Joint Reconstruction

Analytic Certificates, Relational Persistence, and the Doubly Infinite Corner

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Status. Frozen manuscript, version 1.0.0. Finite reconstruction may remain admissible along each marginal direction and nevertheless fail under simultaneous totalization. This paper develops a sufficient theory for survival of the doubly infinite corner, separating mixed incidence, retention, and assembly on the analytic side from relational coherence and strictifier persistence on the comparison side. A certificate-bearing strictifier is treated as paired data (β_C, U_C), retaining both the relational repair and the graph equivalence certifying analytic preservation. Under compatible totalizations, uniform finite relative coercivity, and compact persistence of these paired data, the strictified doubly totalized passage is closable. **Research note – not peer reviewed. Trust, but verify. Always.**

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1 Introduction

Finite reconstruction problems can fail at their total boundary even when every finite stage is valid. A stronger phenomenon occurs when two independent reconstruction directions are present: each marginal totalization may survive while the simultaneous totalization fails.

Let F denote source exposure and R a second finite reconstruction direction. It may happen that

$$(F_\infty, R_r) \in \mathcal{A} \quad \forall r < \infty, \qquad (F_n, R_\infty) \in \mathcal{A} \quad \forall n < \infty,$$

while

$$\boxed{(F_\infty, R_\infty) \notin \mathcal{A}.} \tag{1}$$

The doubly infinite corner therefore contains information not determined by either marginal boundary.

The phenomenon is already present in the infinite-index fusion singularity motivating this series. If source multiplicity and multiplication-face exposure are separated into independent cutoffs, then

$$\boxed{\|m_{n,r}\| = \sqrt{\min(n, r)}.} \tag{2}$$

For every fixed r, complete source exposure remains bounded. For every fixed n, complete face exposure remains bounded. Yet the simultaneous limit is nonclosable.

The analytic problem separates into mixed incidence, retention, and assembly. Mixed incidence alone is insufficient: identical incidence data can support opposite analytic behavior depending only on assembly. This leads to the *Joint Reconstruction Certificate*. If $T_0 = SW_0$ with W_0 closable, S closed, and

$$\|W_0 x\| \le C\left(\|x\|^2 + \|T_0 x\|^2\right)^{1/2}, \tag{3}$$

then T_0 is closable.

A second problem appears when finite realizations are not related by strict embeddings. Correspondence-valued passages may admit compositors whose distinct triple-composition routes disagree by a relative associator Θ. A nontrivial Θ obstructs strict realization of the specified compositor system even when each individual carrier is analytically admissible.

Finite repairs of these relational defects are themselves finite reconstruction data. If $\mathrm{Str}(C)$ denotes the strictifier space of a finite context C, then $\mathrm{Str}(C) \neq \varnothing$ for every C does not imply $\varprojlim_C \mathrm{Str}(C) \neq \varnothing$.

The analytic and relational halves meet through certificate-bearing strictification. At context C, the relevant datum is a pair $\sigma_C = (\beta_C, U_C)$, where β_C strictifies the relational comparison data and U_C identifies the original and strictified effective retained graph carriers by a bounded graph equivalence.

finite solvability does not imply persistence of the data that certify solvability.

The results are realization-sensitive. Failure in the bounded class need not imply failure as a closed unbounded operator; failure of a strict relational realization need not imply failure of the relational system itself; and failure of one retained presentation need not imply failure of the passage. No categorical or K-theoretic classification is asserted.

The present work is the third stage of a finite-to-total reconstruction program. Paper I isolates an infinite-index fusion face whose algebraic multiplication survives while its ordinary Hilbert realization becomes maximally nonclosable. Paper II abstracts the finite-to-total analytic problem and develops graph-controlled reconstruction criteria. Here we introduce a second reconstruction direction and study simultaneous totalization, including the coherence problem that appears when finite realizations are related by correspondences rather than strict embeddings [1, 2].

Section 2 separates marginal from joint totalizability. Section 3 gives the mixed-fusion boundary. Section 4 introduces mixed incidence and assembly dependence. Sections 5 and 6 develop the analytic certificate. Sections 7 to 9 treat relational transport and persistence. Section 10 proves the integrated theorem.

2 Marginal and Joint Totalization

Let Src and Dem be directed systems of finite reconstruction contexts, with formal total stages F_∞ and R_∞. Let

$$\mathcal{A} \subset (\mathsf{Src} \cup \{F_\infty\}) \times (\mathsf{Dem} \cup \{R_\infty\})$$

denote the admissible region for a fixed realization class.

Definition 2.1. The system is *finitely rectangularly admissible* if $(F, R) \in \mathcal{A}$ for all finite F, R. It is *sourcewise totalizable* if $(F_\infty, R) \in \mathcal{A}$ for every finite R, *demandwise totalizable* if $(F, R_\infty) \in \mathcal{A}$ for every finite F, and *jointly totalizable* if $(F_\infty, R_\infty) \in \mathcal{A}$.

Theorem 2.2 (Separation of Marginal and Joint Totalizability)**.** *There exists a downward-closed two-axis reconstruction system which is finitely rectangularly admissible, sourcewise totalizable, and demandwise totalizable, but not jointly totalizable.*

Proof. Let $D_\infty = c_{00}(\mathbb{N})$ and $D_n = \mathrm{span}\{e_1, \ldots, e_n\}$. Let the finite demand stage R_r require retention of the first r coordinate functionals $J_k(x) = x_k$, and let admissible carriers be finite-dimensional vector spaces.

At (D_n, R_r) the demanded data factor through $\mathbb{K}^{\min(n,r)}$, so every finite rectangle is admissible. For fixed finite r, the total source still requires only the first r coordinates. For fixed finite n, every J_k with $k > n$ vanishes on D_n, so complete demand factors through $D_n \cong \mathbb{K}^n$.

If the total context were admissible, there would be a finite-dimensional X and a map $q : c_{00}(\mathbb{N}) \to X$ through which every J_k factors. Then $q(x) = 0$ would imply $J_k(x) = 0$ for all k, hence $x = 0$. Thus q would inject an infinite-dimensional vector space into a finite-dimensional one, a contradiction. □

The finite complexity profile is $r(n, r) = \min(n, r)$. Each marginal supremum is finite, while $\sup_{n,r} r(n, r) = \infty$. The model is intentionally elementary: it proves only that the doubly infinite corner carries genuinely joint information.

3 The Mixed Fusion Boundary

Let $A \subset N$ be the inclusion used in the infinite-index fusion construction, with basic construction N_1, Jones projection e_A, and relative multiplication core $m_0 : N \odot_A N \to N$, $x \odot_A y \mapsto xy$.

We use the orthogonal Jones witness family constructed in Paper I [1]. Let $\{u_i\}_{i \geq 1} \subset N$ be an infinite E_A-orthonormal unitary family and set $p_i = u_i e_A u_i^*$. The properties needed below are

$$p_i p_j = \delta_{ij} p_i, \qquad \mathrm{Tr}_1(p_i) = 1, \qquad m_0(p_i) = 1.$$

Thus the p_i are orthonormal in the natural L^2 geometry while multiplication sends every one to the same target vector.

$$\boxed{\underbrace{p_1, p_2, p_3, \dots}_{\text{orthogonal source}} \overset{m_0}{\longmapsto} \underbrace{1, 1, 1, \dots}_{\text{coherent output}}}$$

Let $F_n = \operatorname{span}\{p_1, \dots, p_n\}$ and let E_r be the orthogonal projection onto F_r. Define $m_{n,r} = m_0 E_r|_{F_n}$.

Theorem 3.1 (Two-Axis Fusion-Face Boundary)**.** *For every finite n, r,*

$$\boxed{\|m_{n,r}\| = \sqrt{\min(n,r)}.} \tag{4}$$

For fixed r, $m_{\infty,r} = m_0 E_r$ is bounded with norm $\sqrt{r}$. For fixed n, $m_{n,\infty} = m_0|_{F_n}$ is bounded with norm $\sqrt{n}$. The simultaneous totalization $m_{\infty,\infty} = m_0|_{\operatorname{span}_{\mathrm{alg}}\{p_i\}}$ is nonclosable.

Proof. For $x = \sum_{i=1}^n a_i p_i$,

$$m_{n,r}(x) = \left(\sum_{i=1}^{\min(n,r)} a_i\right) 1, \qquad \|x\|_2^2 = \sum_{i=1}^n |a_i|^2.$$

Cauchy–Schwarz gives the upper bound. With $q = \min(n,r)$ and $x_q = q^{-1/2} \sum_{i=1}^q p_i$, equality follows. For the simultaneous limit set $\xi_n = n^{-1} \sum_{i=1}^n p_i$. Then $\|\xi_n\|_2^2 = 1/n \to 0$ while $m_{\infty,\infty}(\xi_n) = 1$, proving nonclosability. □

The stronger amplified statement $\operatorname{mul} \overline{\Gamma(m_0)} = L^2(N, \tau)$ was established in Paper I [1]; we use only nonclosability here.

The finite profile $\rho(n,r) = \min(n,r)$ has discrete mixed increment equal to 1 on the diagonal $n = r$ and 0 off it. The diagonal pattern localizes where new source information first meets new face exposure, but it does not explain the singularity. The decisive mechanism is coherent assembly of orthogonal source directions into one target direction.

4 Mixed Incidence and Assembly

The mixed fusion boundary shows that the doubly infinite corner can fail even when both marginal limits remain bounded. We now distinguish which new reconstruction components see which new source increments from how those outputs are assembled.

4.1 Mixed incidence

Let $D_0 \subset D_1 \subset \cdots$ be an increasing algebraic source filtration and let $\delta_r : D_\infty^{\mathrm{alg}} \to Z_r$ denote the rth incremental reconstruction component.

Definition 4.1 (Mixed incidence)**.** The algebraic mixed-incidence map at (n,r) is

$$\mathcal{I}_{n,r}^{\mathrm{alg}} : D_n/D_{n-1} \to Z_r/\delta_r(D_{n-1}), \qquad \mathcal{I}_{n,r}^{\mathrm{alg}}([x]) = [\delta_r(x)]. \tag{5}$$

Proposition 4.2. *The map $\mathcal{I}_{n,r}^{\mathrm{alg}}$ is well defined, and $\mathcal{I}_{n,r}^{\mathrm{alg}} = 0$ iff $\delta_r(D_n) = \delta_r(D_{n-1})$.*

Proof. If $x - x' \in D_{n-1}$, then $\delta_r(x) - \delta_r(x') \in \delta_r(D_{n-1})$. The second assertion is immediate from $D_{n-1} \subseteq D_n$. $\square$

For the fusion witness, $\delta_r(p_i) = \delta_{ri}1$, hence:

Proposition 4.3 (Diagonal incidence in the fusion model)**.** *$\mathcal{I}^{\mathrm{alg}}_{n,r} \neq 0$ iff $n = r$.*

	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$\cdots$
$n = 1$	$\bullet$	0	0	0	$\cdots$
$n = 2$	0	$\bullet$	0	0	$\cdots$
$n = 3$	0	0	$\bullet$	0	$\cdots$
$n = 4$	0	0	0	$\bullet$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

4.2 Identical incidence, different assembly

Let $X = \ell^2(\mathbb{N})$, $D = c_{00}(\mathbb{N})$, and $\delta_r(x) = x_r$. The mixed-incidence maps are diagonal. Retain the increments orthogonally with $J_r^{\oplus}(x) = (x_1, \ldots, x_r)$; the total passage extends to the identity on ℓ^2. Alternatively assemble them additively with $J_r^+(x) = \sum_{k=1}^r x_k$. Then $J_\infty^+(x) = \sum_k x_k$ is nonclosable because $\xi_n = n^{-1}\sum_{k=1}^n e_k \to 0$ while $J_\infty^+(\xi_n) = 1$.

Theorem 4.4 (Assembly Dependence)**.** *There exist two two-axis Hilbert reconstruction systems with identical source filtration, identical reconstruction increments, and identical mixed-incidence maps, such that one joint totalization is bounded and the other is nonclosable. Consequently,*

$$\boxed{\textit{mixed incidence does not determine joint analytic totalization.}}$$

This motivates the factorization

$$\boxed{\text{mixed generation} \longrightarrow \text{retention} \longrightarrow \text{assembly.}}$$

4.3 Gram geometry of assembly

Let $e_1, \ldots, e_n$ be orthonormal retained directions and $T_n e_i = y_i$. Define $G_n = (\langle y_i, y_j \rangle)_{i,j=1}^n$.

Proposition 4.5 (Gram assembly formula)**.** $\|T_n\|^2 = \|G_n\|$.

Proof. For $x = \sum_i a_i e_i$, $\|x\|^2 = a^* a$ and $\|T_n x\|^2 = a^* G_n a$. Take the supremum of the Rayleigh quotient. $\square$

Orthogonal assembly gives $G_n = I_n$ and norm 1. Equal-image assembly gives $G_n = \mathbf{1}_n \mathbf{1}_n^*$ and norm $\sqrt{n}$. This is exactly the geometry of the fusion witness.

Divergence of finite assembly norms is not itself an obstruction: the diagonal operator $D(a_1, a_2, \ldots) = (a_1, 2a_2, 3a_3, \ldots)$ is closed on its natural domain although its finite truncation norms diverge. The positive theory must therefore be graph-theoretic rather than uniformly bounded.

5 The Joint Reconstruction Certificate

Consider

$$D \xrightarrow{W_0} Z \xrightarrow{S} Y. \qquad T_0 = SW_0.$$

Let $D \subset X$ be dense, $W_0 : D \to Z$ closable, and $S : D(S) \subset Z \to Y$ closed with $W_0(D) \subset D(S)$. Equip $Z_S := D(S)$ with graph norm $\|z\|_{Z_S}^2 = \|z\|_Z^2 + \|Sz\|_Y^2$. Let $B_S : Z_S \to Y$, $B_S z = Sz$, and define $W_0^\sharp : D \to Z_S$ by $W_0^\sharp x = W_0 x$. Then $T_0 = B_S W_0^\sharp$.

Proposition 5.1. *If W_0 is closable and S closed, then $W_0^\sharp$ is closable.*

Proof. If $x_n \to 0$ in X and $W_0^\sharp x_n \to z$ in Z_S, then $W_0 x_n \to z$ in Z. Closability of W_0 gives $z = 0$. □

Definition 5.2 (Relative coercivity)**.** The presentation $T_0 = SW_0$ is relatively coercive if there exists $C < \infty$ such that

$$\boxed{\|W_0 x\|_Z \le C\left(\|x\|_X^2 + \|T_0 x\|_Y^2\right)^{1/2}} \tag{6}$$

for every $x \in D$.

Proposition 5.3 (Graph-norm equivalence)**.** *If relative coercivity holds, then*

$$\|x\|_{T_0} \le \|x\|_{W_0^\sharp} \le \sqrt{1 + C^2}\,\|x\|_{T_0}.$$

Proof. Use $\|x\|_{W_0^\sharp}^2 = \|x\|_X^2 + \|W_0 x\|_Z^2 + \|T_0 x\|_Y^2$ and apply (6). □

Theorem 5.4 (Joint Reconstruction Certificate)**.** *Under the hypotheses above, if the presentation is relatively coercive, then T_0 is closable. Moreover,*

$$D(\overline{T}_0) = D(\overline{W_0^\sharp})$$

as subsets of X, and

$$\boxed{\overline{T}_0 = B_S \overline{W_0^\sharp}.} \tag{7}$$

Proof. By Theorem 5.1, $W_0^\sharp$ is closable. Theorem 5.3 identifies the two graph completions. Since B_S is bounded, the closure identity follows. □

5.1 Compatibility failure

Let $X = \ell^2(\mathbb{N})$, $D = c_{00}$, $A_0(a_1, a_2, \ldots) = (a_1, 2a_2, 3a_3, \ldots)$, and $T_0(a) = \sum_k a_k$. Set $W_0 a = (A_0 a, T_0 a)$ and $S(z, c) = c$. Then S is bounded and W_0 is closable because

$$|T_0 a| \le \left(\sum_{k \ge 1} k^{-2}\right)^{1/2} \|A_0 a\|.$$

Yet T_0 is nonclosable on $\xi_n = n^{-1} \sum_{k=1}^n e_k$, while $\|W_0 \xi_n\| \to \infty$.

Theorem 5.5 (Compatibility Failure)**.** *There exist a closable retention map W_0 and bounded assembly S such that SW_0 is nonclosable.*

Within the present architecture, failure may occur in retention (R), assembly (A), or compatibility (C). These are not asserted to exhaust all analytic obstructions.

6 Certificate Equivalence and Cofinal Control

Let $T_0 = SW_0$ satisfy the hypotheses of Section 5. Define the effective retained graph carrier

$$\mathcal{Z}_{\mathrm{eff}} = \overline{W_0^\sharp(D)}^{Z_S}. \tag{8}$$

Suppose the same algebraic passage has a second presentation $T_0 = \widetilde{S}\widetilde{W}_0$.

$$\begin{tikzcd}
D \arrow[r, "W_0^\sharp"] & \mathcal{Z}_{\mathrm{eff}} \arrow[r, "B_S"] \arrow[d, "U"'] & Y \arrow[d, equal] \\
D \arrow[r, "\widetilde{W}_0^\sharp"'] & \widetilde{\mathcal{Z}}_{\mathrm{eff}} \arrow[r, "B_{\widetilde{S}}"'] & Y
\end{tikzcd}$$

Definition 6.1 (Graph equivalence)**.** The two presentations are graph equivalent if there is a boundedly invertible $U : \mathcal{Z}_{\mathrm{eff}} \to \widetilde{\mathcal{Z}}_{\mathrm{eff}}$ with $UW_0^\sharp = \widetilde{W}_0^\sharp$ and $B_{\widetilde{S}}U = B_S$.

Theorem 6.2 (Graph-Equivalent Certificate Invariance)**.** *If two presentations of the same T_0 are graph equivalent and one satisfies the Joint Reconstruction Certificate, then so does the other; both determine the same closed operator $\overline{T}_0$.*

Proof. Graph equivalence gives equivalent retained graph norms, so both completions have the same source domain. Decoder intertwining then gives the same closed passage. □

6.1 Uncontrolled refinement

Take $T_0 = 1$ on $c_{00} \subset \ell^2$. Replace the kth retained coordinate by k identical copies: $W_0'e_k = (1, \dots, 1) \in \mathbb{C}^k$. Let S' average each block. Then $S'W_0' = T_0$ and S' is bounded, but $\|W_0'e_k\| = \sqrt{k}$, so relative coercivity fails. Thus

$$\boxed{T_0 \text{ unchanged}, \qquad \text{certificate destroyed.}}$$

This shows that algebraic cofinality is weaker than certificate cofinality.

6.2 Cofinal systems

Let $\mathcal{Z}_C^{\mathrm{eff}}$ and $\widetilde{\mathcal{Z}}_C^{\mathrm{eff}}$ be directed effective graph systems. Suppose boundedly invertible maps U_C satisfy

$$U_D j_{C,D} = \widetilde{j}_{C,D} U_C, \tag{9}$$

and $\sup_C \|U_C\|, \sup_C \|U_C^{-1}\| < \infty$.

Theorem 6.3 (Cofinal Extension of Certificate Equivalences)**.** *The U_C induce a boundedly invertible map*

$$U_\infty : \mathcal{Z}_{\mathrm{eff},\infty} \xrightarrow{\sim} \widetilde{\mathcal{Z}}_{\mathrm{eff},\infty},$$

with the same uniform two-sided bound.

Proof. Define $U_{\mathrm{alg}}[z_C] = [U_C z_C]$. Compatibility makes it well defined; uniform boundedness and the same argument for the inverses extend it to the completions. □

The full direct-limit bookkeeping is recorded in Section C.

7 Strict and Relational Transport

The analytic theory above assumes sufficiently strict transport between finite presentations. We now separate strict embedded transport from genuinely relational comparison.

Strict transport	**Relational transport**
embeddings	correspondences
mark preserved	comparison data required
associativity inherited	associativity tested
direct totalization	strictifier problem

7.1 Strict marked transport

We use standard Hilbert-module and Morita-theoretic terminology; see [3, 5, 6]. Let $(\mathcal{L}_C, p_C)$ be a directed family of marked von Neumann algebras with faithful normal unital mark-preserving embeddings. Write $q_C = 1 - p_C$ and

$$M_C = p_C \mathcal{L}_C p_C, \qquad N_C = q_C \mathcal{L}_C q_C, \qquad \mathcal{H}_C = p_C \mathcal{L}_C q_C.$$

Assume a common faithful normal concrete realization as an increasing family in $B(K)$, with the marks identified with one projection p. Set

$$\mathcal{L}_\infty = \left(\bigcup_C \mathcal{L}_C \right)''.$$

Theorem 7.1 (Marked Linking Stability)**.** *Under these hypotheses,*

$$p\mathcal{L}_\infty p = \overline{\bigcup_C p\mathcal{L}_C p}^{\,\mathrm{uw}}, \qquad q\mathcal{L}_\infty q = \overline{\bigcup_C q\mathcal{L}_C q}^{\,\mathrm{uw}},$$

and

$$p\mathcal{L}_\infty q = \overline{\bigcup_C p\mathcal{L}_C q}^{\,\mathrm{uw}}.$$

Thus the coefficient algebras, correspondence corner, and marking totalize simultaneously.

Proof. The forward inclusions are immediate. Conversely, by Kaplansky density [12], any $x \in p\mathcal{L}_\infty p$ is the ultraweak limit of a bounded net $px_\lambda p$ with x_λ drawn from finite stages. The other corners are identical. □

Corollary 7.2 (Cofinal fullness stability)**.** *If p is full in $\mathcal{L}_C$ for every C in a cofinal subsystem, then p is full in $\mathcal{L}_\infty$.*

The strict theorem is a control regime. Outside it, finite realizations may be comparable without canonical embeddings into one another.

7.2 Relational comparison systems

Let $A_0, A_1, \ldots, A_n$ be von Neumann algebras and let ${}_{A_i}\mathcal{H}_{ij\,A_j}$ be correspondences. For each $i < j < k$, suppose there is a specified compositor

$$\mu_{ijk} : \mathcal{H}_{ij} \bar{\otimes}_{A_j} \mathcal{H}_{jk} \to \mathcal{H}_{ik}.$$

For relative tensor products and Connes fusion, see [10, 11]. We assume the indicated products exist in the chosen realization class.

For $i < j < k < \ell$ define

$$L_{ijk\ell} = \mu_{ik\ell}(\mu_{ijk} \otimes 1), \qquad R_{ijk\ell} = \mu_{ij\ell}(1 \otimes \mu_{jk\ell}),$$

and, when both maps are unitary onto the same target,

$$\boxed{\Theta_{ijk\ell} = R^*_{ijk\ell} L_{ijk\ell}.} \tag{10}$$

Theorem 7.3 (Relational Associator Obstruction)**.** *If the specified relational comparison system admits a strict linking realization in which every compositor is implemented by multiplication in one common algebra, then $\Theta_{ijk\ell} = 1$ for every composable quadruple.*

Proof. The two routes become $(x_{ij}x_{jk})x_{k\ell}$ and $x_{ij}(x_{jk}x_{k\ell})$. Associativity forces equality. □

Thus $\Theta \neq 1$ obstructs strict realization of the specified compositors, but not relational existence itself.

7.3 Scalar phase model

Take $A_i = \mathbb{C}$, $\mathcal{H}_{ij} = \mathbb{C}$, and $\mu_{ijk}(z \otimes w) = \alpha_{ijk} z w$ with $\alpha_{ijk} \in U(1)$. Then

$$\boxed{\Theta_{0123} = \frac{\alpha_{012}\alpha_{023}}{\alpha_{123}\alpha_{013}}.}$$

Under pairwise rephasing $\beta_{ij} \in U(1)$, the compositor phases transform by $\alpha'_{ijk} = \beta_{ik}^{-1}\beta_{ij}\beta_{jk}\alpha_{ijk}$ and the displayed Θ is unchanged. This proves only rephasing invariance, not completeness under arbitrary strictification.

8 Strictifier Persistence

Even when every finite relational context can be strictified, the strictifiers themselves must persist coherently through refinement.

Let $\mathrm{Str}(C)$ denote the admissible relational strictifiers of a finite context C, with restriction maps $\rho_{D,C} : \mathrm{Str}(D) \to \mathrm{Str}(C)$ satisfying the inverse-system identities. A coherent total strictifier is a family $\beta = (\beta_C)_C \in \varprojlim_C \mathrm{Str}(C)$.

Definition 8.1 (Persistent strictifiability)**.** The system is persistently strictifiable if $\varprojlim_C \mathrm{Str}(C) \neq \varnothing$.

8.1 Finite strictifiability is insufficient

Consider $C_1 \leq C_2 \leq \cdots$ with

$$\mathrm{Str}(C_n) = \{n, n+1, n+2, \ldots\}$$

and identity restrictions on surviving integers.

Proposition 8.2 (Abstract failure of strictifier persistence)**.** *Every* $\mathrm{Str}(C_n)$ *is nonempty, but* $\varprojlim_n \mathrm{Str}(C_n) = \varnothing$.

Proof. A coherent element would be a single integer belonging to every tail $\{n, n+1, \ldots\}$, which is impossible. □

This is an inverse-system counterexample, not yet a concrete correspondence-level realization.

Definition 8.3 (Finite compatible strictifiability)**.** The inverse system is finitely compatibly strictifiable if every finite collection of compatibility equations $\rho_{D,C}(\beta_D) = \beta_C$ admits a simultaneous solution.

8.2 Compact persistence

Assume each $\mathrm{Str}(C)$ is nonempty compact Hausdorff and each restriction map continuous. Let $\mathfrak{P} = \prod_C \mathrm{Str}(C)$ and for $C \leq D$ let $K_{D,C}$ be the closed compatibility locus. Finite compatible strictifiability is exactly the finite-intersection property for these loci.

Theorem 8.4 (Compact Strictifier Persistence)**.** *If every* $\mathrm{Str}(C)$ *is nonempty compact Hausdorff, the restriction maps are continuous, the inverse-system identities hold, and the system is finitely compatibly strictifiable, then* $\varprojlim_C \mathrm{Str}(C)$ *is nonempty compact Hausdorff.*

Proof. Tychonoff gives compactness of $\mathfrak{P}$. The closed compatibility loci have the finite-intersection property, so their total intersection is nonempty; it is precisely the inverse limit. □

Compactness gives existence, not uniqueness. More importantly, a coherent family $(\beta_C)_C$ solves only the relational problem. It need not preserve analytic certificate geometry.

9 Certificate-Bearing Strictification

To couple relational and analytic persistence, we enlarge the finite strictifier datum.

Let $T_C = S_C W_C$ and let $\mathcal{Z}_C^{\mathrm{eff}}$ be the effective retained graph carrier. Suppose $\beta_C \in \mathrm{Str}(C)$ strictifies the finite relational context, with strictified presentation $T_C^{\beta_C} = S_C^{\beta_C} W_C^{\beta_C}$ and effective graph carrier $\mathcal{Z}_{C,\beta_C}^{\mathrm{eff}}$.

Definition 9.1 (Certificate-bearing strictifier datum)**.** A certificate-bearing strictifier datum is a pair

$$\boxed{\sigma_C = (\beta_C, U_C)}$$

where $U_C : \mathcal{Z}_C^{\mathrm{eff}} \xrightarrow{\sim} \mathcal{Z}_{C,\beta_C}^{\mathrm{eff}}$ is boundedly invertible and satisfies

$$U_C W_C^{\sharp} = (W_C^{\beta_C})^{\sharp}, \qquad B_{S_C^{\beta_C}} U_C = B_{S_C}.$$

Its certificate distortion is $\mathrm{dist}(\sigma_C) = \max\{\|U_C\|, \|U_C^{-1}\|\}$.

Proposition 9.2. *If $\sigma_C = (\beta_C, U_C)$ is certificate bearing, then $T_C^{\beta_C} = T_C$.*

Proof. Apply the decoder intertwining to $U_C W_C^\sharp = (W_C^{\beta_C})^\sharp$. □

Theorem 9.3 (Certificate-Preserving Strictification)**.** *If a finite context admits a Joint Reconstruction Certificate and $\sigma_C = (\beta_C, U_C)$ is certificate bearing, then the strictified presentation also admits a Joint Reconstruction Certificate and determines the same closed demanded passage.*

Proof. The two effective graph presentations are graph equivalent, so apply Theorem 6.2. □

For $M \geq 1$ define

$$\mathrm{StrCert}_M(C) = \{\sigma_C = (\beta_C, U_C) : \mathrm{dist}(\sigma_C) \leq M\}. \tag{11}$$

For $C \leq D$, the pair restriction requires both $\rho_{D,C}(\beta_D) = \beta_C$ and

$$\boxed{U_D j_{C,D} = j^\beta_{C,D} U_C.} \tag{12}$$

Thus relational and graph compatibility belong to the same inverse system.

Theorem 9.4 (Compact Certificate-Bearing Strictifier Persistence)**.** *Fix $M < \infty$. If every $\mathrm{StrCert}_M(C)$ is nonempty compact Hausdorff, the pair-restriction maps are continuous and satisfy the inverse-system identities, and every finite subdiagram admits a compatible pair-valued section, then*

$$\varprojlim_C \mathrm{StrCert}_M(C) \neq \varnothing.$$

Proof. Apply Theorem 8.4 to the pair-valued inverse system. □

Any element has the form $\sigma = ((\beta_C, U_C))_C$. By Theorem 6.3, the finite U_C extend to a boundedly invertible total graph equivalence whenever the completed direct limits exist.

Proposition 9.5 (Persistent pair data induce total graph equivalence)**.** *If $\sigma = ((\beta_C, U_C))_C \in \varprojlim_C \mathrm{StrCert}_M(C)$, then the finite U_C extend to*

$$U_{\sigma,\infty} : \mathcal{Z}_{\mathrm{eff},\infty} \xrightarrow{\sim} \mathcal{Z}^\beta_{\mathrm{eff},\infty}$$

with $\|U_{\sigma,\infty}\|, \|U_{\sigma,\infty}^{-1}\| \leq M$, and decoder intertwining survives.

Theorem 9.6 (Certificate-Bearing Strictification Bridge)**.** *Suppose the finite analytic system satisfies one uniform relative-coercivity estimate with constant K, and for some $M < \infty$ the pair-valued spaces $\mathrm{StrCert}_M(C)$ satisfy the preceding compact persistence hypotheses. Then there exists a coherent family $\sigma = ((\beta_C, U_C))_C$ such that $T_C^{\beta_C} = T_C$ and*

$$\|(W_C^{\beta_C})^\sharp x\| \leq M\sqrt{1+K^2}\,\|x\|_{T_C}$$

uniformly in C.

10 Integrated Joint Reconstruction

We now combine the analytic and relational persistence requirements.

Let C be directed. For each C, let $D_C \subset X_C$ be dense, let $W_C : D_C \to Z_C$ be closable, let $S_C : D(S_C) \subset Z_C \to Y_C$ be closed, and define $T_C = S_C W_C$. Assume compatible source, retained, and output transports. Let

$$D_\infty^{\mathrm{alg}} = \varinjlim_C D_C \subset X_\infty$$

be dense, let $W_\infty^{\mathrm{alg}} : D_\infty^{\mathrm{alg}} \to Z_\infty$ be the compatible algebraic total retention, and assume it is closable. Let $S_\infty : D(S_\infty) \subset Z_\infty \to Y_\infty$ be a compatible closed total assembly with $W_\infty^{\mathrm{alg}}(D_\infty^{\mathrm{alg}}) \subset D(S_\infty)$. Set

$$T_\infty^{\mathrm{alg}} = S_\infty W_\infty^{\mathrm{alg}}.$$

Definition 10.1 (Uniform finite relative coercivity)**.** The finite system is uniformly relatively coercive if there exists $K < \infty$ such that

$$\boxed{\|W_C x\|_{Z_C} \le K\big(\|x\|_{X_C}^2 + \|T_C x\|_{Y_C}^2\big)^{1/2}} \tag{13}$$

for every finite context C and every $x \in D_C$.

Lemma 10.2 (Uniform coercivity passes to the total core)**.** *Assume the directed embeddings are isometric. If* (13) *holds, then the same estimate holds on D_∞^{alg} for W_∞^{alg} and T_∞^{alg}.*

Proof. Represent an algebraic-limit vector at a finite stage and use isometric compatibility. □

Theorem 10.3 (Uniform Finite Certificate Totalization)**.** *If D_∞^{alg} is dense, W_∞^{alg} is closable, S_∞ is closed, the retained image lies in $D(S_\infty)$, and the finite system satisfies uniform relative coercivity, then T_∞^{alg} satisfies the Joint Reconstruction Certificate and is closable.*

Proof. By Theorem 10.2, the total algebraic presentation satisfies relative coercivity. Apply Theorem 5.4. □

Suppose now the relational system admits pair-valued certificate-bearing spaces $\mathrm{StrCert}_M(C)$ satisfying Theorem 9.4. Choose

$$\sigma = ((\beta_C, U_C))_C \in \varprojlim_C \mathrm{StrCert}_M(C).$$

Then $T_C^{\beta_C} = T_C$ and the strictified finite system inherits a uniform graph-control bound depending only on K and M.

Assume the strictified finite retention maps admit a compatible closable totalization $W_\infty^{\beta,\mathrm{alg}}$ and the strictified assemblies admit a compatible closed totalization S_∞^β. Define

$$T_\infty^{\beta,\mathrm{alg}} = S_\infty^\beta W_\infty^{\beta,\mathrm{alg}}.$$

uniform finite analytic certificate + compact certificate-bearing persistence +compatible analytic totalization $\Downarrow$ closable strict joint realization

Theorem 10.4 (Integrated Joint Reconstruction)**.** *Let* C *be a directed relational reconstruction system. Assume:*

(i) *each finite context admits a compatible analytic presentation* $T_C = S_C W_C$*;*

(ii) *the finite source system admits a dense algebraic total core* $D_\infty^{\mathrm{alg}} \subset X_\infty$*;*

(iii) *the finite retention maps admit a compatible closable totalization;*

(iv) *the finite assemblies admit a compatible closed totalization;*

(v) *one uniform finite relative-coercivity constant* K *exists;*

(vi) *for some* $M < \infty$*, the pair-valued spaces* $\mathrm{StrCert}_M(C)$ *form a nonempty compact Hausdorff inverse system with continuous restrictions and finite compatible solvability;*

(vii) *the corresponding strictified finite retention and assembly systems admit compatible totalizations with closable total retention and closed total assembly.*

Then there exists

$$\sigma = ((\beta_C, U_C))_C \in \varprojlim_C \mathrm{StrCert}_M(C)$$

such that $T_\infty^{\beta,\mathrm{alg}}$ *satisfies the Joint Reconstruction Certificate. Consequently,*

$$\boxed{T_\infty^{\beta,\mathrm{alg}} \text{ is closable.}} \tag{14}$$

Proof. Pair persistence. Apply Theorem 9.4. *Certificate transport.* The pair-valued restriction law and Theorem 9.5 give a uniformly bounded total graph equivalence. *Finite analytic control.* Apply Theorem 9.6. *Total analytic control.* Uniform finite coercivity passes to the strictified algebraic core. *Closure.* Apply Theorem 10.3. □

The theorem is sufficient, not exhaustive. It identifies one controlled route to a strict joint analytic realization; failure of a hypothesis does not imply disappearance of the underlying algebraic or relational passage.

11 Conclusion and Open Problems

The paper develops a sufficient finite-to-total reconstruction theory for systems with two interacting directions of growth. Analytically, marginal totalizability and mixed incidence do not determine the doubly infinite corner; retention, assembly, and relative coercivity are required. Relationally, strict realization may be obstructed by a relative associator, and finite strictifiers may fail to persist. The two theories meet through certificate-bearing pairs $\sigma_C = (\beta_C, U_C)$.

finite solvability does not imply persistence of the data that certify solvability.

Within the retention–assembly architecture, three visible analytic failure modes are retention failure, assembly failure, and compatibility failure. These are not asserted to be exhaustive. On the relational side, local associator failure and failure of persistence of local repairs are distinct.

Several realization boundaries remain essential:

$$\boxed{\text{bounded} \neq \text{closed-unbounded} \neq \text{nonclosable}},$$

$$\boxed{\text{failure of one certificate presentation} \neq \text{failure of the passage}},$$

and

$$\boxed{\text{failure of strictification} \neq \text{failure of relational existence}}.$$

Open analytic problems include necessity or minimality of relative coercivity, canonical minimal representatives of graph-equivalence classes, and intrinsic finite criteria for closed total assembly. Open relational problems include concrete operator-algebraic realizations of strictifier escape, compactness criteria for $\mathrm{StrCert}_M(C)$, and uniqueness after compact existence.

The pair-valued formulation also suggests the quantitative boundary

$$\kappa(C) = \inf\{M \geq 1 : \mathrm{StrCert}_M(C) \neq \varnothing\}.$$

A system with finite certificate-bearing strictifiers but $\kappa(C) \to \infty$ would exhibit a genuinely integrated relational–analytic escape.

The present theorem reaches a total analytic object by strictification. A direct relational totalization theory may instead retain comparisons themselves in the total object. Likewise, the relative associator Θ may be only the first explicit higher-coherence defect. No cohomological degree, K-theoretic classification, or higher-categorical ontology is claimed here.

The three-paper progression is

$$\boxed{\text{operation} \longrightarrow \text{reconstruction} \longrightarrow \text{joint reconstruction.}}$$

The doubly infinite corner is not determined by its two edges; its survival depends on information transported through the finite interior.

$$\boxed{\textbf{The total passage survives when the data that certify its survival survive with it.}}$$

A Closed and Self-Dual Control Examples

The examples in this appendix show that divergent finite norms, unbounded limiting realizations, and enlargement under completion are not by themselves obstructions.

A.1 Closed unbounded synthesis

Let $D_0 : c_{00}(\mathbb{N}) \to \ell^2(\mathbb{N})$ be $D_0(a_1, a_2, \ldots) = (a_1, 2a_2, 3a_3, \ldots)$. Its finite truncations have norms n, yet D_0 is closable and its closure is the closed diagonal operator with domain $\{a \in \ell^2 : \sum k^2 |a_k|^2 < \infty\}$. Thus divergence of finite synthesis norms does not imply analytic failure.

A.2 Self-dual completion

For foundational self-dual module theory see [3, 4]. Let $A = \ell^\infty(\mathbb{N})$ and let e_n be the coordinate projections. The sequence $\xi = (e_1, e_2, \ldots)$ belongs to the weakly square-summable self-dual completion because $\sum_{n=1}^{N} e_n$ is uniformly bounded and converges weak-* to 1, but it does not lie in the norm module $\ell^2(A)$ because the same sums do not converge in norm. Hence $\ell^2(A) \subsetneq \ell^2_w(A)$.

The appearance of new completion vectors does not itself constitute leakage; the relevant question is whether the completion is canonical in the chosen realization topology.

B Module and Correspondence Gram Geometry

We use standard Hilbert C^*-module and self-dual W^*-module terminology from [3, 5]. The unbounded-operator extension requires additional care: regularity is extra structure in the standard theory [7, 8, 9].

Let P be a C^*-algebra, $\mathcal{E}$ a right Hilbert P-module, and $\zeta_1, \dots, \zeta_n \in \mathcal{E}$. Define $A_n : P^n \to \mathcal{E}$ by $A_n(a_1, \dots, a_n) = \sum_i \zeta_i a_i$, and define the operator-valued Gram matrix $G_n = [\langle \zeta_i, \zeta_j \rangle_P]$.

Proposition B.1 (Module Gram identity). *A_n is adjointable, $A_n^* A_n = G_n$, and*

$$\boxed{\|A_n\|^2 = \|G_n\|.}$$

Proof. The adjoint is $A_n^* \eta = (\langle \zeta_i, \eta \rangle)_i$. Matrix multiplication gives $A_n^* A_n = G_n$, and the C^* identity gives the norm formula. □

Orthogonal increments give $G_n = I_n$; coherent equal-image increments give $G_n = (\mathbf{1}_n \mathbf{1}_n^*) \otimes 1_P$ and $\|A_n\| = \sqrt{n}$.

For module-valued retention and assembly, let $W_0 : D \to \mathcal{Z}$ and $S : D(S) \subset \mathcal{Z} \to \mathcal{Y}$ live in a realization class for which graph closure is available. Define the graph module $\mathcal{Z}_S$ with inner product $\langle z, w \rangle + \langle Sz, Sw \rangle$ and graph lift $W_0^\sharp$.

Theorem B.2 (Module Joint Reconstruction Certificate). *Assume W_0 is closable, S closed, $W_0(D) \subset D(S)$, the graph lift $W_0^\sharp$ is closable in the chosen module realization, and module relative coercivity holds. Then $T_0 = SW_0$ is closable and $\overline{T}_0 = B_S \overline{W_0^\sharp}$.*

Proof. If $x_n \to 0$ and $T_0 x_n \to y$, relative coercivity makes $W_0 x_n$ Cauchy. Hence $W_0^\sharp x_n$ converges in the graph module; closability forces the limit to vanish. Applying the bounded decoder gives $y = 0$. □

The module theorem is deliberately conditional on the chosen unbounded-operator category. The analytic mechanism remains the same: the difficulty is not the number of retained increments, but the geometry of their assembly.

C Technical Direct-Limit Lemmas

This appendix records the directed-limit facts used in the main theorem.

Let $\{H_C, j_{C,D}\}$ be a directed Hilbert system with isometric transition maps, and let H_∞ be the completion of its algebraic direct limit.

Lemma C.1 (Uniformly bounded direct-limit extension). *If compatible bounded maps $T_C : H_C \to K_C$ satisfy $\sup_C \|T_C\| \le M$, then they extend uniquely to a bounded map $T_\infty : H_\infty \to K_\infty$ with $\|T_\infty\| \le M$.*

Proof. Define $T_{\mathrm{alg}}[x_C] = [T_C x_C]$. Compatibility makes this well defined, and the uniform norm bound gives continuity. □

Lemma C.2 (Direct-limit extension of uniform equivalences)**.** *If compatible $U_C : H_C \to \widetilde{H}_C$ are boundedly invertible with $\|U_C\|, \|U_C^{-1}\| \le M$, then they extend to a boundedly invertible U_∞ with the same two-sided bound.*

Lemma C.3 (Passage of uniform finite inequalities)**.** *If compatible maps W_C, T_C satisfy one finite estimate*

$$\|W_C x\| \le K(\|x\|^2 + \|T_C x\|^2)^{1/2}$$

through an isometric directed system, then the same estimate holds on the algebraic direct limit.

Let S_C be compatible closed assemblies and let $G_C = (Z_C)_{S_C}$ be their graph carriers. If the retained and output transports are isometric and intertwine the assemblies, then graph transport is isometric. Consequently the effective graph carriers $\mathcal{Z}_C^{\mathrm{eff}} = \overline{W_C^\sharp(D_C)}$ form a directed Hilbert system.

Lemma C.4 (Decoder intertwining at the limit)**.** *If finite graph equivalences U_C and decoders satisfy $\widetilde{B}_C U_C = B_C$ compatibly and uniformly, then the corresponding total extensions satisfy $\widetilde{B}_\infty U_\infty = B_\infty$.*

For pair-valued strictifier data $\sigma_C = (\beta_C, U_C) \in \mathrm{StrCert}_M(C)$, the restriction law $U_D j_{C,D} = j^\beta_{C,D} U_C$ is exactly the compatibility required by Theorem C.2. Thus the U_C extend to $U_{\sigma,\infty}$ with distortion bounded by M.

Crucially, finite closability does not imply total closability, and finite closed assembly does not imply existence of a closed total assembly. These remain explicit hypotheses of Theorem 10.4.

Proposition C.5 (Direct-Limit Certificate Transport)**.** *Let a directed finite reconstruction system possess compatible effective graph carriers, bounded graph decoders, and a uniform finite relative-coercivity estimate. Suppose a coherent family of certificate-bearing data induces compatible graph equivalences with one uniform distortion bound. Then the graph equivalences and decoders totalize and the finite coercivity estimate passes to the algebraic total core. If, additionally, total retention is closable and total assembly is closed, then the strictified total passage satisfies the Joint Reconstruction Certificate.*

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