

Vacuum Readjustment under Gauge-Cell Attachment

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Adding a gauge cell in its isolated interacting vacuum need not preserve the vacuum of the system receiving it. We establish this distinction on a four-edge, three-cell $SU(2)$ graph, without a representation cutoff. Starting from the edge Casimirs, we derive the shared electric interaction and an exact receiving-defect identity: its squared norm is the product of the added vacuum's electric response and the old state's seam-electric energy, with coefficient $4A^2/3$. The native ground state is not a product across the specified cycle-variable partition, and its restriction to a declared gauge-invariant old-system algebra differs strictly from the old pure vacuum. Near zero seam coupling, the old-vacuum excitation probability has a strictly positive quadratic coefficient. We distinguish the fixed cycle-factor vacuum test from the retained old-edge energy and show that vacuum matching cannot preserve the fixed test. An Abelian control separates this attachment mechanism from specifically non-Abelian effects. These are finite-graph results, not a spacetime localization or continuum mass-gap theorem.

The comparison problem.—A new cell arrives with a vacuum of its own. The old graph has one too. There is no reason to assume that putting the two together gives the vacuum of the joined graph: the shared edge still has to be accounted for. Here that question has a direct answer. We derive the electric coupling from the original edge generators and ask which vacuum comparison it actually changes.

Hamiltonian lattice gauge theory provides a setting in which this distinction can be computed directly from edge rotors [1]; related loop and maximal-tree reductions appear in [2]. Gauge constraints also require care in specifying subsystem observables: a tensor product of cycle-variable spaces is not a tensor product of independent physical gauge systems [4]. Here the partition, the observable algebra, and the vacuum comparison are fixed explicitly. The result identifies what fails under isolated-vacuum insertion, and what must change when the insertion is dressed to match the full vacuum.

Edge-derived model.—Let four edges $u_0, u_1, u_2, u_3 \in SU(2)$ join the same two vertices, with endpoint action $u_i \mapsto au_i b^{-1}$. Use normalized Haar measure and cycles

$$x = u_0 u_1^{-1}, \quad y = u_1 u_2^{-1}, \quad z = u_2 u_3^{-1}. \quad (1)$$

The inverse coordinates are $u_0 = xyzg$, $u_1 = yzg$, $u_2 = zg$, and $u_3 = g$; they preserve product Haar measure. Removing the right endpoint action leaves simultaneous conjugation on (x, y, z) . Thus

$$\begin{aligned} \mathcal{K}_2 &= L^2(SU(2)^2), & \mathcal{K}_z &= L^2(SU(2)), \\ \mathcal{H}_3 &= (\mathcal{K}_2 \otimes \mathcal{K}_z)^{SU(2)}. \end{aligned} \quad (2)$$

with old physical space $\mathcal{H}_2 = \mathcal{K}_2^{SU(2)}$. Partial traces below use $\mathcal{K}_2 \otimes \mathcal{K}_z$, not a postulated physical factorization.

Let L^a, R^a be self-adjoint generators of left and right multiplication, with $C = \sum_a (L^a)^2 = \sum_a (R^a)^2$ and $C\chi_j = j(j+1)\chi_j$. Differentiating the four original edges

induces

$$L_x^a, \quad -R_x^a + L_y^a, \quad -R_y^a + L_z^a, \quad -R_z^a. \quad (3)$$

Consequently, the native electric form is

$$\begin{aligned} q_E[F] &= A \sum_a (\|L_x^a F\|^2 + \|(-R_x^a + L_y^a)F\|^2 \\ &\quad + \|(-R_y^a + L_z^a)F\|^2 + \|R_z^a F\|^2). \end{aligned} \quad (4)$$

For $A, B > 0$ and $\chi = \chi_{1/2}$, add the potential $B(6 - \chi(x) - \chi(y) - \chi(z))$. On the smooth core,

$$H_3 = H_2 \otimes I + I \otimes h_z + V, \quad (5)$$

$$\begin{aligned} H_2 &= 2A(C_x + C_y) - 2AR_x \cdot L_y \\ &\quad + B(4 - \chi(x) - \chi(y)), \\ h_z &= 2AC_z + B(2 - \chi(z)), \quad V = -2AR_y \cdot L_z. \end{aligned} \quad (6)$$

The coefficient of V is fixed by the edge squares, not assigned independently. Retaining only the original three edge contributions on the enlarged carrier instead defines

$$H_{2,\text{edge}} = H_2 \otimes I + AC_z + V. \quad (7)$$

These are different comparisons, even though both are defined on the enlarged graph. The supplement proves that the retained old-edge operator has ground projection $\Pi_{2,\text{edge}} = |\Omega_2 \otimes 1\rangle\langle\Omega_2 \otimes 1|$ on the physical Hilbert space. Its test already registers the isolated product state $\Psi_0 = \Omega_2 \otimes \phi$ through $1 - |\langle 1, \phi \rangle|^2 > 0$. This is not the receiving response measured below by $Q_2 \otimes I$, which vanishes on Ψ_0 .

Write $H(t) = H_0 + tV$, $H_0 = H_2 \otimes I + I \otimes h_z$, with $t = 1$ the native graph. The kinetic forms satisfy

$$q_{E,t}[F] \geq A(2 - \sqrt{2}) \sum_{w=x,y,z} \langle F, C_w F \rangle, \quad 0 \leq t \leq 1. \quad (8)$$

The common closed form domain is $H^1(SU(2)^3)$; the associated operators have compact resolvent and a simple,

smooth, strictly positive ground state. Gauge invariance follows from uniqueness. The self-contained reduction, ellipticity estimate, and domain details are supplied in the Supplemental Material [5].

Exact receiving defect.—Let Ω_2 and ϕ be the normalized positive ground states of H_2 and h_z , with energies E_2, e_z . Define

$$J_\phi f = f \otimes \phi, \quad P_\phi = J_\phi J_\phi^*, \quad Q_\phi = I - P_\phi. \quad (9)$$

Since ϕ is a class function, J_ϕ is gauge equivariant and sends $\mathcal{H}_2$ into $\mathcal{H}_3$.

Theorem 1 (Shared-edge receiving identity). *For smooth $f \in \mathcal{K}_2$, the native off-diagonal block is*

$$K_\phi f := Q_\phi H_3 J_\phi f = -2A \sum_a (R_y^a f) \otimes (L_z^a \phi), \quad (10)$$

with exact norm

$$\|K_\phi f\|^2 = \frac{4A^2}{3} \langle \phi, C_z \phi \rangle \langle f, C_y f \rangle. \quad (11)$$

The resulting first-order operator closes on $D(C_y^{1/2})$. For $B > 0$, its kernel consists precisely of functions independent of y .

Proof. The decoupled Hamiltonian preserves $\text{Ran } J_\phi$. Real positivity and Haar integration give $\langle \phi, L_z^a \phi \rangle = 0$, so the mixed term is orthogonal to that range. Adjoint invariance of the class function makes its electric covariance isotropic:

$$\langle L_z^a \phi, L_z^b \phi \rangle = \frac{\delta_{ab}}{3} \langle \phi, C_z \phi \rangle. \quad (12)$$

This proves the norm identity. Its first factor cannot vanish: otherwise ϕ would be constant, whereas $h_z 1 = B(2 - \chi(z))$ is nonconstant. The domain and kernel follow from the closed y -Casimir form. $\square$

In particular, for $\Psi_0 = \Omega_2 \otimes \phi$,

$$b := V\Psi_0 \neq 0. \quad (13)$$

Indeed, Ω_2 cannot be independent of y : its eigenvalue equation would then contain an uncanceled term $-B\chi(y)\Omega_2$.

Vacuum readjustment at native coupling.—Declare the old observable algebra to be

$$\mathfrak{A}_{\text{old}} = \{O \in \mathcal{B}(\mathcal{K}_2) : [O, U_2(g)] = 0 \ \forall g \in \text{SU}(2)\}, \quad (14)$$

represented on $\mathcal{H}_3$ by $(O \otimes I)|_{\mathcal{H}_3}$. Here U_2 is simultaneous conjugation. In particular,

$$P_2 = |\Omega_2\rangle\langle\Omega_2|, \quad Q_2 = I - P_2 \quad (15)$$

belong to $\mathfrak{A}_{\text{old}}$. This fixes a cycle-factor comparison. It admits a gauge-invariant lift supported on the original edge

set, but is not asserted to preserve the enlarged native old-edge Hamiltonian or to define a spacetime-exterior local algebra. The witness already belongs to the old Hamiltonian's spectral algebra, since $P_2 = s\text{-}\lim_{\tau \rightarrow \infty} e^{-\tau(H_2 - E_2)}$.

Theorem 2 (Strict native vacuum readjustment). *Let Ω_3 be the normalized positive ground state of H_3 . It is not a product across $\mathcal{K}_2 \otimes \mathcal{K}_z$. For $\rho_2 = \text{Tr}_z |\Omega_3\rangle\langle\Omega_3|$,*

$$p_{\text{exc}} := \langle \Omega_3, (Q_2 \otimes I) \Omega_3 \rangle > 0, \quad (16)$$

$$D(\rho_2, P_2) := \frac{1}{2} \|\rho_2 - P_2\|_1 \geq p_{\text{exc}} > 0, \quad (17)$$

$$E_3 < E_2 + e_z. \quad (18)$$

Proof. If $\Omega_3 = \psi \otimes \eta$, its strict positivity permits smooth normalized positive factors. Every individual electric generator has zero expectation in a real factor. Projecting the eigenvalue equation onto η and onto ψ therefore makes each factor a positive eigenfunction of its isolated Hamiltonian. Uniqueness forces $\psi = \Omega_2$ and $\eta = \phi$. But $V\Psi_0 = b \neq 0$ is orthogonal to Ψ_0 , so this product is not an eigenvector. This proves nonfactorization. If $p_{\text{exc}} = 0$, the full vector would belong to $\Omega_2 \otimes \mathcal{K}_z$, again a product. The effect Q_2 gives (17). Finally $\langle \Psi_0, H_3 \Psi_0 \rangle = E_2 + e_z$; equality with the ground energy would make Ψ_0 a ground vector, which it is not. $\square$

Both ρ_2 and P_2 commute with U_2 . Thus the optimal trace-distance effect can itself be chosen in $\mathfrak{A}_{\text{old}}$; the distinction in (17) does not require gauge-noninvariant measurements.

The response is also quantitative near decoupling. Let $E_* = E_2 + e_z$ and let R_* be the inverse of $H_0 - E_*$ on $\Psi_0^\perp$, extended by zero on Ψ_0 . Analytic form perturbation theory [6] gives

$$\Omega(t) = \Psi_0 - tR_*b + O(t^2), \quad (19)$$

$$\langle \Omega(t), (Q_2 \otimes I) \Omega(t) \rangle = t^2 c_{\text{exc}} + O(t^3),$$

$$c_{\text{exc}} = \|R_*b\|^2 > 0. \quad (20)$$

Here b and R_*b lie in $\Omega_2^\perp \otimes \phi^\perp$. This coefficient has the standard entanglement-susceptibility spectral form [3]; our result is its exact edge-derived, untruncated realization. Theorem 2 is proved directly at $t = 1$; it is not an extrapolation of this expansion.

What dressing can preserve.—The nonzero receiving block does not mean that the exact vacuum leaks from its own eigenspace. It also does not forbid a dressed vacuum map.

Corollary (vacuum matching versus the old test). There exists a gauge-equivariant isometry $J_{\text{dr}} : \mathcal{K}_2 \rightarrow \mathcal{K}_2 \otimes \mathcal{K}_z$ with $J_{\text{dr}} \Omega_2 = \Omega_3$. No isometry $\tilde{J}$ with this vacuum-matching property can also satisfy

$$\tilde{J}^*(Q_2 \otimes I) \tilde{J} = Q_2. \quad (21)$$

The incompatibility already holds after restriction to the old physical space.

Proof. A unitary rotation W in $\text{span}\{\Psi_0, \Omega_3\}$, extended by the identity on its orthogonal complement, can send Ψ_0 to Ω_3 . Both vectors are gauge invariant, so W commutes with the gauge action. Then $J_{\text{dr}} = WJ_\phi$ is the required isometry; its explicit rank-two formula is given in [5]. Taking the Ω_2 expectation of (21) would equate $p_{\text{exc}} > 0$ with zero. $\square$

The constructed dressed map preserves a transported copy of the old algebra:

$$J_{\text{dr}}^* W(O \otimes I) W^* J_{\text{dr}} = O. \quad (22)$$

This construction uses the full ground state. It is neither a local attachment rule nor a dynamically intertwining refinement map. On the matched vacuum alone,

$$(I - J_{\text{dr}} J_{\text{dr}}^*) H_3 J_{\text{dr}} \Omega_2 = 0. \quad (23)$$

Abelian control and scope.—For $U(1)$ rotors, $N = -i\partial_\theta$ and $V_{\text{ab}} = -2AN_y N_z$ yield

$$\|K_\phi^{\text{ab}} f\|^2 = 4A^2 \langle \phi, N_z^2 \phi \rangle \langle f, N_y^2 f \rangle. \quad (24)$$

With a nonconstant cosine potential, the isolated vacua have nonzero electric response. Thus attachment-induced readjustment is not uniquely non-Abelian. No susceptibility of a specifically non-Abelian commutator observable is asserted.

What changes is not the existence of a vacuum, but the compatibility of two ways of identifying it. The isolated product is not the native vacuum, and the shared-edge calculation says precisely where its mismatch enters. Neither a dressed vacuum map nor a change in the chosen old observable test can erase that distinction without also

changing the comparison. A spatial localization claim requires a specified exterior measurement algebra and a vacuum-relative comparison, not just nonfactorization; related distinctions occur in photon-localization analyses [7]. This letter constructs no exterior net, continuum refinement limit, physical associator obstruction, or Yang–Mills mass gap. The native shared electric edge forces vacuum readjustment under the specified cycle-factor comparison; exact vacuum matching cannot preserve that fixed vacuum-test observable.

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 - [5] See the accompanying Supplemental Material, v0.3.3, for the full gauge reduction, domains, receiving calculation, perturbative response, observable algebra, dressed isometry, and Abelian control.
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Supplemental Material

Vacuum Readjustment under Gauge-Cell Attachment

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Abstract

This supplement supplies a self-contained four-edge reduction, operator domains, the exact receiving identity, the native-coupling nonfactorization proof, the local perturbation expansion, the declared observable algebra, an explicit dressed vacuum isometry, and the Abelian control. All statements concern a fixed finite graph with full compact-group Hilbert spaces. No representation truncation or continuum passage is used.

S1 Four-edge Gauss reduction and normalization

Let $G = \text{SU}(2)$ and let all Haar measures be normalized. Orient four edges from the right endpoint to the left endpoint, with holonomies u_i transforming as

$$u_i \longmapsto au_i b^{-1}, \quad (a, b) \in G \times G. \quad (\text{S1.1})$$

The kinematical edge space is $L^2(G^4)$. Introduce

$$x = u_0 u_1^{-1}, \quad y = u_1 u_2^{-1}, \quad z = u_2 u_3^{-1}, \quad g = u_3. \quad (\text{S1.2})$$

This is a global smooth bijection with inverse

$$(u_0, u_1, u_2, u_3) = (xyzg, yzg, zg, g). \quad (\text{S1.3})$$

Successive Haar invariance in u_0, u_1, u_2 shows that the change preserves product Haar measure. The right endpoint acts only on g , so right-invariant functions are exactly functions $F(x, y, z)$. The left endpoint acts by simultaneous conjugation on (x, y, z) . Accordingly,

$$\mathcal{H}_3 = L^2(G^3)^{\text{Ad } G}, \quad \mathcal{H}_2 = L^2(G^2)^{\text{Ad } G}. \quad (\text{S1.4})$$

These are invariant subspaces of the kinematical spaces, not an assertion that $\mathcal{H}_3 = \mathcal{H}_2 \otimes L^2(G)^{\text{Ad } G}$.

Take Hermitian fundamental generators $T_a = \sigma_a/2$ and define

$$L^a F(w) = -i \frac{d}{ds} F(e^{isT_a} w) \Big|_{s=0}, \quad R^a F(w) = -i \frac{d}{ds} F(w e^{isT_a}) \Big|_{s=0}. \quad (\text{S1.5})$$

Then $C = \sum_a (L^a)^2 = \sum_a (R^a)^2$ and $C\chi_j = j(j+1)\chi_j$; in particular, $C\chi_{1/2} = \frac{3}{4}\chi_{1/2}$.

Differentiate one original edge by $u_i \mapsto e^{isT_a} u_i$, holding the others fixed. On functions independent of g , the transformations and generators are

Edge	Changed cycle variables	Induced generator D_i^a
u_0	$x \mapsto e^{isT_a} x$	L_x^a
u_1	$x \mapsto x e^{-isT_a}, y \mapsto e^{isT_a} y$	$-R_x^a + L_y^a$
u_2	$y \mapsto y e^{-isT_a}, z \mapsto e^{isT_a} z$	$-R_y^a + L_z^a$
u_3	$z \mapsto z e^{-isT_a}$	$-R_z^a$

The omitted change $g \mapsto e^{isT_a} g$ in the last row acts trivially after the right quotient. As a consistency check,

$$\sum_{i=0}^3 D_i^a = (L_x^a - R_x^a) + (L_y^a - R_y^a) + (L_z^a - R_z^a), \quad (\text{S1.6})$$

which is the remaining diagonal Gauss generator.

The edge Hamiltonian $A \sum_{i,a} (D_i^a)^2$ therefore has form

$$\begin{aligned} q_{E,1}[F] = A \sum_a (&\|L_x^a F\|^2 + \|(-R_x^a + L_y^a)F\|^2 \\ &+ \|(-R_y^a + L_z^a)F\|^2 + \|R_z^a F\|^2). \end{aligned} \quad (\text{S1.7})$$

Expanding on $C^\infty(G^3)$ gives

$$H_{E,1} = 2A(C_x + C_y + C_z) - 2A(R_x \cdot L_y + R_y \cdot L_z). \quad (\text{S1.8})$$

There are two copies of each Casimir. Each mixed coefficient is $-2A$. In particular, removing the y, z cross term for comparison does not remove either copy of AC_z .

Write $\chi = \text{Tr}_{1/2}$ and set

$$\begin{aligned} H_2 &= 2A(C_x + C_y) - 2AR_x \cdot L_y + B(4 - \chi(x) - \chi(y)), \\ h_z &= 2AC_z + B(2 - \chi(z)), \\ H_0 &= H_2 \otimes I + I \otimes h_z, \quad V = -2AR_y \cdot L_z, \\ H(t) &= H_0 + tV. \end{aligned} \quad (\text{S1.9})$$

The native attachment is $t = 1$. The parameter t is an interpolation device, not an additional independently normalized native coupling. The Hamiltonian is a compact-group rotor model on the declared graph in the Hamiltonian lattice-gauge framework [1]; no relation of A, B to a continuum lattice-spacing trajectory is assumed.

S2 The retained old-edge ground state and its spectral test

There are two distinct extensions of the old comparison. The cycle-factor Hamiltonian is $H_2 \otimes I$, while retaining the original u_0, u_1, u_2 edge-energy terms on the four-edge carrier gives

$$H_{2,\text{edge}} = H_2 \otimes I + AC_z - 2AR_y \cdot L_z. \quad (\text{S2.1})$$

The corresponding potential includes only the old two plaquettes $B(4 - \chi(x) - \chi(y))$. The operator is defined by its nonnegative closed edge form. In the original edge coordinates, before endpoint reduction, it is the old three-edge operator acting on (u_0, u_1, u_2) , tensored with the identity on u_3 . The old three-edge operator on $L^2(G^3)$ is uniformly elliptic, with connected configuration manifold, real bounded potential and unique strictly positive ground vector $\tilde{\Omega}_2$. Uniqueness forces invariance under both old endpoint gauge actions, so its gauge-reduced expression is $\Omega_2(x, y)$ and its ground energy is E_2 .

It follows on $L^2(G^4)$ that $H_{2,\text{edge}} \geq E_2 I$ and that its ground eigenspace before gauge reduction is $\tilde{\Omega}_2 \otimes L^2(G)_{u_3}$. Intersecting with the full $G \times G$ gauge-invariant subspace requires the u_3 factor to be bi-invariant: since $\tilde{\Omega}_2$ is already invariant under both endpoint groups, this factor must be constant almost everywhere. Hence on the physical three-cycle carrier,

$$\ker(H_{2,\text{edge}} - E_2) = \mathbb{C}(\Omega_2 \otimes 1_z), \quad \Pi_{2,\text{edge}} = |\Omega_2 \otimes 1_z\rangle\langle\Omega_2 \otimes 1_z|. \quad (\text{S2.2})$$

Here 1_z is the normalized constant function in $L^2(G)$, and $\Omega_2 \otimes 1_z$ is gauge invariant. In particular $H_{2,\text{edge}}(\Omega_2 \otimes 1_z) = E_2(\Omega_2 \otimes 1_z)$, since $C_z 1_z = L_z^a 1_z = 0$.

By contrast, the h_z ground vector ϕ is nonconstant for $B > 0$. Thus the decoupled product $\Psi_0 = \Omega_2 \otimes \phi$ satisfies the two different vacuum-test identities

$$\langle\Psi_0, (Q_2 \otimes I)\Psi_0\rangle = 0, \quad \langle\Psi_0, (I - \Pi_{2,\text{edge}})\Psi_0\rangle = 1 - |\langle 1_z, \phi\rangle|^2 > 0. \quad (\text{S2.3})$$

The second strict inequality is Cauchy–Schwarz with equality only when ϕ is constant, which its nonconstant potential excludes. These identities prevent identifying the cycle-factor vacuum response with the spectral response of the retained old-edge operator. In particular, the strict inequality for the old-edge test exists already at $t = 0$: it must not be attributed to the mixed term V . All later t^2 claims refer to the cycle-factor test $Q_2 \otimes I$.

S3 Closed forms, ellipticity, and the positive vacuum

For $0 \leq t \leq 1$, the kinetic form of (S1.9) is

$$q_{E,t}[F] = 2A \sum_{w=x,y,z} \langle F, C_w F \rangle - 2A \operatorname{Re} \sum_a (\langle R_x^a F, L_y^a F \rangle + t \langle R_y^a F, L_z^a F \rangle). \quad (\text{S3.1})$$

Left and right frames on each group factor differ by an orthogonal adjoint rotation. Pointwise, let a, b, c denote the norms of the three left-gradient vectors. Cauchy–Schwarz bounds the cross terms by ab and bc . The relevant scalar comparison matrix is

$$M_t = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -t \\ 0 & -t & 2 \end{pmatrix}, \quad \operatorname{spec} M_t = \{2, 2 - \sqrt{1+t^2}, 2 + \sqrt{1+t^2}\}. \quad (\text{S3.2})$$

Applying the lower and upper bounds pointwise and integrating yields

$$A(2 - \sqrt{2}) \sum_w \langle F, C_w F \rangle \leq q_{E,t}[F] \leq A(2 + \sqrt{2}) \sum_w \langle F, C_w F \rangle. \quad (\text{S3.3})$$

Equivalently, because the interaction graph in the three gradient variables is a chain, its two orthogonal frame rotations can be removed successively by a block-orthogonal change of tangent frame. There is no cycle holonomy to obstruct this symbol-level comparison.

The potential satisfies

$$0 \leq B(6 - \chi(x) - \chi(y) - \chi(z)) \leq 12B. \quad (\text{S3.4})$$

Thus the full form norm is equivalent to the usual $H^1(G^3)$ norm. Closing from the smooth core gives a nonnegative closed form with domain $H^1(G^3)$ and a self-adjoint Friedrichs operator. Smooth uniform ellipticity on the compact boundaryless manifold gives operator domain $H^2(G^3)$ and compact resolvent. The same statements apply to H_2 and h_z on their factors. Restrictions to diagonal invariants are reducing restrictions of these operators.

The forms are real elliptic diffusion forms plus a real bounded potential. Their heat semigroups are positivity improving on connected G^3 . The lowest eigenfunction can consequently be chosen strictly positive and is unique up to phase. Elliptic regularity makes it smooth. Since the gauge action commutes with the Hamiltonian and preserves positivity, uniqueness makes the normalized positive ground state gauge invariant. These facts justify the use of the full kinematical ground state and its physical invariant restriction without identifying two different vacua.

The principal form stays positive in an open interval about $t = 0$ (indeed the symbol bound is positive for $|t| < \sqrt{3}$). The form family is analytic with common form domain there, so the isolated simple ground eigenvalue and eigenprojection have analytic local branches [2]. No convergence radius reaching $t = 1$ is inferred from the Taylor expansion. Native-coupling statements below use a separate argument.

S4 The receiving operator and its exact domain

Let

$$H_2\Omega_2 = E_2\Omega_2, \quad h_z\phi = e_z\phi, \quad \|\Omega_2\| = \|\phi\| = 1, \quad (\text{S4.1})$$

with both vectors smooth and positive. Define $J_\phi f = f \otimes \phi$, $P_\phi = I \otimes |\phi\rangle\langle\phi|$, and $Q_\phi = I - P_\phi$. On smooth old functions,

$$H_0 J_\phi f = J_\phi (H_2 + e_z) f. \quad (\text{S4.2})$$

For every self-adjoint electric generator $D = -iX$, where X is Haar-divergence-free, a smooth real r obeys

$$\langle r, Dr \rangle = -\frac{i}{2} \int X(r^2) d\mu = 0. \quad (\text{S4.3})$$

Hence $P_\phi V J_\phi = 0$, and

$$Q_\phi H_3 J_\phi f = -2A \sum_a (R_y^a f) \otimes (L_z^a \phi). \quad (\text{S4.4})$$

At interpolation value t , this off-diagonal block is t times (S4.4).

Since ϕ is invariant under conjugation, the Hermitian covariance matrix

$$M_{ab} = \langle L_z^a \phi, L_z^b \phi \rangle \quad (\text{S4.5})$$

commutes with the complexified adjoint representation of $\text{SU}(2)$. That representation is irreducible, so Schur's lemma gives $M_{ab} = m\delta_{ab}$. Taking the trace fixes

$$m = \frac{1}{3} \langle \phi, C_z \phi \rangle. \quad (\text{S4.6})$$

Consequently,

$$\|K_\phi f\|^2 = d_\phi \|C_y^{1/2} f\|^2, \quad d_\phi = \frac{4A^2}{3} \langle \phi, C_z \phi \rangle. \quad (\text{S4.7})$$

The constant d_ϕ is strictly positive for $B > 0$. If it were zero, all z derivatives of ϕ would vanish on connected G , making ϕ constant, in contradiction with its eigenvalue equation and the nonconstant potential.

Domain distinction. The literal product $Q_\phi H_3 J_\phi f$ is an operator expression on the H_3 domain; in particular it is defined for $f \in H^2(G^2)$. Its smooth-core expression (S4.4) has a unique closure as a first-order operator with domain

$$D(K_\phi) = D(C_y^{1/2}) \subset \mathcal{K}_2. \quad (\text{S4.8})$$

This need not impose an x derivative. Formula (S4.7) proves closedness because its graph norm is equivalent to that of $C_y^{1/2}$. It also shows that

$$K_\phi^* K_\phi = d_\phi C_y \quad (\text{S4.9})$$

as nonnegative operators, with the corresponding quadratic-form identity on $D(C_y^{1/2})$. For vectors in this enlarged domain one uses the closed first-order expression, not an undefined application of H_3 . All identities restrict to the diagonal-invariant subspaces.

For $B > 0$, $\ker K_\phi = \ker C_y^{1/2}$, which consists of functions independent of y . The old ground state cannot have that form: if $\Omega_2 = \Omega_2(x)$, then the y kinetic and mixed derivatives vanish, while $-B\chi(y)\Omega_2(x)$ remains in its eigenvalue equation. Thus

$$\epsilon_y := \langle \Omega_2, C_y \Omega_2 \rangle > 0, \quad \epsilon_z := \langle \phi, C_z \phi \rangle > 0, \quad (\text{S4.10})$$

so for $\Psi_0 = \Omega_2 \otimes \phi$,

$$b := V\Psi_0 \neq 0, \quad \|b\|^2 = \frac{4A^2}{3} \epsilon_y \epsilon_z. \quad (\text{S4.11})$$

At $B = 0$, in contrast, $\phi = 1$ and the receiving block vanishes. The positive-response conclusions explicitly use $B > 0$.

S5 Native nonfactorization and perturbative response

S4.1. Nonfactorization at the actual coupling

Suppose the positive native ground state factors as $\Omega_3 = \psi \otimes \eta$. Its factors can be chosen normalized, smooth, and strictly positive. One way to see smoothness is to define

$$\psi(x, y) = \left(\int_G \Omega_3(x, y, z)^2 dz \right)^{1/2}; \quad (\text{S5.1})$$

this is smooth and positive, and the factorization hypothesis makes $\eta = \Omega_3/\psi$ independent of (x, y) .

Using (S4.3), partial inner products of $H_3\Omega_3 = E_3\Omega_3$ give

$$\begin{aligned} H_2\psi &= (E_3 - \langle \eta, h_z \eta \rangle) \psi, \\ h_z \eta &= (E_3 - \langle \psi, H_2 \psi \rangle) \eta. \end{aligned} \quad (\text{S5.2})$$

Each is a positive eigenfunction of its isolated Hamiltonian. A positive eigenfunction cannot be orthogonal to the positive ground eigenfunction, so its eigenvalue is the ground eigenvalue. Simplicity then forces $\psi = \Omega_2$ and $\eta = \phi$.

But $H_3\Psi_0 = (E_2 + e_z)\Psi_0 + b$, with $b \neq 0$ and $\langle \Psi_0, b \rangle = 0$. Hence Ψ_0 is not an eigenvector, a contradiction. This proves nonfactorization against *every* product in the declared kinematical partition, not merely against products containing a prescribed old factor.

Let $Q_2 = I - |\Omega_2\rangle\langle\Omega_2|$. If

$$p_{\text{exc}} = \|(Q_2 \otimes I)\Omega_3\|^2 \quad (\text{S5.3})$$

were zero, Ω_3 would lie in $\Omega_2 \otimes \mathcal{K}_z$ and be a product. Thus $p_{\text{exc}} > 0$. The reduced state is mixed, and in particular differs from the old pure vacuum.

The product trial vector has expectation $E_2 + e_z$. If this equaled E_3 , then

$$\|(H_3 - E_3)^{1/2}\Psi_0\|^2 = 0, \quad (\text{S5.4})$$

so it would be a ground vector. Therefore $E_3 < E_2 + e_z$. This is a variational energy statement on a fixed graph, not a lower bound on an excitation gap or an invariant mass.

S4.2. Local expansion about the decoupled vacuum

Write $E_* = E_2 + e_z$, $P_0 = |\Psi_0\rangle\langle\Psi_0|$, and $Q_0 = I - P_0$. Define the reduced resolvent by

$$R_* := [(H_0 - E_*)|_{\Psi_0^\perp}]^{-1}Q_0. \quad (\text{S5.5})$$

It is bounded and positive on $\Psi_0^\perp$ and is zero on Ψ_0 . The input b is smooth, so the first-order coefficient is unambiguous despite V being unbounded.

Each term of b is the tensor product of a vector orthogonal to Ω_2 and a vector orthogonal to ϕ . Hence

$$b \in \Omega_2^\perp \otimes \phi^\perp. \quad (\text{S5.6})$$

The projection $(I - P_2) \otimes (I - |\phi\rangle\langle\phi|)$ commutes with the decoupled operator H_0 , and therefore with R_* . With the analytic normalized phase chosen positive near zero, perturbation theory gives

$$\begin{aligned} E'(0) &= \langle\Psi_0, V\Psi_0\rangle = 0, \\ \Omega(t) &= \Psi_0 - tR_*b + O(t^2), \\ E(t) &= E_* - t^2\langle b, R_*b\rangle + O(t^3). \end{aligned} \quad (\text{S5.7})$$

The vector remainder is in Hilbert norm. Because R_*b remains in the doubly orthogonal subspace,

$$p_{\text{exc}}(t) = t^2\|R_*b\|^2 + O(t^3). \quad (\text{S5.8})$$

The coefficient is positive: $b \neq 0$ and the reduced resolvent is injective on $\Psi_0^\perp$. For $\rho_2(t) = \text{Tr}_z|\Omega(t)\rangle\langle\Omega(t)|$,

$$\liminf_{t \rightarrow 0} \frac{\frac{1}{2}\|\rho_2(t) - P_2\|_1}{t^2} \geq \|R_*b\|^2 > 0. \quad (\text{S5.9})$$

The first-order partial-trace term vanishes, so the reduced-state difference is $O(t^2)$ in trace norm. In decoupled eigenbases, c_{exc} is the usual boundary-coupling entanglement susceptibility [4]; for the declared cycle tensor partition, $\text{Tr} \rho_2(t)^2 = 1 - 2c_{\text{exc}}t^2 + O(t^3)$. This identification does not equate the kinematical partition entropy with an operational gauge-entanglement measure. This is a small- t result; strictness at native coupling was proved independently in S4.1.

S6 Two old-system comparisons, observable algebra, and vacuum dressing

S5.1. What is being measured

Let $U_2(g)$ implement simultaneous conjugation on $\mathcal{K}_2$. We declare

$$\mathfrak{A}_{\text{old}} = U_2(G)' \subset \mathcal{B}(\mathcal{K}_2). \quad (\text{S6.1})$$

An old observable $O \in \mathfrak{A}_{\text{old}}$ is represented on the full physical space by $(O \otimes I)|_{\mathcal{H}_3}$. This operator preserves $\mathcal{H}_3$ because it commutes with the diagonal action. This declaration fixes a cycle-factor comparison. It admits a gauge-invariant old-edge-set lift in the unquotiented old graph, but is not a proof of exterior AQFT locality or preservation of the native old-edge Hamiltonian. For example, the original shared edge acts as $-R_y + L_z$ on the enlarged graph, rather than as the old terminal derivative $-R_y$; the distinction is precisely retained in the mixed Hamiltonian term.

For any invariant full state Ω , its kinematical partial trace satisfies

$$U_2(g)\rho_2 U_2(g)^* = \rho_2. \quad (\text{S6.2})$$

The reduced state need not be supported in $\mathcal{H}_2$. Nontrivial old conjugation sectors can be paired with new-cell sectors to make a diagonal singlet. Taking the trace on the kinematical factors retains these channels rather than discarding them in advance.

The projector P_2 is itself a global spectral observable of the old Hamiltonian:

$$P_2 = \text{s-}\lim_{\tau \rightarrow \infty} e^{-\tau(H_2 - E_2)}. \quad (\text{S6.3})$$

Thus Q_2 belongs already to the von Neumann algebra generated by bounded spectral functions of H_2 , a smaller algebra sufficient for the strict witness. No identification of the entire invariant commutant with a local generating set is needed for this witness.

Because $\Delta = \rho_2 - P_2$ is self-adjoint, trace class, trace zero, and gauge invariant, its positive spectral projection is an invariant effect. Consequently,

$$\sup_{\substack{0 \leq O \leq I \\ O \in \mathfrak{A}_{\text{old}}}} |\text{Tr}[O(\rho_2 - P_2)]| = \frac{1}{2} \|\rho_2 - P_2\|_1. \quad (\text{S6.4})$$

In particular, the effect Q_2 gives the lower bound by p_{exc} . This is an algebraically gauge-invariant distinction. It is not a claim that a spectral projection can be measured by an operator supported in a specified spacetime exterior. The tensor-factor and regional-algebra distinction is essential in gauge systems [3].

S5.2. A concrete dressed isometry

The positive normalized vectors Ψ_0 and Ω_3 obey

$$0 < c := \langle \Psi_0, \Omega_3 \rangle < 1. \quad (\text{S6.5})$$

Strict positivity gives the lower inequality; the nonfactorization result gives the upper one. Put

$$s = \sqrt{1 - c^2}, \quad \eta = \frac{\Omega_3 - c\Psi_0}{s}, \quad P_\eta = |\eta\rangle\langle\eta|. \quad (\text{S6.6})$$

Then Ψ_0, η are orthonormal invariant smooth vectors. Define

$$W = I + (c - 1)(P_0 + P_\eta) + s(|\eta\rangle\langle\Psi_0| - |\Psi_0\rangle\langle\eta|). \quad (\text{S6.7})$$

On their span, W has matrix $\begin{pmatrix} c & -s \\ s & c \end{pmatrix}$; on the orthogonal complement it is the identity. It is unitary, commutes with the diagonal gauge action, preserves the smooth and Sobolev domains, and satisfies $W\Psi_0 = \Omega_3$. Therefore

$$J_{\text{dr}} = WJ_\phi, \quad J_{\text{dr}}^* J_{\text{dr}} = I, \quad J_{\text{dr}} \Omega_2 = \Omega_3 \quad (\text{S6.8})$$

is an explicit gauge-equivariant vacuum-matching isometry.

Suppose an isometry $\tilde{J}$ both matches that vacuum and preserves the old test under compression:

$$\tilde{J}\Omega_2 = \Omega_3, \quad \tilde{J}^*(Q_2 \otimes I)\tilde{J} = Q_2. \quad (\text{S6.9})$$

Taking the Ω_2 expectation gives $p_{\text{exc}} = 0$, contradicting S4.1. Thus exact matching is incompatible even with preserving this one specified old observable, and a fortiori with preserving the entire fixed old algebra by compression or intertwining.

For the explicit construction, the transported algebra instead obeys

$$J_{\text{dr}}^* W(O \otimes I) W^* J_{\text{dr}} = O \quad (O \in \mathcal{B}(\mathcal{K}_2)). \quad (\text{S6.10})$$

For invariant O , both sides have the appropriate physical restrictions. This construction preserves a transported copy of the algebra; it does not classify every possible embedding, and does not restore the original old marginal.

Let $P_{\text{dr}} = J_{\text{dr}} J_{\text{dr}}^*$. The matched vacuum satisfies

$$(I - P_{\text{dr}}) H_3 J_{\text{dr}} \Omega_2 = 0, \quad (\text{S6.11})$$

because it is an exact eigenvector in the receiving range. There is no assertion that $(I - P_{\text{dr}}) H_3 J_{\text{dr}}$ vanishes on every old vector. The construction uses the exact full vacuum and is not a source-local map, a refinement-compatible family, or a dynamical intertwiner. It demonstrates existence of a dressed vacuum map without importing any of those additional properties.

S7 Abelian control and the theorem boundary

For $G = U(1)$ with angles $\theta_x, \theta_y, \theta_z$, use $N_w = -i\partial_{\theta_w}$ and the same four-edge change of variables. The electric operator becomes

$$H_{E,1}^{\text{ab}} = 2A(N_x^2 + N_y^2 + N_z^2) - 2A(N_x N_y + N_y N_z). \quad (\text{S7.1})$$

Take the nonnegative potential

$$B(3 - \cos \theta_x - \cos \theta_y - \cos \theta_z), \quad B > 0. \quad (\text{S7.2})$$

Its amplitude convention is declared here rather than identified with an $SU(2)$ fundamental trace. The isolated added-cell ground state is smooth, positive, and nonconstant, and has $\langle \phi, N_z \phi \rangle = 0$. The same receiving calculation gives

$$K_\phi^{\text{ab}} f = -2A(N_y f) \otimes (N_z \phi), \quad \|K_\phi^{\text{ab}} f\|^2 = 4A^2 \langle \phi, N_z^2 \phi \rangle \langle f, N_y^2 f \rangle. \quad (\text{S7.3})$$

The old ground state has nonzero y response for the same nonconstant-potential reason. The native nonfactorization and small-coupling arguments therefore apply to this control model as well. The receiving mechanism is not a discriminator between Abelian and non-Abelian gauge groups.

For $SU(2)$ one can separately define the bounded invariant multiplication observable

$$D_{\text{NA}}(x, y) = \frac{1}{2} \|xy - yx\|_{\text{HS}}^2, \quad (\text{S7.4})$$

which vanishes identically in the Abelian control. No sign or nonzero coefficient for its seam-coupling susceptibility has been calculated here.

Established within the declared finite model. The edge normalization, positive elliptic operator setup, receiving norm identity, strict native nonfactorization, old-vacuum spectral witness, local quadratic excitation coefficient, and the explicit dressing/old-test incompatibility follow from the calculations above. The graph is finite, but its rotor Hilbert spaces are not finite dimensional.

Not supplied by these calculations. No exterior-local measurement algebra, point-localization obstruction, source-derived dressed map, compatible refinement tower, uniform continuum estimate, Type III reconstruction, physical associator or gerbe class, Poincare reconstruction, or Yang–Mills mass gap is constructed. The positive numbers at one fixed graph do not constitute lower bounds uniform in graph refinement or physical rescaling. The canonical associator of Connes fusion is not identified with a physical obstruction. Photon-localization work is cited only as context for the need to distinguish measurement locality from subsystem factorization [5].

Scope of review. This supplement contains mathematical derivations for a consolidated working manuscript. Its accompanying consistency checks are internal checks, not independent peer review, a formal proof-assistant certificate, or an exhaustive priority analysis.

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