

Spectral-Action Emergence and the Critical Dimension of Relativistic Stability

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Abstract

We formulate a spectral-shape mechanism for the critical scaling of relativistic matter. In a regular spatial Dirac-type spectral triple of spectral dimension d , the first-order matter operator has kinetic order $s = 1$. We extract the corresponding fermionic density exponent $\rho^{1+1/d}$ from local Weyl asymptotics, equivalently from heat-kernel or Dixmier-trace data, rather than treating it as independent semiclassical input. If the gauge interaction is not externally prescribed but arises from nontrivial inner fluctuations of an almost-commutative or genuinely noncommutative triple, with leading dynamics supplied by the spectral action, then the static quadratic gauge operator is Laplace-type and therefore second order. Its near-diagonal Green kernel has singularity order $\lambda = d - 2$ for $d > 2$, with the logarithmic replacement in $d = 2$. The emergently generated matter–gauge system therefore has spectral scaling shape

$$(d, s, \lambda) = (d, 1, d - 2).$$

The critical diagonal $s = \lambda$ is attained if and only if $d = 3$. Thus three spatial dimensions are unique: only there does first-order matter coupled to its own second-order static gauge mediator sit exactly at the exponent-level stability threshold. The novelty is not the known dimension-dependent exponent arithmetic, but the forcing mechanism: spectral-action emergence collapses an apparent three-parameter scaling shape to a one-parameter family. We also explain why, in observable-first formulations where causal structure is reconstructed rather than assumed, the static-kernel analysis should be regarded as downstream of a separate, conditional gap-to-cone-rigidity theorem.

1 Introduction

The stability of relativistic matter is critical in a precise scaling sense. In three spatial dimensions, the first-order relativistic kinetic energy and the Coulomb singularity both produce the density exponent

$$\rho^{4/3}.$$

By contrast, nonrelativistic Schrödinger kinetic energy produces $\rho^{5/3}$, while the Coulomb singularity remains at $\rho^{4/3}$. The nonrelativistic theory therefore possesses an exponent-level coercive buffer, whereas the relativistic theory lies on the critical diagonal. The analytic realization of this distinction belongs to the classical stability-of-matter theory, including the relativistic estimates of Lieb and Yau and the generalized fermionic kinetic inequalities of Daubechies [9, 7, 10].

The purpose of this note is not to rediscover that exponent table, and it is not to replace the analytic estimates of Lieb–Yau, Daubechies, Lieb–Oxford, Lieb–Thirring, Hardy–Littlewood–Sobolev,

or related many-body inequalities [9, 7, 8, 6, 10]. Those estimates remain indispensable. The question here is mechanistic: if the interaction is generated by the same spectral geometry that propagates the matter, is the interaction singularity order λ independent data at all?

In an almost-commutative or genuinely noncommutative spectral geometry, nontrivial inner fluctuations of a real spectral triple generate gauge potentials [2]. The spectral action then supplies the leading Yang–Mills curvature term [3, 4]. After static reduction, the leading quadratic gauge operator is Laplace-type and therefore second order. In d spatial dimensions, its near-diagonal Green kernel behaves as

$$G(x, y) \sim \text{dist}(x, y)^{-(d-2)}, \quad d > 2,$$

with the logarithmic replacement in $d = 2$. Hence

$$\lambda = d - 2.$$

Meanwhile, Dirac matter is first order, so

$$s = 1.$$

The spectral scaling shape is therefore not a general triple (d, s, λ) . Under the emergence hypotheses considered here, it collapses to

$$(d, s, \lambda) = (d, 1, d - 2).$$

The criticality condition $s = \lambda$ becomes

$$1 = d - 2,$$

and therefore

$$d = 3.$$

Three spatial dimensions are thus not merely a dimension in which relativistic Coulomb matter happens to be critical. They are the unique dimension in which first-order matter and its own second-order static gauge mediator are exactly balanced at the exponent level.

The note has three technical tasks. First, we extract the kinetic density exponent from the local spectral asymptotics of the triple. Second, we identify the second-order static gauge mediator generated by inner fluctuations and the spectral action. Third, we combine these two facts into the spectral-shape collapse theorem and its dimension-three corollary. A final section records a separate logical issue: when causal structure is itself emergent, a static interaction kernel is physically legitimate only after sufficient spectral rigidity has made the reconstructed causal cone sharp. That gap-dependent layer is stated conditionally and is not used in the proof of the main theorem.

2 Spectral scaling shapes

Definition 1 (Spectral scaling shape). *Let d be the spatial spectral dimension, let $s > 0$ be the order of the matter kinetic operator, and let $\lambda \geq 0$ be the short-distance singularity order of a static Riesz-type interaction kernel,*

$$V(x, y) \sim \text{dist}(x, y)^{-\lambda}.$$

The triple

$$(d, s, \lambda)$$

is called the spectral scaling shape of the matter–interaction system.

A kinetic operator of order s produces the fermionic density exponent

$$1 + \frac{s}{d}.$$

A Riesz interaction of order λ produces the local singularity exponent

$$1 + \frac{\lambda}{d}.$$

At the level of homogeneous dilation, the interaction statement follows from the mean-spacing estimate $\ell \sim \rho^{-1/d}$: the interaction scale per particle is $\ell^{-\lambda} \sim \rho^{\lambda/d}$, and hence the corresponding density scale is $\rho^{1+\lambda/d}$.

Definition 2 (Critical diagonal). *A shape (d, s, λ) is called subcritical, critical, or supercritical according as*

$$s > \lambda, \quad s = \lambda, \quad s < \lambda.$$

Equivalently, the kinetic density exponent is respectively larger than, equal to, or smaller than the interaction singularity exponent.

The point of the present note is that in a self-generated spectral-action model, s and λ are not independent choices.

3 The kinetic exponent from spectral asymptotics

Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular spatial Dirac-type spectral triple of spectral dimension d . Equivalently, one may take the spatial restriction of an almost-commutative spacetime triple. In a noncompact or semifinite setting, all counting statements below are understood locally, after insertion of a positive localization element $a \in \mathcal{A}$ and use of the appropriate trace or weight.

For $s > 0$, define the positive kinetic operator

$$K_s := |D|^s.$$

Assume the local heat asymptotic

$$\mathrm{Tr}(ae^{-tK_s}) \sim C_{d,s} t^{-d/s} \tau_D(a), \quad t \downarrow 0,$$

where τ_D is the measure functional determined by the triple. By a Tauberian passage, this is equivalent to the local Weyl law

$$\mathrm{Tr}(a \mathbf{1}_{[0,\Lambda]}(K_s)) \sim C'_{d,s} \Lambda^{d/s} \tau_D(a), \quad \Lambda \rightarrow \infty.$$

For a measurable regular spectral triple, the same dimension functional is encoded by the Dixmier trace of $a|D|^{-d}$; these are alternative expressions of the same high-energy spectral dimension [1, 4].

Lemma 1 (Weyl extraction of the fermionic kinetic exponent). *Suppose $K_s = |D|^s$ satisfies the local Weyl law above. Then the homogeneous fermionic ground-state energy associated with K_s has density exponent*

$$\rho^{1+s/d}.$$

More precisely, in a local region of spectral volume V , filling the first N one-particle levels gives the asymptotic scaling

$$E_s(N, V) \asymp V \left(\frac{N}{V} \right)^{1+s/d}$$

at the exponent level.

Proof. The counting law gives

$$N(\Lambda) \sim CV\Lambda^{d/s}.$$

The sum of occupied one-particle energies below Λ satisfies

$$E(\Lambda) \sim \int_0^\Lambda \mu \, dN(\mu) \sim C_s V \Lambda^{1+d/s}.$$

Solving the counting relation for the Fermi level gives

$$\Lambda \sim \left(\frac{N}{V} \right)^{s/d}.$$

Substitution yields

$$E_s(N, V) \sim C_{d,s} V \left(\frac{N}{V} \right)^{1+s/d}.$$

Writing $\rho = N/V$ gives the claimed density exponent. $\square$

Remark 1. *The Pauli principle supplies the fermionic occupancy rule; the spectral triple supplies the state-counting exponent. The lemma is an exponent extraction from Weyl data, not a replacement for a sharp Lieb–Thirring or Daubechies inequality. Constants, lower-order terms, localization errors, and admissible classes of densities remain analytic questions.*

For a Dirac-type matter operator, $s = 1$. Hence the spectral kinetic exponent is

$$1 + \frac{1}{d}.$$

4 Nontrivial inner fluctuations and static gauge dynamics

A necessary qualification comes first. A purely commutative real spin spectral triple does not by itself produce a nontrivial internal Yang–Mills sector through inner fluctuation: for the canonical commutative geometry, the real-structure contribution cancels the would-be internal gauge fluctuation. Nontrivial gauge sectors require an almost-commutative product

$$C^\infty(M) \otimes \mathcal{A}_F$$

or a genuinely noncommutative algebra. This is the setting assumed from now on [2, 4].

Let

$$(\mathcal{A}, \mathcal{H}, D, J, \Gamma)$$

be such a real even spectral triple, interpreted either spatially or as the static restriction of a spacetime triple. An inner fluctuation has the form

$$D_A = D + A + \epsilon' J A J^{-1}, \quad A = \sum_j a_j [D, b_j] = A^*,$$

where ϵ' is the KO-dimensional sign. The finite internal algebra $\mathcal{A}_F$ determines the gauge group, representation content, and multiplicity data. It does not alter the manifold spectral dimension or the differential order of the leading gauge kinetic operator.

The bosonic spectral action is

$$S_{\text{bos}}(D_A; \Lambda) = \text{Tr}(f(D_A^2/\Lambda^2)).$$

Under the standard regularity and heat-kernel hypotheses, its asymptotic expansion contains a Yang–Mills term of the form

$$C_F \int \text{tr}(F_{\mu\nu} F^{\mu\nu}),$$

with coefficients depending on the cutoff moments and internal representation data [3, 4]. At quadratic order in the gauge potential, and after gauge fixing and static reduction, the principal operator is Laplace-type. Its spatial differential order is therefore

$$q = 2.$$

Higher spectral-action terms and lower-order curvature couplings may modify coefficients or introduce suppressed higher-derivative corrections; they do not alter the leading two-derivative static scaling considered here.

Proposition 1 (Static Green-kernel order). *Let L_2 be the positive gauge-fixed static Laplace-type operator arising from the leading quadratic spectral-action gauge sector on a d -dimensional spatial spectral geometry. Then, for $d > 2$, the near-diagonal Green kernel satisfies*

$$G_2(x, y) \sim C_d \text{dist}_D(x, y)^{-(d-2)}.$$

For $d = 2$, the corresponding singularity is logarithmic. Consequently, the Riesz interaction order is

$$\lambda = d - 2$$

for $d > 2$.

Proof. A Laplace-type operator has the local short-time heat-kernel form

$$p_t(x, y) \sim (4\pi t)^{-d/2} \exp\left(-\frac{\text{dist}_D(x, y)^2}{4t}\right) \sum_{k \geq 0} t^k u_k(x, y),$$

with $u_0(x, x)$ equal to the appropriate bundle identity, modulo the standard geometric normalization [5]. Formally,

$$G_2(x, y) = \int_0^\infty p_t(x, y) \, dt.$$

An infrared qualification is required for the full integral: one may remove zero modes on a compact geometry, introduce a small mass regulator, or work with a reduced Green operator. None of these choices affects the near-diagonal singularity, which is governed entirely by the short-time part of the heat kernel. Integrating the leading short-time term gives

$$G_2(x, y) \sim C_d \text{dist}_D(x, y)^{2-d}$$

for $d > 2$, and logarithmic behavior for $d = 2$. □

Remark 2 (Universality across internal algebras). *The internal algebra changes the gauge group, the number of gauge components, coupling normalizations, and matter representations. Under the hypotheses above, it changes multiplicative and matrix-valued data, not the exponent pair $(s, \lambda) = (1, d - 2)$. The spectral-shape collapse is therefore universal across admissible finite internal algebras at the level of principal differential order.*

5 The spectral-shape collapse theorem

Theorem 1 (Spectral-action collapse of the scaling shape). *Let $(\mathcal{A}, \mathcal{H}, D, J, \Gamma)$ be a regular d -dimensional spatial Dirac-type spectral triple, or the static spatial restriction of a regular almost-commutative spacetime triple. Assume:*

- (i) *matter propagation is governed at principal order by the first-order Dirac operator D ;*
- (ii) *the gauge sector is generated by nontrivial inner fluctuations of an almost-commutative or genuinely noncommutative real spectral triple;*
- (iii) *the leading bosonic gauge dynamics are supplied by the spectral action and have a gauge-fixed static quadratic operator of Laplace type;*
- (iv) *the local Weyl and heat-kernel asymptotics used in Lemma 1 and Proposition 1 hold.*

Then, for $d > 2$, the emergently generated matter–gauge system has spectral scaling shape

$$\boxed{(d, s, \lambda) = (d, 1, d - 2)}.$$

In $d = 2$, the interaction component is logarithmic rather than a positive-order Riesz singularity.

Proof. The Dirac principal symbol is first order, hence $s = 1$. By Lemma 1, the corresponding fermionic kinetic density exponent is $1 + 1/d$. Nontrivial inner fluctuations generate the gauge potential, while the leading spectral-action curvature term produces a second-order static gauge operator. Proposition 1 then gives $\lambda = d - 2$ for $d > 2$. Combining the two principal orders yields $(d, 1, d - 2)$. $\square$

Corollary 1 (Dimension-three criticality). *Under the hypotheses of Theorem 1, the critical diagonal $s = \lambda$ is attained if and only if*

$$\boxed{d = 3}.$$

Thus three is the unique spatial dimension in which emergently interacting first-order matter is exactly critical at the exponent level.

Proof. The theorem gives $s = 1$ and $\lambda = d - 2$. Therefore

$$s = \lambda \iff 1 = d - 2 \iff d = 3.$$

$\square$

The corollary is the central interpretation of the mechanism. The criticality of relativistic Coulomb matter in three spatial dimensions is not being introduced as an isolated arithmetic coincidence. It is the unique fixed point of the emergence law

$$(d, s, \lambda) = (d, 1, d - 2).$$

6 Dimension dependence

For emergently generated first-order matter, the kinetic density exponent is

$$p_{\text{kin}}(d) = 1 + \frac{1}{d},$$

while the static interaction exponent for $d > 2$ is

$$p_{\text{int}}(d) = 1 + \frac{d-2}{d} = 2 - \frac{2}{d}.$$

Their difference is

$$p_{\text{kin}}(d) - p_{\text{int}}(d) = \frac{3-d}{d}.$$

Hence the sign changes only at $d = 3$.

d	s	interaction kernel	λ	density comparison	regime
2	1	$\log \text{dist}^{-1}$	logarithmic	$\rho^{3/2}$ vs. $\rho \log \rho$	subcritical
3	1	dist^{-1}	1	$\rho^{4/3}$ vs. $\rho^{4/3}$	critical
4	1	dist^{-2}	2	$\rho^{5/4}$ vs. $\rho^{3/2}$	supercritical
$d \geq 5$	1	$\text{dist}^{-(d-2)}$	$d-2$	$1 + 1/d$ vs. $2 - 2/d$	supercritical

In $d = 1$, the Green function of a second-order static operator grows linearly with separation rather than defining a positive-order local Riesz singularity. The present short-distance Riesz classification is therefore naturally stated for $d \geq 2$, with the logarithmic case handled separately at $d = 2$.

The table is a scaling diagnosis, not by itself a stability or instability theorem. At $d \geq 4$, the interaction exponent exceeds the first-order kinetic exponent before coupling constants, statistics, charge neutrality, ultraviolet regularization, or many-body structure are examined. Those further data determine the actual theorem available for a particular model.

7 The nonrelativistic buffer as a decoupling scar

In three spatial dimensions, Schrödinger–Coulomb matter has shape

$$(d, s, \lambda) = (3, 2, 1).$$

Its kinetic and interaction exponents are therefore

$$1 + \frac{2}{3} = \frac{5}{3}, \quad 1 + \frac{1}{3} = \frac{4}{3}.$$

Thus $s > \lambda$, and the theory possesses the familiar exponent-level coercive buffer.

The present framework gives that buffer a structural interpretation. In the relativistic spectral geometry, the first-order Dirac operator both propagates matter and, through its noncommutative inner fluctuations, generates the gauge field whose spectral action supplies the static mediator. The interaction is autochthonous to the same geometry.

The nonrelativistic limit breaks that order-level identification. The matter dispersion becomes second order, while the static interaction continues to descend from the second-order gauge sector. In $d = 3$ this produces

$$s = 2, \quad \lambda = d - 2 = 1.$$

The gap between $5/3$ and $4/3$ may therefore be read as a *decoupling scar*: the matter kinetic operator is no longer the first-order operator from which the interaction-generating geometry is spectrally reconstructed.

This interpretation does not alter the nonrelativistic stability proof. It explains why a coercive exponent buffer appears after the nonrelativistic reduction but disappears when the full first-order matter geometry is retained.

8 Relation to stability-of-matter theory

The dimension-dependent comparison of kinetic and Coulomb exponents belongs to the established stability-of-matter landscape; see, for example, Lieb and Seiringer [10]. The novelty claimed here is therefore mechanistic, not arithmetical.

Theorem 1 does not prove stability or instability. It does not replace:

- Lieb–Thirring inequalities and their generalizations;
- Daubechies-type relativistic kinetic estimates;
- Lieb–Yau relativistic stability bounds;
- Lieb–Oxford control of indirect Coulomb energy;
- Hardy–Littlewood–Sobolev estimates;
- localization, ultraviolet, and charge-neutrality arguments;
- model-specific control of negative spectral subspaces and no-pair projections.

Rather, the theorem identifies the scaling regime in which such inequalities must operate once the interaction is required to arise from the same spectral geometry as the matter field. The mechanism is

$$\text{Dirac principal order} \implies s = 1,$$

combined with

$$\text{inner fluctuations} + \text{spectral-action dynamics} \implies \text{second-order static mediation} \implies \lambda = d - 2.$$

The apparent three-parameter family collapses to

$$(d, 1, d - 2).$$

9 Conditional cone rigidity in an observable-first theory

The main theorem is a static spectral-shape theorem. In a formulation with an assumed background causal geometry, the static Green kernel is obtained by an ordinary elliptic reduction of the gauge sector. One starts with a causal or Euclideanized field theory, fixes a static slice, and extracts the Gauss-law kernel.

The logical order is different in an observable-first theory where the causal cone is itself to be reconstructed from correlation, localization, and transport data. There, the static reduction is physically meaningful only after the observable dynamics have become sufficiently rigid to support a sharp effective cone.

A Lieb–Robinson estimate can provide a finite propagation velocity or an approximate causal region in lattice and quasi-local many-body systems [13]. It does not, by itself, imply that a cone reconstructed from the observables is sharp at all scales. A spectral gap can provide additional rigidity: under the hypotheses of a gap-to-clustering theorem, connected ground-state correlations decay exponentially with a correlation length controlled by the inverse gap [11, 12]. Gapless sectors instead often exhibit persistent power-law tails.

These facts motivate, but do not prove here, the following conditional architectural order:

$\begin{aligned} \text{kinematic substrate} &\longrightarrow \text{spectral gap} \longrightarrow \text{cone rigidity and leakage control} \\ &\longrightarrow \text{well-posed static reduction} \longrightarrow \text{critical scaling diagnosis} \end{aligned}$

The box is a statement of dependency, not a theorem established in this note. In particular, exponential clustering alone is not identified with a no-light-cone-leakage theorem. A full observable-first result would require a separate argument converting the spectral gap, covariance, transport bounds, and sector-correct localization into reconstructable sharp cones or worldline structure. The present paper uses none of those claims in Theorem 1; it records them only to clarify how the static spectral-shape mechanism would sit inside a larger emergent causal program.

10 Conclusion

The critical relativistic exponent in three spatial dimensions is often presented as a fact about placing a first-order Dirac kinetic operator beside a Coulomb potential. The spectral-action mechanism gives a sharper interpretation.

When the interaction is externally prescribed, (d, s, λ) may be treated as three independent scaling parameters. When the interaction is generated from the same noncommutative spectral geometry that propagates the matter, the principal orders are linked. Dirac propagation forces $s = 1$. Nontrivial inner fluctuations generate the gauge sector, and the leading spectral-action curvature term forces a second-order static mediator. Its Green kernel forces $\lambda = d - 2$. Therefore

$$(d, s, \lambda) = (d, 1, d - 2).$$

The critical diagonal is reached precisely when

$$1 = d - 2,$$

that is, precisely in

$$d = 3.$$

Three spatial dimensions are therefore unique at the exponent level: they are the only dimension in which first-order matter coupled to its own second-order static gauge sector is neither underpowered nor overpowered by the interaction it generates. In this restricted and explicit sense, criticality is the spectral fingerprint of an autochthonous interaction.

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