}_{\text{fin}}} \mathfrak{X}_D^{\text{good}, \mathfrak{v}} \neq \emptyset. \quad (\text{E8})$$

Every element is a full-scale cofinal physical higher pseudosection whose vacuum type is the compatible coefficient-line class $\mathfrak{v}$.

Proof. Apply the finite-intersection argument of Theorem IV.5 to the fixed-type subsystem. Directed common extension and the seed imply nonemptiness of every factor. In the compact product, each full bonding equation is closed because the target is Hausdorff and restriction is continuous. A common finite extension realizes any finite set of bonding equations. Type preservation follows from Theorem E.3; it is not inserted as an objectwise label. No countable cofinal chain and no scalar-cell replacement are used. $\square$

6. The explicit fixed-stratum recursive twisted type

For the recursively physicalized family of Appendices O and P, each finite depth diagram inherits the shell-parity grade, coefficient lines, multiplication maps, canonical vacuum correspondences, and the pulled-back reassociator

$$\Omega_D(g, h, k) = \Omega_1(g, h, k), \quad [\Omega_1]_{\text{band}} \neq 0. \quad (\text{E9})$$

Its lifting two-presheaf is the one constructed in Appendix V. The relative component of the separated-shell comparison system is chosen trivial, hence

$$\boxed{\text{Type}_{\text{vac}}(\mathfrak{D}_D) = (0, [\Omega_1]_{\text{band}})} \quad (\text{E10})$$

for every finite diagram in the fixed-stratum depth system.

Theorem E.6 (Banded obstruction theorem for the fixed-stratum depth family). *The recursive coefficient-line vacuum depth system admits no band-local global Hilbert trivialization. Equivalently, there is no normalized $\lambda : \mathbb{Z}_2^2 \rightarrow U(1)$ satisfying*

$$\delta\lambda = \Omega_1. \quad (\text{E11})$$

Its compatible diagram-level type is $(0, [\Omega_1]_{\text{band}})$, while its local lifting two-presheaf is nonempty on every finite rooted depth chart.

Proof. For every normalized band cochain,

$$(\delta\lambda)(1, 1, 1) = 1,$$

whereas $\Omega_1(1, 1, 1) = -1$. Thus the band class cannot be removed by an admissible global coefficient tensorator. Local nonemptiness follows from the rooted tensorator of Appendix V, and compatibility under restriction follows from Theorem E.3. $\square$

7. Why the band qualification is necessary

For the standard finite depth simplex with vertices $0 < 1 < \dots < N$, write

$$\hat{\Omega}(i, j, k, \ell) = \Omega_1(|k \rightarrow \ell|, |j \rightarrow k|, |i \rightarrow j|).$$

Define the arrow-dependent two-cochain

$$\Lambda(i, j, k) = \begin{cases} 1, & i = 0, \\ \hat{\Omega}(0, i, j, k), & i > 0. \end{cases} \quad (\text{E12})$$

Proposition E.7 (Explicit local trivialization on a finite depth simplex). *The cochain (E12) satisfies*

$$\delta\Lambda = \hat{\Omega}.$$

Hence it is an object of the local lifting two-groupoid $\text{Triv}_{\Omega_1}(D_N)$, and the pulled-back coefficient law has an ordinary Hilbert realization on that finite chart.

Proof. If $i > 0$, the identity is the cocycle equation for $\hat{\Omega}$ evaluated on $(0, i, j, k, \ell)$. If $i = 0$, all terms except $\Lambda(j, k, \ell)$ are normalized to one, and the same identity follows directly. $\square$

Warning E.8 (Local lifting charts and band-local equivalences differ). The theorem above does not assert nontriviality in the ordinary cohomology of the bare directed depth nerve. That nerve is contractible. The rooted cochain is background natural but depends on the absolute base depth and fails shell stationarity, as shown in Proposition F.8. It is therefore a valid object of the local lifting two-presheaf but not a global monoidal equivalence of the stationary repeated-shell coefficient law.

8. Unified output

The repaired logical chain is

$$\begin{aligned} &\text{complete physicalized finite diagram} \longrightarrow \text{canonical coefficient-line scale bordisms} \\ &\longrightarrow (\mathcal{L}_D^\Omega, \mathbf{Triv}_{\Omega_D}) \longrightarrow \text{Type}_{\text{vac}}(\mathfrak{D}_D) = (\mathfrak{t}_{\text{sc}}^{\text{vac}}, [\Omega_D]_{\text{band}}). \end{aligned}$$

The same object is used in the good-diagram loci, restriction maps, fixed-type inverse systems, and explicit depth realization. Under the hypotheses of Theorem E.5, it passes to a full cofinal inverse-limit higher pseudosection; in the explicit model it passes only to the compatible fixed-stratum depth system. At no stage is the coefficient-line descent replaced by an independent scalar-cell pseudosection.

Appendix F: Center-background covariance and physical admissibility of the band

1. Purpose and the correct symmetry statement

The four-character seed is not invariant under the center involution on a single fixed magnetic-background fiber. Indeed, the fundamental Wilson multiplier is center odd. The correct finite statement is therefore *covariance over the two center-twisted background fibers*, not a false commutator relation on one Hamiltonian. This appendix constructs that background completion, proves equivariance of the complete physicalized package and of the canonical scale-bordism correspondence, and uses it to define physical admissibility.

The original recursive branch is the untwisted-background fiber of this completion. Hence none of its finite matrices or coercive constants is changed.

2. The center involution and the two seed fibers

On the retained character space $\mathcal{K} = \text{span}\{\chi_0, \chi_{1/2}, \chi_1, \chi_{3/2}\}$, define the unitary center involution

$$Z\chi_j = (-1)^{2j}\chi_j, \quad Z = \text{diag}(1, -1, 1, -1). \quad (\text{F1})$$

Let X be the compression of multiplication by the fundamental character, so that

$$H_{\text{mag}}^{(0)} = I - \frac{1}{2}X.$$

Then

$$ZH_{\text{el}}Z = H_{\text{el}}, \quad ZXZ = -X. \quad (\text{F2})$$

For $\epsilon \in \mathbb{Z}_2$, define the center-background fiber

$$H^{[\epsilon]} = H_{\text{el}} + I - \frac{(-1)^\epsilon}{2}X. \quad (\text{F3})$$

Thus $H^{[0]}$ is the seed Hamiltonian of Appendix A, whereas $H^{[1]}$ has the opposite fundamental Wilson sign. Equation (F2) gives the exact covariance law

$$\boxed{ZH^{[\epsilon]}Z = H^{[\epsilon+1]}.} \quad (\text{F4})$$

Let E_0 be the common ground energy and set

$$P^{[\epsilon]} = \mathbf{1}_{\{E_0\}}(H^{[\epsilon]}), \quad \mathcal{V}^{[\epsilon]} = \text{Ran } P^{[\epsilon]}.$$

Functional calculus in (F4) yields

$$P^{[\epsilon+1]} = ZP^{[\epsilon]}Z, \quad Z : \mathcal{V}^{[\epsilon]} \xrightarrow{\cong} \mathcal{V}^{[\epsilon+1]}. \quad (\text{F5})$$

3. Equivariance of the complete physicalized seed package

Let $F^{W,[0]} = F^W$ be the Wilson channel of Appendix A and define

$$F^{W,[\epsilon]} = Z^\epsilon F^{W,[0]}. \quad (\text{F6})$$

Define $T^{[\epsilon]}$, $G^{[\epsilon]}$, $F^{\text{phys},[\epsilon]}$, the channel and ultraviolet projections, and the mixed block exactly as in Appendix A, using the fiber $H^{[\epsilon]}$ and projection $P^{[\epsilon]}$.

Theorem F.1 (Center covariance of the complete seed package). *For $\epsilon \in \mathbb{Z}_2$,*

$$T^{[\epsilon+1]} = ZT^{[\epsilon]}, \quad G^{[\epsilon+1]} = G^{[\epsilon]}, \quad (\text{F7})$$

$$F^{\text{phys},[\epsilon+1]} = ZF^{\text{phys},[\epsilon]}, \quad p_{\text{ch}}^{[\epsilon+1]} = Zp_{\text{ch}}^{[\epsilon]}Z, \quad (\text{F8})$$

$$p_{\text{uv}}^{[\epsilon+1]} = Zp_{\text{uv}}^{[\epsilon]}Z, \quad K^{[\epsilon+1]} = ZK^{[\epsilon]}Z. \quad (\text{F9})$$

The physical magnetic recompression, every block lower bound, the mixed-block norm, the Gram bounds, and the coercive constant are identical in the two fibers.

Proof. Equations (F4), (F5), and (F6) give

$$(I - P^{[\epsilon+1]})F^{W,[\epsilon+1]} = Z(I - P^{[\epsilon]})F^{W,[\epsilon]}.$$

Taking adjoints proves equality of the Gram operators, after which the physicalized-channel identity follows by functional calculus. All projections and mixed blocks are obtained from these operators by products, adjoints, and spectral compression, so they transform by conjugation. Unitary conjugation preserves spectra, singular values, norms, and quadratic form lower bounds. $\square$

4. The recursive center-background groupoid

At depth n , let

$$\mathcal{B}_n = (\mathbb{Z}_2)^n$$

be the set of shellwise center backgrounds. For $\epsilon = (\epsilon_1, \dots, \epsilon_n)$, set

$$\mathcal{H}_{n,\epsilon} = \mathcal{K}^{\otimes n}, \quad H_{n,\epsilon} = \sum_{m=1}^n \left(H^{[\epsilon_m]} \right)^{(m)}, \quad P_{n,\epsilon} = \bigotimes_{m=1}^n P^{[\epsilon_m]}. \quad (\text{F10})$$

For $\eta \in (\mathbb{Z}_2)^n$, define

$$Z_\eta = \bigotimes_{m=1}^n Z^{\eta_m}, \quad \eta \cdot \epsilon = \epsilon + \eta.$$

Then

$$Z_\eta H_{n,\epsilon} Z_\eta^{-1} = H_{n,\eta \cdot \epsilon}, \quad Z_\eta P_{n,\epsilon} Z_\eta^{-1} = P_{n,\eta \cdot \epsilon}. \quad (\text{F11})$$

The anchored coefficient spaces and all remaining physicalized package components are defined fiberwise as in Appendix C. Theorem F.1 tensorizes, so the complete recursive package is a functor on the action groupoid

$$\mathcal{B}_n / (\mathbb{Z}_2)^n.$$

All fibers carry the same quantitative profile and the same two coercive bounds.

For a refinement adding a background word $\tau \in (\mathbb{Z}_2)^k$, define

$$L_\tau = \mathcal{V}^{[\tau_1]} \hat{\otimes} \dots \hat{\otimes} \mathcal{V}^{[\tau_k]}.$$

The canonical correspondence becomes

$$\mathbb{J}_{n;k}^{\epsilon;\tau} : \mathcal{H}_{n,\epsilon} \hat{\otimes} L_\tau \longrightarrow \mathcal{H}_{n+k,\epsilon \oplus \tau}. \quad (\text{F12})$$

It is the same tensor-concatenation correspondence as before.

Theorem F.2 (Center naturality of the canonical scale bordism). *For $\eta \in (\mathbb{Z}_2)^n$ and $\theta \in (\mathbb{Z}_2)^k$,*

$$\boxed{Z_{\eta \oplus \theta} \mathbb{J}_{n;k}^{\epsilon;\tau} = \mathbb{J}_{n;k}^{\epsilon + \eta; \tau + \theta} \left(Z_\eta \hat{\otimes} Z_\theta |_{L_\tau} \right)}. \quad (\text{F13})$$

The equality holds on the full correspondence domain and on the complete vacuum restriction.

Proof. Both sides apply the same tensor-factor center involutions and then concatenate the same factors. Tensor concatenation is natural with respect to tensor products of unitaries. Equation (F5) gives the vacuum restriction. $\square$

5. Why the refinement grade is physical

The elementary Wilson-shell bordism is marked by the fundamental Wilson multiplier X on the newly added shell. Its center parity is the exact operator identity

$$ZXZ^{-1} = -X. \quad (\text{F14})$$

Thus an elementary marked refinement has grade one. Tensor composition of k marked shell refinements has grade

$$|a_{n;k}| = \bar{k} \in \mathbb{Z}_2. \quad (\text{F15})$$

This is the grade used in Appendix D. It is not inferred from a preferred vacuum vector.

Definition F.3 (Center-background-natural marked equivalence). A marked equivalence between two recursive systems is a family of unitaries on every background fiber which:

- (i) intertwines the complete physicalized package;
- (ii) is natural for the background transformations Z_η ;
- (iii) intertwines the canonical correspondences $\mathbb{J}_{n;k}^{\epsilon;\tau}$;
- (iv) preserves the center parity of every marked Wilson-shell multiplier.

This definition concerns object and arrow data. It does not, by itself, restrict a scalar higher gauge to depend only on the center grade.

Proposition F.4 (Invariance of the center grade). *A center-background-natural marked equivalence cannot convert an odd marked shell refinement into an even one. Consequently it preserves the grade functor $|\cdot| : \mathfrak{R}_{\text{bat}} \rightarrow \mathbb{Z}_2$.*

Proof. Let Y be a marked operator and suppose $ZYZ^{-1} = (-1)^q Y$. Naturality of a marked unitary U with respect to the center action and intertwining of the marking give

$$Z'(UYU^{-1})(Z')^{-1} = U(ZYZ^{-1})U^{-1} = (-1)^q UYU^{-1}.$$

Hence the parity q is invariant. Apply this to $Y = X$, for which $q = 1$ by (F14), and use additivity under tensor composition. $\square$

6. Shell-local, stationary higher equivalence

Center-background naturality preserves the grade, but it does not force a scalar higher gauge to be grade only. The additional restriction used in this article is a locality and stationarity condition on binary shell junctions. It is stated separately so that no implication from center naturality alone is asserted.

Write $a_{n;k} : n \rightarrow n + k$ for the batch bordism adding k identical separated Wilson shells. A normalized scalar higher gauge on composable batch arrows is a family

$$\Lambda_{n;k,\ell} \in U(1), \quad (a_{n;k}, a_{n+k;\ell}),$$

with $\Lambda_{n;0,\ell} = \Lambda_{n,k,0} = 1$.

Definition F.5 (Shell-local stationary coefficient-monoidal gauge). A normalized scalar higher gauge Λ is *physically local at the shell junction* if:

- (i) it is independent of the center-background word and hence is natural under every Z_η ;
- (ii) it is stationary under insertion of a common prefix of separated shells,

$$\Lambda_{n+r;k,\ell} = \Lambda_{n;k,\ell}; \tag{F16}$$

- (iii) it respects the canonical neutral-pair identifications in the fixed coefficient band,

$$\Lambda_{n;k+2,\ell} = \Lambda_{n;k,\ell}, \quad \Lambda_{n;k,\ell+2} = \Lambda_{n;k,\ell}. \tag{F17}$$

Condition (F16) says that a local junction gauge cannot depend on the absolute depth at which two identical shell bordisms are composed. Condition (F17) says that adjoining a canonically center-neutral pair, whose coefficient object is identified with the tensor unit, cannot change that local gauge.

Remark F.6 (Coefficient locality does not erase physical shells). The neutral-pair condition concerns only the scalar automorphism of the coefficient-line multiplication. It does not identify the physical bordism $\mathbb{J}_{n;2}$ with an identity arrow, remove its Hamiltonian factors, or alter any finite package. Those data remain in the scale-bordism correspondence. The condition says only that a local gauge of the fixed $\mathbb{Z}_2$ coefficient law cannot distinguish two coefficient objects that are canonically identified in that law.

The two requirements are the discrete separated-shell form of locality, translation stationarity, and monoidality under disjoint shell union. They exclude phases that use the global history of the exhaustion rather than the coefficient objects meeting at the junction.

Theorem F.7 (Characterization of physically local higher gauges). *For a normalized scalar higher gauge Λ , the following are equivalent:*

- (i) Λ is shell-local, stationary, and coefficient-monoidal in the sense of Definition F.5;
- (ii) there is a unique normalized cochain $\lambda : \mathbb{Z}_2^2 \rightarrow U(1)$ such that

$$\Lambda_{n;k,\ell} = \lambda(\bar{\ell}, \bar{k}) \tag{F18}$$

for every n, k, ℓ .

Thus the physically local higher gauges are exactly the pullbacks of normalized cochains on the fixed coefficient band.

Proof. Stationarity removes the dependence on n . The two neutral-pair identities then reduce k and ℓ independently modulo two. Hence

$$\lambda(\bar{\ell}, \bar{k}) := \Lambda_{0;k,\ell}$$

is well defined, normalized, unique, and gives (F18). Conversely, every pullback of a normalized band cochain is background independent, stationary in n , and unchanged when either shell count is increased by two. $\square$

Proposition F.8 (The rooted contraction is nonlocal in shell depth). *The arrow-dependent contracting cochain of Proposition E.7 is background natural but is not shell stationary. In particular, for two successive elementary odd shells,*

$$\Lambda_{0;1,1}^{\text{root}} = 1, \quad \Lambda_{1;1,1}^{\text{root}} = \Omega_1(1, 1, 1) = -1. \tag{F19}$$

Therefore it is a valid local chart trivialization of the bare depth simplex, but not an admissible higher equivalence of the stationary shell coefficient law.

Proof. The first equality is the rooted normalization at the initial object. For a junction based at depth one, formula (E12) evaluates the pulled-back cocycle on the three odd increments $0 \rightarrow 1 \rightarrow 2 \rightarrow 3$, giving -1 . The cochain is scalar and background independent, so this example also shows explicitly that center-background naturality alone does not imply band locality. $\square$

Theorem F.9 (Physical admissibility of the center band). *The center parity of the fundamental Wilson marking supplies an invariant $\mathbb{Z}_2$ grade on the recursive shell bordisms. If higher equivalences are additionally required to be shell-local, stationary, and coefficient-monoidal, then:*

- (i) *object and arrow equivalences preserve the grade functor;*
- (ii) *admissible scalar higher gauges are exactly normalized band cochains;*
- (iii) *the rooted arrow-dependent contraction does not define an admissible global higher equivalence, although it remains available on each unbanded finite lifting chart.*

Consequently $[\Omega_1]_{\text{band}}$ is intrinsic relative to this declared center-natural and shell-local higher-equivalence category.

Proof. Part (i) is Proposition F.4. Part (ii) is Theorem F.7. Part (iii) is Proposition F.8. The remaining obstruction is therefore exactly the banded cohomology class computed in Appendices D and E. $\square$

7. Scope of the result

The grade is forced by the genuine center parity of the marked Wilson shell. The restriction on higher gauges is a separate locality, stationarity, and coefficient-monoidality requirement; it is not a consequence of center covariance alone. This distinction is essential. If that higher-locality requirement is deliberately forgotten, the rooted contracting cochains strictify every finite depth simplex.

The theorem does not claim that the objectwise finite Hamiltonian selects the nontrivial reassociator Ω_1 rather than the strict law Ω_0 . Those are two distinct global coefficient laws for the same finite analytic family.

The construction is the pure $SU(2)$ prototype. Incorporating Standard Model matter requires replacing the isolated center band by the appropriate diagonal subgroup of $SU(3) \times SU(2) \times U(1)$ that acts trivially on the chosen matter representations. No such full matter-coupled extension is asserted here.

Appendix G: Central-twist invariance of the complete physicalized package

1. Purpose and scope

Appendices O–Q construct two center-banded realizations of the same recursively physicalized $SU(2)$ family: the strict control and the nontrivial coefficient reassociation whose induced vacuum associator has class $[\Omega_1]_{\text{band}}$. This appendix proves that the central associator changes only the higher scale-gluing data. It does not change any objectwise finite Hamiltonian, physicalized Wilson channel, Gram operator, magnetic recompression, diagonal block, mixed block, vacuum-overlap singular value, or coercive constant.

The statement is exact for the displayed family and, more generally, for any change of scale lift by unit-modulus central factors. It does not assert that an arbitrary noncentral categorical deformation is analytically invisible, nor that microscopic dynamics selects one central extension over another.

2. The objectwise package

At depth n , write the complete physicalized package as

$$\mathfrak{P}_n = \left(H_n, E_{0,n}, P_{\text{vac},n}, F_n^W, T_n, G_n, F_n^{\text{phys}}, B_{\text{mag},n}^{\text{phys}}, P_{\text{ch},n}, P_{\text{res},n}, P_{\text{UV},n}, K_n \right). \quad (\text{G1})$$

Let

$$\mathfrak{S}^{\text{str}} \quad \text{and} \quad \mathfrak{S}^{\text{tw}}$$

denote respectively the strict control and the center-banded twisted lift of Appendix P. By construction, they have the same objects and the same projective scale comparisons. Their only difference is the reassociation law of the center-graded coefficient lines and the vacuum associator induced from it by triple composition of the same canonical bordisms.

Proposition G.1 (Exact objectwise identity). *For every depth n ,*

$$\boxed{\mathfrak{P}_n^{\text{tw}} = \mathfrak{P}_n^{\text{str}} = \mathfrak{P}_n}. \quad (\text{G2})$$

In particular,

$$H_n^{\text{tw}} = H_n^{\text{str}}, \quad P_{\text{vac},n}^{\text{tw}} = P_{\text{vac},n}^{\text{str}}, \quad K_n^{\text{tw}} = K_n^{\text{str}}.$$

Proof. The banded coefficient reassociation of Appendix P modifies neither the depth objects nor the finite operators attached to them. It modifies only the higher reassociation cell induced when the same projective scale comparisons are composed threefold. Every entry of (G1) is defined from the objectwise Hamiltonian, complete ground projection, raw Wilson map, and their finite compressions. These data are identical in the strict and twisted realizations. $\square$

3. Unitary covariance of physicalization

The preceding proposition is already sufficient for the displayed family. The following covariance statement shows that even local changes of coefficient chart or central phase cannot alter the quantitative profile.

Let $\mathfrak{E}$ be a finite coefficient space, let

$$F^W : \mathfrak{E} \longrightarrow \mathcal{H}, \quad T = (I - P_{\text{vac}})F^W, \quad G = T^*T,$$

and assume $G > 0$. For a unitary $W \in U(\mathfrak{E})$, define

$$\tilde{F}^W = F^W W.$$

Lemma G.2 (Coefficient-unitary covariance). *The transformed physicalization satisfies*

$$\tilde{T} = TW, \quad \tilde{G} = W^*GW, \quad \tilde{F}^{\text{phys}} = F^{\text{phys}}W. \quad (\text{G3})$$

Consequently,

$$\tilde{P}_{\text{ch}} = \tilde{F}^{\text{phys}}(\tilde{F}^{\text{phys}})^* = P_{\text{ch}}. \quad (\text{G4})$$

For every self-adjoint finite operator B ,

$$(\tilde{F}^{\text{phys}})^* B \tilde{F}^{\text{phys}} = W^*(F^{\text{phys}})^* B F^{\text{phys}} W. \quad (\text{G5})$$

Hence all spectral lower bounds of the recompressed operator are unchanged.

Proof. The first two identities are immediate. Functional calculus gives

$$(W^*GW)^{-1/2} = W^*G^{-1/2}W,$$

and therefore

$$\tilde{F}^{\text{phys}} = TW(W^*GW)^{-1/2} = TG^{-1/2}W = F^{\text{phys}}W.$$

The remaining statements follow by direct substitution. $\square$

Corollary G.3 (Central phase invariance). *If $W = zI$ with $|z| = 1$, then*

$$\tilde{G} = G, \quad \tilde{P}_{\text{ch}} = P_{\text{ch}},$$

and every coefficient-space recompression is unitarily identical to the original one. In a one-dimensional coefficient channel it is numerically identical.

4. Vacuum comparison and overlap invariance

Let $a : x \rightarrow y$ be a scale arrow. On a finite lifting chart, the strict and twisted vacuum lifts differ by a central phase:

$$U_a^{\text{vac,tw}} = z_a U_a^{\text{vac,str}}, \quad z_a \in U(1). \quad (\text{G6})$$

The compressed comparison obeys the same relation,

$$C_a^{\text{vac,tw}} = z_a C_a^{\text{vac,str}}.$$

Proposition G.4 (Exact overlap invariance). *For every scale arrow,*

$$(C_a^{\text{vac,tw}})^* C_a^{\text{vac,tw}} = (C_a^{\text{vac,str}})^* C_a^{\text{vac,str}}. \quad (\text{G7})$$

Thus the complete singular-value lists and every uniform vacuum-overlap lower bound are identical in the strict and twisted branches.

Proof. The phase cancels between the adjoint and the comparison. □

For the end-to-end family,

$$\boxed{\nu_*^{\text{tw}} = \nu_*^{\text{str}} = 1.} \quad (\text{G8})$$

5. Analytic defect covariance

The central extension also leaves the analytic curvature norms unchanged. Work on a finite lifting chart and write

$$\mathbb{T}_a^{\text{tw}} = z_a \mathbb{T}_a^{\text{str}}.$$

For a composable pair (b, a) , choose the comparison lift in the twisted chart as

$$U_{\chi_{b,a}}^{\text{tw}} = \frac{z_{ba}}{z_b z_a} U_{\chi_{b,a}}^{\text{str}}. \quad (\text{G9})$$

This is exactly the central adjustment required to compare two lifts of the same projective composite.

Proposition G.5 (Phase covariance of analytic refinement defects). *Let*

$$\mathfrak{D}_{b,a}^{\text{str}} = U_{\chi_{b,a}}^{\text{str}} \mathbb{T}_b^{\text{str}} \mathbb{T}_a^{\text{str}} - \mathbb{T}_{ba}^{\text{str}}.$$

Then

$$\boxed{\mathfrak{D}_{b,a}^{\text{tw}} = z_{ba} \mathfrak{D}_{b,a}^{\text{str}}} \tag{G10}$$

Consequently,

$$\|\mathfrak{D}_{b,a}^{\text{tw}}\| = \|\mathfrak{D}_{b,a}^{\text{str}}\|. \tag{G11}$$

The analytic part of the renormalization curvature is therefore identical in the two central lifts.

Proof. Substitution of (G9) gives

$$\begin{aligned} \mathfrak{D}_{b,a}^{\text{tw}} &= \frac{z_{ba}}{z_b z_a} U_{\chi_{b,a}}^{\text{str}} (z_b \mathbb{T}_b^{\text{str}}) (z_a \mathbb{T}_a^{\text{str}}) - z_{ba} \mathbb{T}_{ba}^{\text{str}} \\ &= z_{ba} \mathfrak{D}_{b,a}^{\text{str}}. \end{aligned}$$

Since $|z_{ba}| = 1$, the norm is unchanged. □

Remark G.6 (What can change in the total curvature). The equality above concerns the analytic defect contribution. A declared total curvature functional may assign a separate positive cost to nontrivial central compositors or associators. Therefore the strict and twisted branches can have identical analytic curvature and different higher-coherence costs. This difference is a property of the chosen branch-cost functional, not of the finite Hamiltonian spectrum.

6. Invariance of the full quantitative profile

Let

$$\mathbf{q}_* = (g_*, G^*, c_{\text{ch},*}, c_{\text{res},*}, c_{\text{UV},*}, \varepsilon_{\text{sr},*}, \varepsilon_{\text{mix},*}, \rho_{\text{sr},*}, \rho_*, \nu_*)$$

be the common profile of Appendix O.

Theorem G.7 (Central-twist invariance of the complete physicalized profile). *For the strict control and the nontrivial center-banded twisted realization of the end-to-end recursive family,*

$$\boxed{\mathbf{q}_*^{\text{tw}} = \mathbf{q}_*^{\text{str}} = \mathbf{q}_*}. \quad (\text{G12})$$

More explicitly, at every depth:

- (i) *the Gram spectra are identical;*
- (ii) *the physical magnetic recompressions are identical up to coefficient unitary conjugacy;*
- (iii) *the channel, residual, and ultraviolet projections are identical;*
- (iv) *the channel, residual, and ultraviolet lower bounds are identical;*
- (v) *the complete mixed blocks and their norms are identical;*
- (vi) *the vacuum-overlap singular values are identical;*
- (vii) *the analytic refinement-defect norms are identical.*

Therefore both branches satisfy

$$\boxed{\widehat{H}_n \geq 0.24975(I - P_{\text{vac},n})} \quad (\text{G13})$$

and the stronger direct estimate

$$\boxed{\widehat{H}_n \geq 3.093219423836 \dots (I - P_{\text{vac},n})} \quad (\text{G14})$$

for every n .

Proof. The objectwise identities follow from Theorem G.1. Coefficient-chart covariance is Theorem G.2; overlap invariance is Theorem G.4; analytic defect covariance is Theorem G.5. The package and direct lower bounds are those of Theorem C.8. $\square$

Corollary G.8 (Two end-to-end fixed-stratum depth globalizations with one analytic package). *The recursively physicalized family has:*

- (i) *a strict center-banded control depth globalization;*
- (ii) *a non-strictifiable center-banded twisted depth globalization of diagram-level type*

$$(0, [\Omega_1]_{\text{band}});$$

- (iii) *one common complete physicalized package and one common positive coercive profile.*

The two branches differ only in their global banded lifting data.

7. Finite spectral invisibility of the central extension

Theorem G.9 (No objectwise spectral discriminator). *Let Φ be any invariant determined solely by the objectwise finite physicalized packages $\{\mathfrak{P}_n\}$. Then*

$$\Phi(\mathfrak{S}^{\text{tw}}) = \Phi(\mathfrak{S}^{\text{str}}). \quad (\text{G15})$$

In particular, finite spectra, ground ranks, finite spectral gaps, Gram condition numbers, physical magnetic eigenvalues, block constants, and mixed norms cannot distinguish the strict and twisted lifts.

Proof. The two inputs to Φ are identical by Theorem G.1. □

Warning G.10 (No microscopic selection theorem). The theorem above has a two-sided interpretation. It proves that the central associator cannot destroy the finite physicalized estimates. It also proves that the finite Hamiltonians of the displayed model do not select the twisted extension uniquely. Any such selection must use additional scale-sensitive or global information beyond the objectwise finite spectra.

8. Stability under common objectwise perturbations

Suppose the same finite perturbation family B_n is applied to both central lifts, and suppose the hypotheses of Appendix M preserve the exact moved vacuum sectors and the complete physicalized package. Since central phases still commute with all objectwise operators, every statement of this appendix continues to hold with H_n replaced by $H_n + B_n$. Thus central-twist invariance is compatible with the moving-vacuum stability theory whenever the same objectwise perturbation is used in both branches.

9. Final conclusion

The repaired constructive statement is therefore stronger than coexistence of an analytic example and a separate cocycle example. The same recursive physicalized Wilson-character family supports both the strict and the nontrivial center-banded twisted scale-bordism realization, and the complete finite coercive profile survives the twist exactly. The associator is analytically invisible objectwise but remains globally visible in the banded vacuum descent.

Appendix H: Strict trivial-cocycle control and paired banded globalizations

1. Purpose

This appendix constructs the trivial-reassociation control of the same recursive physicalized family and proves complete strictification in the fixed normalized $\mathbb{Z}_2$ band. It introduces no second Hamiltonian model.

2. A global strict representative

Let

$$\mathcal{V} = \text{Ran } P_{\text{vac}}^{(2)}$$

be the one-shell vacuum line and

$$\mathcal{V}_n = \mathcal{V}^{\otimes n}.$$

Choose temporarily a unit vector $e \in \mathcal{V}$. For the batch arrow $a_{n;k} : n \rightarrow n+k$, define

$$J_{n;k}^e : \mathcal{V}_n \longrightarrow \mathcal{V}_{n+k}, \quad J_{n;k}^e(\xi) = \mathbb{J}_{n;k}(\xi \otimes e^{\otimes k}). \quad (\text{H1})$$

Set $J_{n;0}^e = I_{\mathcal{V}_n}$.

Proposition H.1 (Exact strict composition). *For all $k, \ell \geq 0$,*

$$J_{n+k;\ell}^e J_{n;k}^e = J_{n;k+\ell}^e. \quad (\text{H2})$$

Moreover, the projective class of $J_{n;k}^e$ is the canonical comparison $\overline{U}_{n;k}^{\text{vac}}$ of Appendix P.

Proof. For $\xi \in \mathcal{V}_n$,

$$J_{n+k;\ell}^e J_{n;k}^e \xi = (\xi \otimes e^{\otimes k}) \otimes e^{\otimes \ell} = \xi \otimes e^{\otimes (k+\ell)}.$$

The projective statement follows because the nonzero maps between two one-dimensional Hilbert spaces form one projective class. $\square$

Thus the vacuum lines admit a strict unitary comparison functor after one phase convention has been chosen.

3. The phase convention is gauge, not vacuum data

If $e' = ze$, then $J_{n;k}^{e'} = z^k J_{n;k}^e$. With $W_n(z) = z^n I_{V_n}$,

Proposition H.2 (Gauge equivalence of strict representatives).

$$J_{n;k}^{e'} = W_{n+k}(z) J_{n;k}^e W_n(z)^{-1}.$$

Hence the chosen unit section supplies only a representative; the invariant vacuum datum is the spectral line $\mathcal{V}$, not a distinguished vector.

Proof. The right side is $z^{n+k} z^{-n} J_{n;k}^e = z^k J_{n;k}^e$. □

4. The trivial center-banded extension

Let

$$\Omega_0(g, h, k) = 1 \quad (g, h, k \in \mathbb{Z}_2). \quad (\text{H3})$$

The corresponding skeletal banded category $\mathbb{Z}_2^{\Omega_0}$ has the same objects, central automorphisms, tensor product, and strict unit as the extension in Appendix P, but its associator is the identity. Pulling this band along the shell-parity grade produces the strict scale-bordism category

$$\mathfrak{R}_{\text{bat}}^{\Omega_0}.$$

Theorem H.3 (End-to-end strict recursively physicalized branch). *The recursively physicalized family of Appendix O admits a global strict center-banded realization over $\mathfrak{R}_{\text{bat}}^{\Omega_0}$ with the following properties:*

- (i) *its vacuum comparisons are the maps $J_{n;k}^e$ of (H1);*
- (ii) *they compose strictly as in (H2);*
- (iii) *every vacuum compositor and associator is the identity;*
- (iv) *its diagram-level vacuum type is*

$$\boxed{\text{Type}_{\text{vac}}(\mathfrak{D}_D^{\text{str}}) = (0, [\Omega_0]_{\text{band}}) = (0, 0)} \quad (\text{H4})$$

for every finite demand diagram D ;

(v) *it carries exactly the complete physicalized package and uniform bounds of Appendix O.*

In particular,

$$\widehat{H}_n \geq 0.24975(I - P_{\text{vac},n}) \quad (\text{H5})$$

and

$$\widehat{H}_n \geq 3.093219423836 \dots (I - P_{\text{vac},n}) \quad (\text{H6})$$

for every depth n .

Proof. Strict composition is Theorem H.1. The trivial associator satisfies the pentagon identically. The relative component is the same trivial separated- shell component used in Appendix Q, while the banded associator class is zero, which proves (H4). The complete physicalized package and both lower bounds are those of Theorem C.8; they are independent of the chosen phase convention by Theorem H.2 and Appendix R. $\square$

5. Strictification completeness in the fixed $\mathbb{Z}_2$ band

We restrict to the canonical rank-one vacuum system of this article: the objects are $\mathcal{V}_n$, the projective arrows are induced by $\mathbb{J}_{n,k}$, the relative scale class vanishes, and all scalar gauges are shell local, translation stationary, coefficient monoidal, and natural over the center-background completion. For normalized cochains set

$$(\delta\mu)(h, g) = \mu(h)\mu(g)\mu(h + g)^{-1}, \quad (\text{H7})$$

$$(\delta\lambda)(k, h, g) = \frac{\lambda(h, g)\lambda(k, h + g)}{\lambda(k + h, g)\lambda(k, h)}. \quad (\text{H8})$$

Lemma H.4 (Normalized $\mathbb{Z}_2$ cochains). *Every normalized 2-cochain λ on $\mathbb{Z}_2$ satisfies $\delta\lambda = 1$. Every normalized 2-cocycle β is a coboundary: if $q = \beta(1, 1)$, choose $r^2 = q$, put $\mu(0) = 1$, $\mu(1) = r$, and then $\delta\mu = \beta$.*

Proof. Normalization leaves only the value at $(1, 1)$. At the only nonautomatic triple,

$$(\delta\lambda)(1, 1, 1) = \lambda(1, 1)\lambda(1, 0)[\lambda(0, 1)\lambda(1, 1)]^{-1} = 1.$$

The second assertion follows from $(\delta\mu)(1, 1) = r^2 = q$. Hence, for normalized representatives,

$$[\Omega]_{\text{band}} = 0 \iff \Omega = 1. \quad (\text{H9})$$

$\square$

After Ω is trivial, write the residual local composition on the unique batch arrows $a_{i,j} : i \rightarrow j$ as

$$U_{j,k}U_{i,j} = \beta(k, j, i)U_{i,k}. \quad (\text{H10})$$

The ordinary cocycle equation on $0 \rightarrow i \rightarrow j \rightarrow k$ is

$$\beta(k, j, i)\beta(k, i, 0) = \beta(j, i, 0)\beta(k, j, 0). \quad (\text{H11})$$

Define the center-background-natural scalar arrow gauge

$$r(i, j) = \beta(j, i, 0)^{-1} \quad (\text{H12})$$

(with normalized identity values) and set $U'_{i,j} = r(i, j)U_{i,j}$. Equation (H11) gives $U'_{j,k}U'_{i,j} = U'_{i,k}$. This gauge changes only local compositors; it does not change the coefficient band or its associator class.

Theorem H.5 (Strictification completeness for the canonical $\mathbb{Z}_2$ -banded vacuum system). *For the canonical constant-rank coefficient-line vacuum descent system in the fixed normalized $\mathbb{Z}_2$ band, the following are equivalent:*

- (i) $[\Omega]_{\text{band}} = 0$;
- (ii) *an ordinary center-background-natural Hilbert descent exists;*
- (iii) *the higher descent two-presheaf has a band-local global section equivalent to the strict control constructed from $J_{n;k}^e$.*

The only alternative class is $[\Omega_1]_{\text{band}} \neq 0$, for which no such strictification exists.

Proof. By Theorem H.4, (i) gives $\Omega = 1$; the explicit contraction above then gives strict lifts, proving (ii). Write those lifts as $U'_{i,j} = \rho(i, j)J_{i;j-i}^e$. Strict composition implies $\rho(i, k) = \rho(j, k)\rho(i, j)$. With $w_i = \rho(0, i)$, one has $\rho(i, j) = w_j w_i^{-1}$, so the center-natural object unitaries $W_i = w_i I_{V_i}$ satisfy

$$U'_{i,j} = W_j J_{i;j-i}^e W_i^{-1}.$$

Thus (ii) implies (iii), while (iii) implies (ii). Any strict descent has identity associator, so (ii) implies (i). Finally, $H^3(B\mathbb{Z}_2; U(1)) \cong \mathbb{Z}_2$, and the nonzero class is excluded by Theorem I.11. $\square$

6. The strict and twisted globalizations

Let

$$\mathfrak{S}^{\text{str}} \quad \text{and} \quad \mathfrak{S}^{\text{tw}}$$

denote the strict realization of Theorem H.3 and the twisted realization of Theorem D.8. They have the same depth objects, the same canonical pairwise bordism correspondences, the same canonical projective comparisons, and the same complete objectwise physicalized package. Their banded types are

$$\text{Type}_{\text{vac}}(\mathfrak{S}^{\text{str}}) = (0, 0), \quad \text{Type}_{\text{vac}}(\mathfrak{S}^{\text{tw}}) = (0, [\Omega_1]_{\text{band}}). \quad (\text{H13})$$

Theorem H.6 (Inequivalence of the paired banded globalizations). *The strict and twisted coefficient-module realizations are not equivalent through any admissible center-band-preserving pseudonatural equivalence. Their unbanded projective quotients are identical, but their banded associator classes are distinct.*

Proof. A band-preserving equivalence changes the associator by the coboundary of a normalized band 2-cochain and therefore preserves its class in $H^3(B\mathbb{Z}_2; U(1))$. The strict class is zero, while $[\Omega_1]_{\text{band}} \neq 0$ by Theorem E.6. Hence no such equivalence exists. Both classes become invisible after passage to the projective quotient, which is the strict category $\mathfrak{R}_{\text{bat}}$. $\square$

7. What the paired theorem does and does not select

Theorem H.7 (Paired-globalization theorem). *One complete recursively physicalized $\text{SU}(2)$ family admits two inequivalent center-banded fixed-stratum depth globalizations:*

$$\boxed{\mathfrak{S}^{\text{str}} \quad \text{of type } (0, 0), \quad \mathfrak{S}^{\text{tw}} \quad \text{of type } (0, [\Omega_1]_{\text{band}}).} \quad (\text{H14})$$

They have identical objectwise finite packages, identical projective transport, identical analytic defect norms, and identical uniform coercive constants, but they are inequivalent in the center-banded scale-bordism category.

Proof. Existence of the two branches is Theorems D.8 and H.3. Objectwise and analytic equality is Theorem G.7. Banded inequivalence is Theorem H.6. $\square$

The theorem proves existence and persistence of both globalizations. It does not prove that the finite Hamiltonian dynamically chooses one of them. By Theorem G.9, no invariant depending only on the objectwise physicalized packages can do so. Any selection between the two must therefore use additional global, band-sensitive, boundary, or reconstruction data.

Appendix I: Coefficient-line-valued higher Hilbert descent and the reconstruction obstruction

1. Why global ordinary lifts are not the twisted object

The twisted vacuum datum must retain its coefficient lines. It cannot be encoded by globally chosen ordinary unitaries satisfying $U_b U_a = v_{b,a} U_{ba}$ together with a nontrivial scalar associator. Ordinary operator multiplication is strictly associative and would force $\delta v = 1$, hence a trivial associator. The correct object is therefore a higher descent system of local Hilbert realizations of the coefficient-line category, not one global tuple of ordinary lifted maps.

Let D_N be the rooted finite depth simplex with objects $0 < 1 < \dots < N$, and let $a_{ij} : i \rightarrow j$ denote its unique arrow. The center grade is

$$|a_{ij}| = \overline{j - i} \in \mathbb{Z}_2.$$

Pull the pointed coefficient category $\mathcal{B}^\Omega$ of Appendix D back along this grade. The resulting coefficient-line system

$$\mathcal{L}_D^\Omega = |\cdot|_D^* \mathcal{B}^\Omega$$

assigns a formal invertible line L_{ij} to every arrow and has reassociator

$$\widehat{\Omega}_D(i, j, k, \ell) = \Omega(|a_{k\ell}|, |a_{jk}|, |a_{ij}|). \quad (\text{I1})$$

The line parenthesizations are part of the datum and are not replaced by chosen global unit vectors.

2. The lifting two-groupoid and its rooted contractions

A *rooted finite simplicial chart* E is a finite ordered simplex together with its distinguished least vertex $r = r(E)$. Morphisms of rooted charts are root-preserving face inclusions.

Cochains below are multiplicative and normalized: identity or degenerate values are one, and a normalized zero-cochain satisfies $h(r) = 1$.

Definition I.1 (Coefficient-line lifting two-groupoid). For a rooted finite simplicial chart E , let $\text{Triv}_\Omega(E)$ be the normalized two-groupoid whose:

- (i) objects are normalized arrow-dependent two-cochains Λ satisfying

$$\delta\Lambda = \widehat{\Omega}_E; \quad (\text{I2})$$

- (ii) one-morphisms $\kappa : \Lambda \rightarrow \Lambda'$ are normalized arrow one-cochains satisfying

$$\Lambda' = (\delta\kappa)\Lambda; \quad (\text{I3})$$

- (iii) two-morphisms $h : \kappa \Rightarrow \kappa'$ are normalized object zero-cochains satisfying

$$\kappa' = (\delta h)\kappa. \quad (\text{I4})$$

The differential conventions are

$$(\delta\kappa)(i, j, k) = \frac{\kappa(i, k)}{\kappa(i, j)\kappa(j, k)}, \quad (\text{I5})$$

$$(\delta h)(i, j) = \frac{h(j)}{h(i)}. \quad (\text{I6})$$

For a two-cochain c ,

$$(\delta c)(i, j, k, \ell) = \frac{c(i, k, \ell)c(i, j, k)}{c(i, j, \ell)c(j, k, \ell)}. \quad (\text{I7})$$

In indices, (I2) is the module-pentagon equation

$$\Lambda(i, k, \ell)\Lambda(i, j, k) = \widehat{\Omega}_E(i, j, k, \ell)\Lambda(i, j, \ell)\Lambda(j, k, \ell). \quad (\text{I8})$$

Proposition I.2 (Local coefficient-line trivializations exist). *For every finite depth simplex D_N and for $\Omega \in \{\Omega_0, \Omega_1\}$,*

$$\text{Triv}_\Omega(D_N) \neq \emptyset.$$

For Ω_1 , one may take the rooted cochain

$$\Lambda_N(i, j, k) = \begin{cases} 1, & i = 0, \\ \widehat{\Omega}_N(0, i, j, k), & i > 0. \end{cases}$$

Proof. For Ω_0 , take $\Lambda = 1$. For Ω_1 , the cocycle identity evaluated on $(0, i, j, k, \ell)$ gives $\delta\Lambda_N = \widehat{\Omega}_N$, exactly as in Proposition E.7. $\square$

Proposition I.3 (Explicit connectedness of lifting objects). *Let E be a rooted finite simplex and let $\Lambda, \Lambda' \in \text{Triv}_\Omega(E)$. Put*

$$c(i, j, k) = \frac{\Lambda'(i, j, k)}{\Lambda(i, j, k)}.$$

Then $\delta c = 1$, and the normalized rooted one-cochain

$$\kappa_{\Lambda, \Lambda'}(i, j) = \begin{cases} 1, & i = r, \\ c(r, i, j)^{-1}, & r < i < j \end{cases} \quad (\text{I9})$$

satisfies

$$\Lambda' = (\delta\kappa_{\Lambda, \Lambda'})\Lambda.$$

Consequently any two objects of $\text{Triv}_\Omega(E)$ are joined by a one-morphism.

Proof. Since $\delta\Lambda' = \delta\Lambda = \widehat{\Omega}_E$, multiplicativity of the differential gives $\delta c = 1$. If $i = r$, then

$$(\delta\kappa)(r, j, k) = \kappa(r, k)[\kappa(r, j)\kappa(j, k)]^{-1} = c(r, j, k).$$

For $r < i < j < k$, the cocycle equation $(\delta c)(r, i, j, k) = 1$ gives

$$c(i, j, k) = \frac{c(r, j, k)c(r, i, j)}{c(r, i, k)}.$$

Substitution of (I9) into (I5) yields exactly this value. Thus $\delta\kappa = c$, proving the assertion. $\square$

Proposition I.4 (Explicit connectedness of transition groupoids). *Let $\kappa, \kappa' : \Lambda \rightarrow \Lambda'$ be parallel one-morphisms in $\text{Triv}_\Omega(E)$, and put*

$$q(i, j) = \frac{\kappa'(i, j)}{\kappa(i, j)}.$$

Then $\delta q = 1$. The normalized rooted zero-cochain

$$h_{\kappa, \kappa'}(r) = 1, \quad h_{\kappa, \kappa'}(i) = q(r, i) \quad (i > r) \quad (\text{I10})$$

satisfies

$$\kappa' = (\delta h_{\kappa, \kappa'})\kappa.$$

It is the unique normalized two-morphism from κ to κ' .

Proof. The two transition equations imply $\delta q = 1$. On $r < i < j$, this says

$$q(r, j) = q(r, i)q(i, j).$$

Hence

$$(\delta h)(i, j) = \frac{h(j)}{h(i)} = \frac{q(r, j)}{q(r, i)} = q(i, j),$$

and the same formula is immediate when $i = r$. If h' is another normalized solution, then h'/h is constant on the connected vertex set and equals one at r ; therefore $h' = h$. $\square$

Theorem I.5 (Contractibility of each finite lifting two-groupoid). *For every rooted finite depth simplex E , the two-groupoid $\text{Triv}_\Omega(E)$ is nonempty and biequivalent to the terminal two-groupoid. Equivalently, it is connected on objects, every transition groupoid is connected, and normalized parallel transitions admit a unique modification.*

Proof. Nonemptiness is Proposition I.2 after relabeling the root. Object connectedness is Proposition I.3; connectedness and uniqueness in each hom-groupoid are Proposition I.4. $\square$

Proposition I.6 (Strict restriction two-presheaf). *For each finite demand diagram D , root-preserving face restriction defines a strict contravariant two-functor*

$$\mathbf{Triv}_{\Omega_D} : \text{Chart}_*(D)^{\text{op}} \longrightarrow \mathbf{2Gpd}, \quad E \longmapsto \text{Triv}_{\Omega_D|_E}(E). \quad (\text{I11})$$

It restricts Λ , κ , and h pointwise and satisfies the identity and composition laws on the nose.

Proof. Cochain restriction commutes with δ , preserves the distinguished root and all normalization conditions, and is compatible with pointwise multiplication. Therefore it preserves objects, one-morphisms, two-morphisms, and both levels of composition strictly. $\square$

Warning I.7 (No stack or gerbe theorem is asserted). Equation (I11) defines a locally nonempty and locally contractible two-presheaf on the declared category of rooted finite charts. No Grothendieck topology, effective two-descent theorem, stackification, or continuum gerbe is constructed in this article. Those additional steps belong to the later continuum reconstruction problem.

3. Local Hilbert realizations with coefficient lines retained

The vacuum restriction of the canonical scale bordism is the unitary correspondence

$$\mathbb{U}_{ij}^{\text{vac}} : \mathcal{V}_i \hat{\otimes} L_{ij} \longrightarrow \mathcal{V}_j. \quad (\text{I12})$$

This is the invariant arrow datum. No unit vector in L_{ij} is part of (I12).

An object $\Lambda \in \text{Triv}_\Omega(E)$ determines a strong coefficient-line trivialization

$$T_\Lambda : \mathcal{L}_E^\Omega \longrightarrow \mathcal{L}_E^0 \quad (\text{I13})$$

that is the identity on the underlying formal lines and whose tensorator on a composable pair is Λ . Equation (I2) is precisely the coherence equation making T_Λ strong monoidal. Composing T_Λ with the phase-free vacuum restrictions of $\mathbb{J}$ gives an ordinary Hilbert realization on the chart:

$$\mathfrak{H}_E^\Lambda = \mathbb{J}_E^{\text{vac}} \circ T_\Lambda. \quad (\text{I14})$$

In this chartwise realization the ordinary Hilbert maps compose strictly. The nontrivial associator has not disappeared; it is stored in the tensorator of T_Λ and therefore in the transition data between different lifting charts.

If $\kappa : \Lambda \rightarrow \Lambda'$, then κ defines a monoidal natural equivalence

$$T_\Lambda \Longrightarrow T_{\Lambda'}$$

and hence a unitary equivalence

$$\mathfrak{E}_\kappa : \mathfrak{H}_E^\Lambda \Longrightarrow \mathfrak{H}_E^{\Lambda'}.$$

A two-morphism $h : \kappa \Rightarrow \kappa'$ gives the corresponding central modification $\mathfrak{E}_\kappa \Rightarrow \mathfrak{E}_{\kappa'}$. Propositions I.3 and I.4 prove that all chartwise Hilbert realizations and their transition choices are coherently related.

Definition I.8 (Coefficient-line-valued higher vacuum descent two-presheaf). The *coefficient-line-valued higher vacuum descent two-presheaf* of class $[\Omega]_{\text{band}}$ is

$$\mathbf{Vac}_{\text{Hilb}}^\Omega : E \longmapsto \left\{ \mathfrak{H}_E^\Lambda : \Lambda \in \text{Triv}_\Omega(E) \right\}, \quad (\text{I15})$$

equipped with the transition equivalences $\mathfrak{E}_\kappa$, their central modifications, and the strict restriction two-functors. It is a locally nonempty and locally contractible two-presheaf of ordinary Hilbert realizations of the formal coefficient-line system.

Theorem I.9 (Existence and local contractibility of twisted higher Hilbert descent). *The canonical vacuum-line scale bordisms of Appendix O and the coefficient system $\mathcal{B}^{\Omega_1}$ determine the two-presheaf $\mathbf{Vac}_{\text{Hilb}}^{\Omega_1}$. Every finite rooted depth chart has an ordinary Hilbert realization; any two realizations are joined by an explicit unitary transition, and any two parallel transitions are joined by a unique normalized central modification.*

Proof. Nonemptiness is Proposition I.2. The canonical vacuum restrictions supply the strict realization of $\mathcal{L}_E^0$, and composition with T_Λ gives (I14). The rooted formulas (I9) and (I10) give the transition and modification claims. Proposition I.6 gives strict compatibility under root-preserving restriction. $\square$

4. Obstruction to a band-local global Hilbert trivialization

Definition I.10 (Band-local ordinary vacuum descent). A *band-local ordinary vacuum descent* is a compatible global choice in $\mathbf{Vac}_{\text{Hilb}}^\Omega$ whose coefficient-line trivialization is shell local, translation stationary, and coefficient monoidal. By Theorem F.7, this is equivalent to factorization through the fixed grade: there is a normalized $\lambda : \mathbb{Z}_2^2 \rightarrow U(1)$ such that

$$\Lambda(i, j, k) = \lambda(|a_{jk}|, |a_{ij}|)$$

on every finite chart. Equivalently, the twisted coefficient law is removed by one normalized tensorator on the fixed center band before the Hilbert maps are evaluated.

Theorem I.11 (Higher Hilbert reconstruction obstruction). *For the twisted recursive branch:*

- (i) *the locally nonempty and locally contractible higher descent two-presheaf $\mathbf{Vac}_{\text{Hilb}}^{\Omega_1}$ exists;*
- (ii) *it has no band-local ordinary vacuum descent;*
- (iii) *its obstruction is exactly*

$$[\Omega_1]_{\text{band}} \neq 0 \quad \text{in} \quad H^3(B\mathbb{Z}_2; U(1)).$$

Proof. Part (i) is Theorem I.9. By Theorem F.7, a shell-local stationary global trivialization would supply a normalized band two-cochain λ with $\delta\lambda = \Omega_1$. Every normalized λ satisfies $(\delta\lambda)(1, 1, 1) = 1$, while $\Omega_1(1, 1, 1) = -1$. Hence such a trivialization does not exist, proving (ii) and (iii). $\square$

Corollary I.12 (Local rank one does not give a band-local global vacuum line). *Although every $\mathcal{V}_n$ is one dimensional, the twisted system has no single shell-local stationary choice of coefficient-line trivializations that glues all local vacuum lines into an ordinary strict vacuum line. Its invariant finite-chart datum is the higher descent two-presheaf $\mathbf{Vac}_{\text{Hilb}}^{\Omega_1}$.*

Warning I.13 (Unbanded and band-local statements differ). The rooted depth nerve is contractible. The arrow-dependent cochains Λ_N therefore provide ordinary Hilbert realizations on finite charts. They are background natural but fail shell stationarity, so they do not define global equivalences of the stationary repeated-shell coefficient law. The theorem obstructs a single trivialization descending from the fixed coefficient band; it does not deny unmarked strictification of the depth category.

5. Compatibility with the finite package and continuum handoff

Every local realization uses the same objectwise Hamiltonians, complete ground projections, Wilson channels, Gram operators, magnetic recompressions, and mixed blocks. Transitions in $\text{Triv}_{\Omega}(E)$ act only on coefficient-line trivializations. Hence the two coercive bounds remain

$$\widehat{H}_n \geq 0.24975(I - P_{\text{vac},n})$$

and

$$\widehat{H}_n \geq 3.093219423836 \dots (I - P_{\text{vac},n}).$$

Theorem I.14 (Strict/twisted reconstruction alternative). *Within the fixed normalized $\mathbb{Z}_2$ coefficient band:*

- (i) *if $[\Omega]_{\text{band}} = 0$, the higher descent two-presheaf has a band-local global section and is equivalent to the explicit strict control of Appendix S;*
- (ii) *if $[\Omega]_{\text{band}} = [\Omega_1]_{\text{band}} \neq 0$, no band-local global section exists, while the coefficient-line-valued higher Hilbert descent two-presheaf $\mathbf{Vac}_{\text{Hilb}}^{\Omega_1}$ exists and is locally nonempty and locally contractible on every finite depth chart.*

The two alternatives have identical complete finite analytic shadows.

Proof. The zero-class statement is Theorem H.5. The nonzero statement is Theorem I.11. Finite-shadow equality is Appendices R and T. □

The explicit twisted datum passed forward as a fixed-stratum witness is

$$\boxed{\mathfrak{D}_{\Pi, \sigma_0}^{\text{tw}} = \left(\mathfrak{R}_{\text{depth}}^{\Omega_1}(\sigma_0), \{\mathcal{H}_n, H_n, P_{\text{vac}, n}, \mathfrak{P}_n\}, \{\mathbb{J}_a\}, \mathbf{Vac}_{\text{Hilb}}^{\Omega_1}, [\Delta_\zeta, \tau]_{\text{ref/adm}}, [\Omega_1]_{\text{band}}, \mathbf{q}_*, \kappa_* \right)}. \quad (\text{I16})$$

This is a fixed-stratum finite-depth higher descent object, not a full-scale cofinal system and not a continuum vacuum. A later theorem must extend or reconstruct compatible data along a genuinely cofinal multiparameter family, including tube thinning and cutoff removal, then establish effective two-descent or stackification for the limiting chart system, construct a limit of the local Hilbert realizations, prove analytic convergence, and prove no leakage of the uniform threshold. If the abstract full-scale extension hypotheses are verified with the same coefficient law, the analogous handoff replaces $\mathfrak{R}_{\text{depth}}(\sigma_0)$ by the resulting $\mathfrak{R}_{\text{cof}}$; that transfer is conditional and is not established by the tensor-depth witness alone.

Appendix J: Spectral indistinguishability of banded globalizations

The strict and twisted branches are inequivalent in the center-banded category, but the forgetful passage to finite analytic data erases that difference. For a compatible physical family $\mathfrak{S}$, either full-scale cofinal or fixed-stratum depth-exhaustive, define

$$\text{Sh}_{\text{fin}}(\mathfrak{S}) = \left(\{\mathfrak{P}_n\}_{n \geq 1}, \{\overline{U}_a^{\text{vac}}\}_a, \{\text{sv}(C_a^{\text{vac}})\}_a, \{\|\mathfrak{D}_{b,a}^{(\nu)}\|\}_{a,b,\nu} \right), \quad (\text{J1})$$

where $\mathfrak{P}_n$ is the complete finite physicalized package. This shadow forgets the coefficient band, compositor phases, and banded associator class.

Definition J.1 (Finite-package invariant). An invariant Φ is a *finite-package invariant* if

$$\Phi(\mathfrak{S}) = \widehat{\Phi}(\text{Sh}_{\text{fin}}(\mathfrak{S}))$$

for some map $\widehat{\Phi}$, with no continuity or locality assumption.

Proposition J.2 (Equality of strict and twisted finite shadows). *For the paired globalizations,*

$$\boxed{\text{Sh}_{\text{fin}}(\mathfrak{S}^{\text{str}}) = \text{Sh}_{\text{fin}}(\mathfrak{S}^{\text{tw}})}. \quad (\text{J2})$$

Proof. The objectwise packages and projective comparisons are identical by Theorem VII.2; overlap singular values and analytic defect norms agree by Theorems G.4 and G.5. $\square$

Theorem J.3 (No finite-package discriminator). *Every finite-package invariant satisfies*

$$\Phi(\mathfrak{S}^{\text{str}}) = \Phi(\mathfrak{S}^{\text{tw}}).$$

Hence no deterministic, statistical, or randomized rule receiving only the complete finite analytic shadow can select exactly one of the paired branches.

Proof. Apply the same map $\hat{\Phi}$ to the identical inputs in (J2). A randomized rule is included by taking $\hat{\Phi}$ to be probability-measure valued. $\square$

Corollary J.4 (Covered analytic invariants). *The two branches agree on all invariants built from finite spectra, resolvents, functional calculus, ground ranks, excitation gaps, coercive constants, Gram data, physical-channel projections, magnetic recompressions, block constants, mixed-block norms, vacuum-overlap singular values, and finite or asymptotic operations on analytic refinement-defect norms.*

Theorem J.5 (Necessary band sensitivity). *Any discriminator between $\mathfrak{S}^{\text{str}}$ and $\mathfrak{S}^{\text{tw}}$ must use information absent from Sh_{fin} , such as the center band, coefficient multiplication or reassociator data, band-sensitive boundary conditions, or the distinction between strict and coefficient-line higher-descent reconstruction.*

Proof. This is the contrapositive of Theorem J.3. $\square$

Warning J.6 (No conclusion of physical equivalence). The result is a limitation of finite spectral information, not an equivalence of global vacua:

$$[\Omega_0]_{\text{band}} = 0, \quad [\Omega_1]_{\text{band}} \neq 0.$$

Accordingly, any later numerical ordering of the pair must enter through an explicitly band-sensitive convention or additional reconstruction data, not through the finite Hamiltonian shadow alone.

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