

Scale Bordisms and Wilson Observable Algebras

Relative Filling, Čech Descent,
and Normal Realization in $3 + 1$ Dimensions

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Abstract

Wilson traces retain less information than the open transports from which they are formed. We study this distinction on finite electric–magnetic graph frusta, retaining boundary modules, exterior charge partners, complete grounds, and coefficient lines. Normal $*$ -homomorphic overlap maps give a W^* -algebra of compatible observable families, while line-valued implementers can retain transport invisible to their adjoint actions. Physical spacetime overlap identities use the full Gauss–Ward identity on the declared domains. A native finite refinement supplies exact inherited Wilson transport and a positive-overlap vacuum comparison. We explain how the retained exterior factors enter that comparison, evaluate a specified longitude seam cancellation, and distinguish it from the meridional obstruction and the linked source’s carry associator. An elementary flat-bundle criterion states precisely when a line-decorated parallel comparison exists. Relative fillings and changes of presentation are tracked at the level of both operators and coefficients. A separate bounded-completion theorem applies when the full represented residuals satisfy its summability hypotheses. The results separate finite observable descent from identification of the full physical coefficient system and from a continuum refinement limit.

1 Wilson existence, comparison, and completion

A finite Wilson operator is a bounded function of an ordered holonomy word. Its existence precedes spectral analysis. The question addressed here is how that operator is compared across a cut, a physical spacetime cover, and a change of scale while the boundary coefficient data remain present.

The construction has three distinct stages. The electric–magnetic graph frustum fixes the open-bank actions and their Hilbert modules. Admitted coefficient prisms and physical Ward identities then compare their local realizations. A common normal receiving algebra and quantitative control of the represented residuals supply limits. The paper evaluates the finite module and coefficient mechanisms and proves the normal descent and completion statements at their stated hypotheses.

The charged coefficient example is the finite $PSU(3)$ bulk with exposed $SU(3)$ ports specified in [7]. Its complete ground, charged contractions, compensating lines, and prescribed selector interaction are retained together. We compute the selector’s transgressed multiplier on the actual endpoint module. The earlier cyclic linked-flux associator of [4] is a different class: its cyclic detector prevents identification by ordinary cyclic pullback. The source’s selected fixed-nerve realization is used at its own scope.

Three ordinary algebras also remain distinct: the algebra generated by Wilson multipliers, the boundary coefficient comparison algebra, and an algebra enlarged by electric operators. Each is explicitly evaluated below. A coefficient phase can disappear under adjoint transport while remaining visible in closed-history holonomy. Multiplying the Wilson operator itself by that phase would instead change the observable.

The main distinction is operational. An ordinary Wilson family is tested in its normal observable algebra. An implementing comparison is additionally tested on its boundary module, coefficient transport, and seam composition. Equal adjoint actions need not give equal implementing comparisons. Section 7 collects the retained-carrier and marked-transport tests; the appendices preserve detailed source calculations without making every auxiliary construction a condition of the main results.

Conventions and scope. Scalar Hilbert inner products and charge forms are linear in the first argument. Right Hilbert-module inner products are linear in the second argument; their scalarization in our convention is $\langle T, S \rangle_{\text{Hilb}} = \text{Tr}(S^* T)$. All module, coefficient, and overlap comparisons use their declared source and target. A coisometric charged insertion is not a scale isometry. Source-dependent statements are cited with their selected branch and domain; the general normal-algebra statements are proved here. Neither a four-dimensional continuum construction nor a mass-gap theorem follows from finite coefficient coherence alone.

1.1 Main result and its finite applications

The main result separates ordinary observable descent from the extra information carried by its implementers. The hypotheses below concern one declared observable diagram; the finite examples following it verify specified parts of that structure rather than silently identifying distinct physical sources.

Theorem 1.1 (Observable descent with retained coefficient comparisons). *Fix an admitted finite cover and a diagram of von Neumann algebras with normal unital overlap $*$ -homomorphisms. All compatible observable families form a von Neumann algebra $\mathcal{D}_s$. Specified Wilson families satisfying the overlap equations generate a von Neumann subalgebra $\mathcal{M}_s^W \subseteq \mathcal{D}_s$. Normal supported scale $*$ -homomorphisms induce maps $\Phi_a : \mathcal{D}_s \rightarrow \mathcal{D}_t$ whenever their actual overlap squares commute. Such a map restricts to $\mathcal{M}_s^W \rightarrow \mathcal{M}_t^W$ if and only if its image of the source unit and of every admitted source Wilson generator belongs to $\mathcal{M}_t^W$. This additional condition is not implied by overlap covariance.*

Suppose, in addition, that local unitary implementers have central projective overlap multipliers. Their adjoint maps obey the ordinary cocycle law. A coefficient prism with the stated lateral seam certificate rephases the endpoint multipliers without changing the target adjoint maps. Neither operation asserts triviality of the coefficient class or an identification of endpoint modules.

For physical collar presentations satisfying the common-domain and splitting hypotheses of Section 5.4, the boundary Ward pairings agree on overlaps. For a scale lineage, a limiting observable follows separately if its full represented residuals satisfy the boundedness and summability hypotheses of Theorem 8.1 in a specified common normal realization.

Proof. The normal-algebra and supported-map assertions are Theorem 5.3; restriction to the Wilson-generated subalgebras is Proposition 5.4. Central adjoint cancellation is the calculation in (36); the rephasing assertion is Proposition 3.1. The physical Ward assertion is Proposition 5.2, and the

final assertion is Theorem 8.1. Each uses its own displayed hypotheses. In particular the Ward conclusion is an equality of prepared pairings, not a construction of unrelated operator overlap maps. $\square$

The theorem packages standard normal-closure and transport arguments for the declared physical comparisons. The concrete contribution is their evaluation with retained bank modules, selected charged maps, inverse coefficient lines, and native Wilson words. The matrix-module, flat-monodromy, and bounded-completion arguments are tools; they are not asserted to be new general theorems of operator algebra or descent.

The finite applications have separate source contracts:

Claim and setting		Inputs, proof status, and boundary
Normal observable descent		Supplied normal overlap homomorphisms and compatible Wilson families; proved here in Section 5. Restriction of scale maps to specified Wilson subalgebras additionally requires Proposition 5.4. Physical effective descent is imported only on the named branch and nerve.
Charged endpoint package		The composed source's bulk form, selection and external port prescription are imported. Section 2.2 evaluates its complete ground, coisometric maps, kernels and compensating-line square.
Exact finite refinement		The cap-8 to cap-16 scalar Gauss/copy carrier, native word map and Haar identities are imported. Proposition 6.1 proves inherited Wilson intertwining. Its vacuum-state estimate uses an imported positive-overlap certificate.
Physical Ward crossing		The prepared domains, insertion identifications and splitting laws are source inputs. Proposition 5.2 proves their cancellation; Section 5.5 evaluates the elementary neutral cut.
Retained regional carrier		Native gluing unitaries and exterior counts come from [1], Development Notes X–Y and AR. Section 7 retains their charge partners; its conjugated map is not a second independently constructed route.
Coefficient comparisons	compar-	The product selector and cyclic linked carry are separately prescribed systems. Their displayed multiplier, slant, cyclic detector and seam calculations are evaluated here; equality of their full physical diagrams is not asserted.
Analytic completion		A common normal receiving algebra, a uniform operator bound and summable full represented residuals are hypotheses of Section 8. A physical cross-scale family satisfying them is not supplied by the finite example.

The abstract theorem is dimension-independent. The stated $3 + 1$ application uses the program's positive-thickness spatial graph windows and their spacetime collar/history comparisons. These finite constructions do not by themselves provide a four-dimensional continuum field or its ultraviolet estimates. The optional operator-repair models in Appendix E are diagnostics of additional proposals and are not hypotheses of Theorem 1.1.

1.2 Finite and classical Wilson operators

Fix a compact Lie group G and a finite-dimensional unitary representation ρ of dimension d_ρ . On a finite oriented graph, put $X = G^{E^+}$. For a closed edge word $\gamma = e_1^{\sigma_1} \cdots e_N^{\sigma_N}$, define

$$w_{\gamma, \rho}(U) = \frac{1}{d_\rho} \text{Tr } \rho(U_{e_1}^{\sigma_1} \cdots U_{e_N}^{\sigma_N}). \quad (1)$$

Vertex gauge transformations cancel along the internal vertices and conjugate the product at its basepoint. Thus $w_{\gamma, \rho}$ is gauge invariant and continuous, with $\|w_{\gamma, \rho}\|_\infty \leq 1$. Multiplication by this function is bounded on the finite-graph Gauss-invariant Haar carrier. No bordism, vacuum choice, or spectral gap is needed for this statement.

A finite cellular tube can retain a family of such words, their transverse placement, and their relative markings. For example, a specified combination

$$w_T = \sum_{j=1}^N c_j w_{\gamma_j, \rho_j}, \quad \sum_j |c_j| \leq 1, \quad (2)$$

is again a bounded gauge-invariant observable. Its support is the support of its actual edge dependence. Declaring a containing tube does not establish dependence throughout its interior. The finite construction and its support convention must be stated together.

For a classical connection on a smooth manifold, parallel transport along a curve is available whenever its pullback is canonically defined and integrable. In a unitary frame, solve

$$U'(s) = -a(s)U(s), \quad U(0) = 1, \quad a = \rho_*(\gamma^* A) \in L^1([0, 1], \mathfrak{u}(d_\rho)). \quad (3)$$

The solution is absolutely continuous and unitary. Its normalized trace is a classical Wilson loop. A useful sufficient regularity condition near a smooth embedded curve in four dimensions is $A \in H_{\text{loc}}^s$ with $s > 3/2$; the corresponding threshold on a three-dimensional spatial slice is $s > 1$, by the codimension trace theorem [12]. These sufficient conditions concern restriction to prescribed curves. They do not characterize every weak connection admitting holonomy.

A distributional quantum field need not satisfy this trace condition. A positive-thickness definition must therefore provide an actual finite cellular construction, a gauge-covariant smoothing, or another justified extension of holonomy. Positive-flow-time regularization is a useful comparison: its ultraviolet finiteness has an all-orders perturbative analysis [13, 14]. A scale coordinate alone does not produce the smoothing equation or its estimates. Heat-type smoothing also requires care if exact regional support is demanded.

2 The native frustum module and selector

2.1 Why the cut geometry requires a module

The physical order of construction starts with the electric–magnetic graph frustum and its retained boundary actions. A Hilbert carrier chosen after taking all Wilson traces has already forgotten part of this data. We therefore extract the local module mechanism directly from the native cut word in [6], subsection *Gauge invariance across the cut*. This calculation concerns its declared finite representation sector. It does not replace the full frustum’s incidence data, physical form, or complete selection.

Let h_L run from an incoming endpoint to the left bank, b from the left bank to the right bank, and h_R from the right bank to an outgoing endpoint. Independent frames act by

$$h_L \mapsto u_L h_L u_i^{-1}, \quad b \mapsto u_R b u_L^{-1}, \quad h_R \mapsto u_f h_R u_R^{-1}.$$

Thus $h_R b h_L \mapsto u_f (h_R b h_L) u_i^{-1}$. The seam datum is retained before the endpoints are closed. Setting $b = 1$ is a frame choice with residual diagonal matching, not removal of the seam.

The bank actions determine the coefficient module. Fix a unitary representation sector ρ , and let V_L, V_R be its nonzero finite-dimensional endpoint copies. The open seam matrix belongs to

$$E_{RL} = \text{Hom}(V_L, V_R).$$

It has left algebra $A_R = B(V_R)$, right algebra $A_L = B(V_L)$, actions $a_R T a_L$, and right algebra-valued inner product

$$\langle T, S \rangle_{A_L} = T^* S.$$

This is a finite self-dual Hilbert A_L -module, with adjointable left A_R -action. Its norm is the operator norm, not the Hilbert–Schmidt norm used for an ordinary L^2 coefficient carrier. A scalar Hilbert realization is obtained by tracing the module inner product, with its arguments reversed for the convention of Section 1. The module keeps the endpoint algebra actions before that scalar choice.

The matrix algebras here describe the retained finite endpoint coefficient sector. They are not being identified with the full physical regional W^* -algebras. Their role is determined by the two banks: $T = \rho(b)$ transforms as

$$T \mapsto \rho(u_R) T \rho(u_L)^*.$$

Infinitesimally, for $X_R, X_L \in \mathfrak{g}$,

$$\delta_{X_R, X_L} T = \rho_*(X_R) T - T \rho_*(X_L). \quad (4)$$

This is the electric boundary action on the open magnetic coefficient. It is fixed by orientation and endpoint covariance.

Seam composition is balanced composition. For three banks V_0, V_1, V_2 , retain $E_{21} = \text{Hom}(V_1, V_2)$ and $E_{10} = \text{Hom}(V_0, V_1)$. Matrix composition induces a unitary Hilbert-module identification

$$E_{21} \widehat{\otimes}_{B(V_1)} E_{10} \cong E_{20}, \quad T \otimes S \longmapsto TS. \quad (5)$$

Indeed, $(Ta)S = T(aS)$ proves balancing, and

$$\langle T \otimes S, T' \otimes S' \rangle_{B(V_0)} = S^* T^* T' S' = \langle TS, T' S' \rangle_{B(V_0)}$$

proves preservation of the module inner product. Every rank-one map $|v_2\rangle\langle v_0|$ factors through a unit vector in V_1 , so the map is onto.

At the common bank the electric variation is

$$T \mapsto T \rho(e^{-tX}), \quad S \mapsto \rho(e^{tX}) S.$$

Its first derivative is

$$(-T \rho_*(X)) S + T(\rho_*(X) S) = 0.$$

Thus the shared electric action cancels on the same composition that forms the magnetic word. The module relation retains this matching at the intermediate algebra. Taking separate traces before composition does not: in general $\text{Tr}(TS)$ is not determined by $\text{Tr}(T)$ and $\text{Tr}(S)$.

The compact-group Haar matching projector in [6] imposes the invariant common-bank action in a scalar Hilbert realization. Equation (5) proves the finite matrix-module composition itself; it does not silently identify an arbitrary Haar-projected tensor product with a relative tensor product over an unspecified physical W^* -algebra. Such an identification must retain the actual representations, inner products, and normalization.

A closed observable forgets part of the module. With the return edge included, the Wilson observable takes the normalized trace of the complete ordered word. Internal bank frames then cancel. Before this closure, a center carry at one bank is still visible. If ρ has triality b_ρ , then

$$\rho(\omega^m I_3)T = \omega^{mb_\rho}T, \quad \rho(\omega^m I_3)T\rho(\omega^m I_3)^* = T.$$

Independent-bank transport can therefore retain a coefficient character even though simultaneous conjugation is neutral. This explains why replacing the open bank module by bare class functions loses the data tested in Section C.1. It does not yet identify a particular one-dimensional ground intertwiner line inside the full module.

For the full frustum, repeat the bank construction on its actual oriented incidences and retain the magnetic face words, external ports, and joint electric actions. The physical selection acts on that represented system; the Hamiltonian and complete-ground projection must come from its actual electric and magnetic forms. Neither the boundary cancellation above nor associativity of matrix composition determines that ground projection. In particular, strict elementary matrix associativity does not produce the nonneutral linked associator: the selected carry lines and their native multiplication maps remain necessary.

Across physical spacetime overlaps the corresponding represented actions are compared using the full prepared Ward identity of Section 5.4. The finite bank identity supplies its local orientation mechanism; the graph domains, collar splitting, and actual insertion identifications supply its physical scope.

The explanatory order is consequently: retain the frustum's open boundary actions and magnetic words; construct their modules and seam composition; perform the declared physical and ground selections; and then compare the resulting coefficient lines and ordinary Wilson observables. Abstract Hilbert-space isomorphism or equality of scalar phases skips the structure that makes those comparisons meaningful.

2.2 The declared native charged ground module

The composed geometric Paper I [7], supplied revision 108, Sections 3.9.1–3.9.5, specifies a physical source that makes the preceding structure concrete. Its bulk global form is $PSU(3)$, while the exposed boundary retains the covering group $SU(3)$. The exposed ports and their compensation lines are prescribed boundary data. They are not consequences of the quotient bulk alone. Its actual selection and charged maps are specified below; Appendix C explains why bare class-function cutoffs cannot replace them.

The electric–magnetic source and its selection. On its declared finite graph $Q = PSU(3)^E$, let P^G average only internal vertices. The scalar bulk form is

$$h_{\text{bulk}}[u] = \sum_e c_e \|\nabla_e u\|^2 + \int_Q \sum_f \beta_f \left(1 - \frac{\chi_{\text{ad}}(g_f)}{8}\right) |u|^2 dq, \quad c_e > 0, \quad \beta_f \geq 0.$$

The edge and face weights are assigned from the spatial geometry before spectral inspection. The physical face list retains the distinct fine strip faces $U = a_1 a_*^{-1}$ and $V = a_* a_2^{-1}$; their composite UV does not acquire a second elementary face weight. Comparison faces and collapsed caps are not inserted as physical energy terms.

Set $V_p = \mathbb{C}^3$, retain an inert reference $\bar{V}_{\text{ref}} \cong \overline{\mathbb{C}^3}$, and take

$$E_0 = \mathbb{C}, \quad E_1 = V_p, \quad E_2 = V_p^*, \quad P_j^K = P^G \otimes I_{E_j \otimes \bar{V}_{\text{ref}}} \otimes P_+.$$

Here P_+ selects equal collar copies and $\mathcal{C}_+ = \text{Ran } P_+$ is one dimensional in the declared two-copy row. The physical carrier is consequently

$$\mathcal{H}_j^K = \text{Ran } P^G \otimes E_j \otimes \bar{V}_{\text{ref}} \otimes \mathcal{C}_+.$$

The exposed $SU(3)$ action acts on the edge variables through the quotient and on E_j with center character ω^j . It acts trivially on the reference. Averaging that exposed action as an additional singlet condition would change this source.

Positive ellipticity on the connected compact edge product and the bounded real face potential give the scalar operator a unique normalized positive ground u_0 . Uniqueness and positivity make it internally gauge invariant. The source's closed-form argument supplies the domain and spectral assertions. Since the coefficient and collar factors are inert after selection,

$$H_j = H_{\text{bulk}}^G \otimes I, \quad \mathcal{V}_j = \text{Ran } P_{0,j} = \mathbb{C} u_0 \otimes E_j \otimes \bar{V}_{\text{ref}} \otimes \mathcal{C}_+. \quad (6)$$

The complete fiber ground dimensions are 3, 9, 9. Their multiplicities cannot be discarded by choosing one scalar vacuum vector.

Identifying $E_j \otimes \bar{V}_{\text{ref}}$ with $\text{Hom}(V_{\text{ref}}, E_j)$ gives the ground coefficient a right $B(V_{\text{ref}})$ -module structure with inner product T^*S . The scalar product $\text{Tr}(S^*T)$ recovers the physical coefficient Hilbert realization. Thus the concrete source supplies both the scalar Hilbert realization and its finite module interpretation. The shape is the bulk ground factor together with the charged endpoint and reference actions, not just its total dimension.

Endpoint notation used throughout. Write $\mathcal{V}^{\text{fix}} = \mathcal{V}_1$ for the fixed fundamental endpoint and $\mathcal{V}^{\text{gr}} = (\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2)$ for the graded endpoint package, with its declared insertion maps. The separately marked linked source is denoted by its lines $L_{m,a}$ and carry K_m . A selection rule on $\mathcal{V}^{\text{fix}}$ does not remove the other members of $\mathcal{V}^{\text{gr}}$. Conversely, the existence of $\mathcal{V}^{\text{gr}}$ does not identify its topology, selected lines or seam multiplication with those of the linked source.

The selected line maps are nonzero and have kernels. With a fixed unit volume tensor, the source maps are

$$\begin{aligned} t_0(1 \otimes e_i) &= e_i, \\ t_1(e_i \otimes e_l) &= 2^{-1/2} \sum_k \varepsilon_{kil} e^k, \\ t_2(e^k \otimes e_l) &= 3^{-1/2} \delta_{kl}. \end{aligned}$$

They satisfy $t_j t_j^* = I_{E_{j+1}}$, with cyclic endpoint labels, and

$$\rho_{j+1}(g) t_j = t_j (\rho_j(g) \otimes \rho_V(g)).$$

Their physical extensions T_j act as t_j on the endpoint and inserted port and as the identity on the common bulk, reference, and collar. Direct tensor calculation gives

$$P_{j+1}^K T_j = T_j (P_j^K \otimes I_V), \quad P_{0,j+1} T_j = T_j (P_{0,j} \otimes I_V). \quad (7)$$

The same tensor identity intertwines the Hamiltonians on their operator domains. On form domains it means $h_{j+1}[T_j \psi, \phi] = (h_j \otimes I_V)[\psi, T_j^* \phi]$; it does not assert preservation of every quadratic energy by a map with a kernel. On the complete insertion ground, T_j is a nonzero coisometry of norm one. Its full kernel dimensions are 0, 18, 24: the coefficient kernels $0, \text{Sym}^2 V, \mathfrak{sl}(V)$ have dimensions 0, 6, 8, and the reference contributes a factor of three. No kernel is removed to turn these contractions into embeddings.

The lines CT_j therefore pass the exact selection and nonvanishing tests of Section A.4 on this declared native source. Their ground restrictions are right $B(V_{\text{ref}})$ -linear. This is a concrete realization of surviving charged maps, not merely equality of their center characters.

An actual compensating line and its square. The same source supplies longitude and comparison lifts

$$C = \text{diag}(1, \omega, \omega^2), \quad Se_i = e_{i+1 \bmod 3}, \quad CS = \omega SC.$$

The comparator line L_h has longitude holonomy ω^{-1} and comparison-loop holonomy one, with its declared trivialization on the return seam. In overlapping unit frames,

$$(CSC^{-1}) \otimes \ell_{h,1} = (\omega S) \otimes (\omega^{-1} \ell_{h,0}) = S \otimes \ell_{h,0}.$$

This verifies the actual line-valued comparator $\mathcal{I}_h = S \otimes \ell_h$. On endpoint E_j , use its representation and line power L_h^j . Equivariance at $g = S$ gives the concrete insertion square

$$\rho_{j+1}(S) t_j = t_j (\rho_j(S) \otimes S). \quad (8)$$

Input compensation lines multiply to L_h^{j+1} ; at the cyclic endpoint use the source's normalized reduction of L_h^3 . The same square holds on $\mathcal{V}_j$, because the maps act on the retained coefficient factors and preserve the common scalar ground. This proves the square for the source's common within-level bulk. It does not identify Hamiltonians on independently weighted graphs.

The target line also changes what the scalar detector measures. The canonical mixed scalar contribution is

$$\omega^{a_\ell b_h m} \text{hol}_{\ell^{a_\ell}}(L_h^{mb_h}) = \omega^{a_\ell b_h m} \omega^{-a_\ell b_h m} = 1.$$

This is the explicit cancellation in [7], Theorem 6.1. That source obtains a prescribed nontrivial higher history phase only after retaining its additional selector coupling. It cannot be identified with the nonneutral linked class of Original Paper III merely because both use clock-shift phases.

What determines the bulk ground. The charged factors above are inert, but the scalar ground is an interacting electric-magnetic object. For the named strip, conditional averaging over a_* leaves a nonzero coupling from inherited functions to new directions:

$$K^* K = M_{\sigma^2(UV)}, \quad \sigma^2(g) = \frac{\beta_U^2 + \beta_V^2}{64} + \frac{\beta_U \beta_V}{256} \chi_{\text{ad}}(g).$$

This is the source's Proposition 3.8, with unchanged inherited electric weights and its stated spectator scope. It explains why the fine ground need not be an inherited or separately factorized ground even though $UV = a_1 a_2^{-1}$. The module in (6) uses the ground of that actual full Hamiltonian.

The complete ground, charged maps, kernels, and line-valued square are thus explicit on this source. The following selector construction adds its specified higher coupling; physical spacetime overlaps use the prepared Ward comparisons of Section 5.4.

2.3 The selector state line and the cyclic comparison

The additional selector in [7], Sections 6.8–6.10, supplies an actual line action, not just a numerical period. For the source's relative history complex (M, N) , let B and s be its compatible degree-two and degree-one $\mathbb{Z}_3$ -cocycles. Write $\omega = e^{2\pi i/3}$ and fix $k \in \mathbb{Z}_3$. A closed relative two-chain y has a state line $\mathcal{L}_y = \mathbb{C}|y\rangle$. An admitted relative three-chain D , with $\partial D = y_1 - y_0$, acts by

$$F_D|y_0\rangle = \omega^{k\langle B \smile s, D \rangle} |y_1\rangle. \quad (9)$$

The cocycle condition makes this independent of adding an admitted four-boundary; addition of histories gives composition. Relative trivializations are retained on the lateral seam. If $B' = B + \delta r$ and $s' = s + \delta t$, then

$$B' \smile s' - B \smile s = \delta \kappa, \quad \kappa = r \smile s + B \smile t + r \smile \delta t.$$

Thus $\Gamma_y = \omega^{k\langle \kappa, y \rangle}$ gives $F'_D \Gamma_{y_0} = \Gamma_{y_1} F_D$. This is the operational dependence on the representative, with the source's relative seam conditions understood.

For each already certified native map T_D , the tensor product $T_D \otimes F_D$ preserves its established selection and ground-intertwining identities. It also preserves the full kernel, up to tensoring with the source line, and its operator norm. This follows because F_D is unitary between one-dimensional spaces. It creates no new physical history or form-intertwining certificate. In particular, the native kernels calculated above do not disappear when the selector is installed.

Two coefficient classes on different groups. On the marked product of the longitude, comparison, and selector directions, the source's ordered finite-group representative is

$$A = (\mathbb{Z}_3)^3, \quad \alpha_k(x, y, z) = \omega^{k x_\ell y_h z_\Sigma}.$$

The earlier linked-flux source instead uses the cyclic carry representative

$$\Omega_m(a, b, c) = \omega^{ma \lfloor (b+c)/3 \rfloor}, \quad a, b, c \in \{0, 1, 2\}, \quad m = 1, 2.$$

Both are normalized three-cocycles with trivial coefficient action, valued in $U(1)$. Equal values on selected marked cycles do not identify their coefficient systems.

Proposition 2.1 (Cyclic restriction separates the two classes). *For every homomorphism $f_v : \mathbb{Z}_3 \rightarrow A$, $f_v(a) = av$, the pullback $f_v^* \alpha_k$ is a coboundary. It is therefore not cohomologous to Ω_m for $m = 1, 2$.*

Proof. Set $c_v = kv_\ell v_h v_\Sigma$ modulo 3. Then $f_v^* \alpha_k(a, b, c) = \omega^{c_v abc}$. The normalized two-cochain $q_v(a, b) = \omega^{-c_v ab^2}$ satisfies

$$(\delta q_v)(a, b, c) = \frac{q_v(b, c)q_v(a, b+c)}{q_v(a+b, c)q_v(a, b)} = \omega^{-2c_v abc} = \omega^{c_v abc}.$$

All arguments in this calculation are in $\mathbb{Z}_3$. Reduction modulo 3 does not change the exponents.

For any normalized three-cocycle η on $\mathbb{Z}_3$, the detector

$$I_3(\eta) = \prod_{b=0}^2 \eta(1, b, 1)$$

is unchanged by a three-coboundary: the two ratios in $\prod_b (\delta q)(1, b, 1)$ each telescope around the cyclic group. For the carry cocycle, $I_3(\Omega_m) = \omega^m \neq 1$. Thus its class cannot equal the trivial cyclic restriction of α_k . $\square$

This conclusion does not mean that α_k itself is trivial. For $k \neq 0$, the alternating bar cycle formed from the three independent generators e_ℓ, e_h, e_Σ evaluates to ω^k : among its six ordered terms only $[e_\ell | e_h | e_\Sigma]$ contributes. Its information is genuinely in the three independent directions, even though all cyclic restrictions vanish.

Consequence for the native comparison. The selector line is now realized on the actual native charged modules, with its gauge comparison and lateral trivialization. But no comparison obtained merely by restricting the three product directions along a cyclic group homomorphism identifies its class with the linked carry class. Such an identification would require a different, explicitly specified correspondence or transgression, together with its line maps and lateral-seam identifications. This calculation leaves that possibility open; it rules out this particular pullback identification even after scalar rephasing.

The distinction also matters for observables. Tensoring an existing operator with the identity of $\mathcal{L}_y$ leaves its local action unchanged, and a scalar in a transport cancels under adjoint transport. Closed-history holonomy can nevertheless remain ω^k . Hence ordinary adjoint observables can descend while the selector line retains nontrivial higher information. Neither adjoint cancellation nor equality of a single closed-history phase identifies the two coefficient bands.

2.4 Selector transgression and the native boundary module

There is a direct comparison with the native clock-shift record that does not use cyclic restriction. Retain the two independent directions $Q = (\mathbb{Z}_3)_\ell \times (\mathbb{Z}_3)_h$, and fix selector winding m . For the chosen orientation, define the central slant representative by

$$(\tau_{me_\Sigma} \alpha_k)(x, y) = \frac{\alpha_k(me_\Sigma, x, y) \alpha_k(x, y, me_\Sigma)}{\alpha_k(x, me_\Sigma, y)}, \quad x, y \in Q.$$

Here Q is embedded with zero selector coordinate. Substitution gives the explicit degree-two coefficient

$$\beta_n(x, y) = \omega^{nx_\ell y_h}, \quad n = km \pmod{3}. \quad (10)$$

This is a transgression along the retained selector cycle, not a restriction of all three directions to one cyclic group.

Proposition 2.2 (Native realization of the transgressed multiplier). *For $n \neq 0$, the cocycle β_n is nontrivial on Q . On the native fundamental endpoint $E_1 = V$, the actual clock and shift give a projective unitary action*

$$R_n(x) = S^{nx_h} C^{x_\ell}, \quad R_n(x) R_n(y) = \beta_n(x, y) R_n(x + y).$$

Acting on the output of $\text{Hom}(V_{\text{ref}}, V)$, these maps are adjointable unitary right- $B(V_{\text{ref}})$ -module maps. They retain the complete fundamental ground module of Section 2.2.

Proof. The degree-two cocycle identity follows by expanding the bilinear exponent in $\beta_n(x, y)\beta_n(x + y, z) = \beta_n(y, z)\beta_n(x, y + z)$. Its antisymmetric ratio is

$$\frac{\beta_n(x, y)}{\beta_n(y, x)} = \omega^{n(x_\ell y_h - y_\ell x_h)}.$$

At $x = e_\ell, y = e_h$ this is $\omega^n \neq 1$. Every scalar two-coboundary on an abelian group is symmetric, so β_n cannot be a coboundary. The relation $CS = \omega SC$ gives the displayed projective multiplication exactly.

For a module vector $T : V_{\text{ref}} \rightarrow V$, set $U_n(x)T = R_n(x)T$. Then $U_n(x)(Ta) = (U_n(x)T)a$ and

$$\langle U_n(x)T, U_n(x)T' \rangle_{B(V_{\text{ref}})} = T^* R_n(x)^* R_n(x) T' = T^* T'.$$

The adjoint is left multiplication by $R_n(x)^*$. Extending by identities on the bulk ground and collar factors preserves the complete ground and the declared selection. These operations are powers of the source's coefficient clock and comparator shift; no replacement of its bulk Hamiltonian or ground is involved. $\square$

The actual compensating line. In the source's canonical mixed record, the comparison winding y_h with displacement n has target line $L_h^{ny_h}$. Its longitude transport is

$$\text{hol}_{\ell^{x_\ell}}(L_h^{ny_h}) = \omega^{-nx_\ell y_h} = \beta_n(x, y)^{-1}.$$

Thus (10) agrees with the native uncompleted lift multiplier, and the inherited target line supplies precisely its inverse. This is the line-valued cancellation already certified in [7], Section 6.7. It is not a scalar trivialization of β_n on Q : the target lines and their transport are essential. Their completed interchange factor is one. Consequently, recovering this native lift multiplier does not derive the additional selector coupling from the completed ordinary maps.

Why the module shape matters. On the complete fundamental ground, the right-module presentation and the scalar Hilbert presentation are

$$\mathcal{V}_1 \simeq \text{Hom}(V_{\text{ref}}, V), \quad \mathcal{V}_1^{\text{Hilb}} \simeq V \otimes \overline{V_{\text{ref}}}, \quad \dim \mathcal{V}_1 = 9.$$

The linear-first scalar product in the second presentation is $\text{Tr}(T'^* T)$. For $n = 1, 2$, the nine matrices $S^{nb} C^a$, $a, b \in \mathbb{Z}_3$, form a trace-orthogonal basis of $B(V)$. Indeed, a nonzero shift has zero trace, while $\text{Tr}(C^a) = 0$ for $a = 1, 2$. Hence the von Neumann algebra generated by these boundary coefficient comparisons is

$$\{U_n(x) : x \in Q\}'' = B(V) \otimes I_{\overline{V_{\text{ref}}}} \simeq M_3(\mathbb{C}), \quad (11)$$

and its commutant is $I_V \otimes B(\overline{V_{\text{ref}}})$. The latter is the scalar-Hilbert realization of the opposite right coefficient algebra. The nine-dimensional carrier therefore retains a three-dimensional multiplicity space; the comparison algebra is not all of $B(\mathcal{V}_1^{\text{Hilb}}) \simeq M_9(\mathbb{C})$.

Moreover, $\text{Ad } U_n(x) \text{ Ad } U_n(y) = \text{Ad } U_n(x + y)$, because the multiplier is scalar. This is an explicit ordinary adjoint action with a nontrivial projective coefficient class. Equation (11) describes the

boundary comparison algebra, not an identification of the Wilson-generated algebra. Physical spacetime passage still uses the actual Ward-compatible overlap maps described in Section 5.4.

This comparison supplies the selector slant, its native coefficient-module action, and the inherited inverse line on the marked two-direction record. It leaves the earlier cyclic carry class distinct and makes no identification of all higher history diagrams. In particular, the original charged maps and their kernels are retained; the coefficient action above does not convert the native coisometries into unitary transport.

3 Coefficient sewing on admitted prisms

The coefficient two-cocycle, its admitted comparison cells, and its lateral seam are the inputs here. A filling identifies endpoint multipliers; it does not identify their carriers or bound their physical observable residual. The relative mapping-fiber and differential-geometric comparisons are recorded separately in Appendix E.

3.1 The native coefficient prism and its seam filling

The October source insertion [6], Section `sec : scale`, supplies a coefficient comparison on a declared common prism. It works with an actual transgressed two-cocycle and retains the lateral boundary and seam certificate. This gives a filling of an endpoint coefficient difference; the source keeps it distinct from the relative mapping-fiber datum of Original Paper II.

Write phases in $\mathbb{R}/\mathbb{Z}$. Let $\theta \in Z^2(X; \mathbb{R}/\mathbb{Z})$, let $f_0, f_1 : Y \rightarrow X$ be the admitted endpoint comparison maps, and put $\theta_r = f_r^* \theta$. Suppose their chain comparison has

$$f_{1\#} - f_{0\#} = \partial H + H\partial + S, \quad b(c) = \theta(Hc), \quad r(z) = \theta(Sz). \quad (12)$$

Here H raises chain degree by one, c is a one-chain, z a two-chain, and S retains the lateral boundary. Evaluation on z gives

$$\theta_1 - \theta_0 = \delta b + r. \quad (13)$$

Indeed, $\theta(\partial Hz) = 0$ by $\delta\theta = 0$, and $\theta(H\partial z) = (\delta b)(z)$. This is the source's prism identity.

On a simplicial comparison prism, write $x_i = f_0(v_i)$, $y_i = f_1(v_i)$. When the mixed simplices below are admitted, the staircase prism is

$$H[v_0, \dots, v_n] = \sum_{q=0}^n (-1)^q [x_0, \dots, x_q, y_q, \dots, y_n]. \quad (14)$$

Interior faces cancel, leaving the endpoint and boundary prisms. In particular,

$$b_{01} = \theta[x_0, y_0, y_1] - \theta[x_0, x_1, y_1]. \quad (15)$$

Repeated vertices use normalized degeneracies. For two subordinate choices of a cover-refinement map, the mixed simplices lie in the common comparison nerve whenever the refined intersection is nonempty. A general physical bordism must supply its actual mixed cells and lateral faces.

If the seam certificate provides a one-cochain a with $r = \delta a$, set

$$\beta = b + a, \quad \theta_1 - \theta_0 = \delta\beta, \quad t_{ij} = e^{2\pi i \beta_{ij}}. \quad (16)$$

This convention incorporates the lateral term into the primitive. Without that certificate, r remains in (13), including its evaluation on closed two-cycles.

Proposition 3.1 (Action of the coefficient filling). *Under these admitted prism and seam identities, let $\zeta_r = e^{2\pi i \theta_r}$. Then $\zeta_1 = (\delta t)\zeta_0$. If endpoint overlap implementers obey $u_{ij}^r u_{jk}^r = \zeta_{r,ijk} u_{ik}^r$, the rephased target implementers*

$$\tilde{u}_{ij}^1 = t_{ij}^{-1} u_{ij}^1 \quad (17)$$

have multiplier ζ_0 and the same adjoint observable maps as u_{ij}^1 . Thus the target's ordinary compatible-family observable algebra is unchanged by this rephasing. The statement does not identify the two endpoint carriers or their observable algebras.

Proof. Exponentiate (16). The rephased multiplier is $(t_{ij} t_{jk} t_{ik}^{-1})^{-1} \zeta_{1,ijk} = \zeta_{0,ijk}$. Scalar phases cancel in the adjoint action, so the ordinary target overlap equations are unchanged. $\square$

The filling also respects the mixed scale square. For a physical arrow with local maps $J_{a,i}$, suppose

$$u_{ij}^t J_{a,j} = J_{a,i} (u_{ij}^s \otimes \ell_{a,ij}).$$

For arbitrary scalar endpoint rephasings q_{ij}^r , set

$$\tilde{u}_{ij}^r = q_{ij}^r u_{ij}^r, \quad \tilde{\ell}_{a,ij} = q_{ij}^t (q_{ij}^s)^{-1} \ell_{a,ij}. \quad (18)$$

Both sides acquire the same factor q_{ij}^t . The physical arrow is neither inverted nor reconstructed from the phases. These are the supplied coefficient overlap maps; their higher cells remain part of the datum. Use the gauge and spacetime overlaps of the admitted common diagram, retaining its mixed prisms. Proposition 5.1 then transports the Wilson residual through the same squares.

The primitive trivializes an endpoint *difference*. Either endpoint class, the global coefficient band, and represented vacuum descent can remain nontrivial. Two primitives differ by a closed one-cochain whose flat coefficient information can survive in represented descent. At each fixed endpoint, Theorem 5.3 gives the same ordinary W^* -algebra before and after rephasing when its normal-map hypotheses hold. Across endpoints, the actual physical arrow, observable comparison, and analytic estimates retain their own roles.

This is the coefficient sewing supplied by a filled transgression prism. An assertion that the relative primitive τ of Original Paper II evaluates instead to $U - I$, or to another physical operator correction, is additional data. Coefficient sewing and ordinary Wilson descent do not require that particular identification.

3.2 The mixed double prism for the two Čech directions

Gauge-chart refinement and spacetime-cover refinement give two paths around a comparison square. A filling along each edge does not by itself specify their comparison. Pull the square back to a common admitted nerve Y , with four maps $f_{ab} : Y \rightarrow X$, $a, b \in \{0, 1\}$, into the coefficient comparison complex. Here a records gauge refinement and b spacetime refinement; these indices do not denote physical scale arrows. Let $\theta_{ab} = f_{ab}^* \theta$, using the same closed two-cocycle θ as above. All edge data below are understood after this common pullback.

For each oriented edge $e : p \rightarrow q$, suppose the prism and seam certificate give

$$f_{q\#} - f_{p\#} = \partial H_e + H_e \partial + S_e, \quad \theta S_e = \delta a_e, \quad \beta_e = \theta H_e + a_e.$$

The evaluations in this display have the degrees of Section 3.1. Write A for the path $00 \rightarrow 10 \rightarrow 11$, and B for $00 \rightarrow 01 \rightarrow 11$. Sum their prisms and seam primitives to obtain H_A, a_A and H_B, a_B . Then

$$\beta_A = \theta H_A + a_A, \quad \beta_B = \theta H_B + a_B, \quad \eta = \beta_A - \beta_B, \quad \delta\eta = 0. \quad (19)$$

The last equality follows because both fillings have coboundary $\theta_{11} - \theta_{00}$. Closedness alone leaves a possible flat coefficient difference.

Proposition 3.2 (Mixed coefficient comparison). *Suppose the admitted square supplies a degree-two chain prism $K : C_n(Y) \rightarrow C_{n+2}(X)$ and its retained lateral term $S_\square$, with*

$$H_A - H_B = \partial K - K\partial + S_\square. \quad (20)$$

Suppose also that its seam comparison provides a zero-cochain c satisfying

$$\theta S_\square + a_A - a_B = \delta c \quad \text{on one-chains of } Y. \quad (21)$$

Then the two edge fillings are related by the specified zero-cochain

$$\eta = \delta\kappa, \quad \kappa(v) = c(v) - \theta(Kv). \quad (22)$$

Consequently, with $t_A = e^{2\pi i\beta_A}$, $t_B = e^{2\pi i\beta_B}$, and $k_i = e^{2\pi i\kappa(v_i)}$,

$$t_{A,ij} t_{B,ij}^{-1} = k_j k_i^{-1}, \quad u_{ij}^A = k_i u_{ij}^B k_j^{-1}, \quad (23)$$

where $u^A = t_A^{-1} u^{11}$ and $u^B = t_B^{-1} u^{11}$. Both target presentations therefore induce the same ordinary adjoint descent equations.

Proof. Evaluate (20) on a one-chain z . Closedness gives $\theta(\partial Kz) = 0$, since Kz is a three-chain. Thus

$$\eta(z) = -\theta(K\partial z) + (\theta S_\square + a_A - a_B)(z) = (\delta(c - \theta K))(z).$$

Exponentiating gives the first identity in (23). Substitution into the two rephasings gives the second. Central scalars cancel under the adjoint action. $\square$

The formula can be evaluated directly on admitted mixed simplices. At a vertex v_i , put $x_{ab,i} = f_{ab}(v_i)$. With no lateral or seam terms, take

$$K(v_i) = [x_{00,i}, x_{10,i}, x_{11,i}] - [x_{00,i}, x_{01,i}, x_{11,i}], \quad \kappa_i = -\theta(K(v_i)). \quad (24)$$

Its boundary is the gauge-then-spacetime path minus the spacetime-then-gauge path. On an edge, triangulate the cube with coordinates gauge, spacetime, and nerve-edge. Its six oriented tetrahedra are

$$K[v_0, v_1] = \sum_{\pi \in S_3} \text{sgn}(\pi) [x_{000}, x_{e_{\pi(1)}}, x_{e_{\pi(1)} + e_{\pi(2)}}, x_{111}],$$

where $x_{abi} = f_{ab}(v_i)$ and e_1, e_2, e_3 are the coordinate vectors. The boundary is $(H_A - H_B)[v_0, v_1] + K(v_1) - K(v_0)$, verifying the sign in (20). Every displayed simplex must belong to the admitted comparison complex. A drawing of a square does not supply these cells or the seam comparison.

This is where the two Čech nerves interact at the level of the filling: the double prism compares their two refinement orders, and its evaluated phase is a local central modification. Without this comparison, the class $[\eta] \in H^1(Y; \mathbb{R}/\mathbb{Z})$ need not vanish. Ordinary adjoint descent forgets these scalar phases even in that case; represented coefficient descent can retain them. The proposition records the actual comparison when supplied, without identifying endpoint carriers, inverting physical arrows, or providing a physical Wilson refinement estimate.

3.3 Coherence of the phase comparisons

The local central comparison of Proposition 3.2 must itself compose. Let P, Q, R be three admitted refinement paths with the same endpoints on a common nerve Y . Their fillings satisfy $\delta\beta_P = \delta\beta_Q = \delta\beta_R$. Suppose the retained double prisms and seam comparisons give zero-cochains

$$\beta_P - \beta_Q = \delta\kappa_{PQ}, \quad \beta_Q - \beta_R = \delta\kappa_{QR}, \quad \beta_P - \beta_R = \delta\kappa_{PR}.$$

Their composition defect is

$$\lambda_{PQR} = \kappa_{PQ} + \kappa_{QR} - \kappa_{PR}, \quad \delta\lambda_{PQR} = 0. \quad (25)$$

Thus the defect is constant on each connected component of Y , with values in $\mathbb{R}/\mathbb{Z}$. For $k_{PQ} = e^{2\pi i \kappa_{PQ}}$, coherent composition means $k_{PQ}k_{QR} = k_{PR}$, equivalently $\lambda_{PQR} = 0$. Pairwise exactness proves only the displayed closedness. Even on a one-point nerve, all three filling differences can vanish while the chosen comparisons $\kappa_{PQ} = \kappa_{QR} = 0$, $\kappa_{PR} = \alpha \neq 0$ fail this composition equation. Compatible choices can remove such a defect in this example; its point is that the retained choices must be checked.

Proposition 3.3 (Coherence from the third prism). *Use the data of Proposition 3.2 for each pair, with $\kappa_{PQ} = c_{PQ} - \theta K_{PQ}$. Suppose a supplied degree-three prism L and lateral comparison T obey*

$$K_{PQ} + K_{QR} - K_{PR} = \partial L + L\partial + T. \quad (26)$$

Then on vertices,

$$\lambda_{PQR} = c_{PQ} + c_{QR} - c_{PR} - \theta T. \quad (27)$$

In particular, the seam coherence identity $c_{PQ} + c_{QR} - c_{PR} = \theta T$ on vertices implies coherent composition of the local phases.

Proof. Evaluate (26) on a vertex v . Here $\partial v = 0$, and $\theta(\partial L v) = (\delta\theta)(Lv) = 0$. Substituting in (25) gives (27). All values are taken in $\mathbb{R}/\mathbb{Z}$, so its vanishing is exactly the exponentiated composition equation. $\square$

There is a concrete cube calculation behind this statement. Consider three successive refinement operations, labelled 1, 2, 3, whose mixed comparison cells are admitted. These may include two refinements in one Čech direction and one in the other; they do not introduce a third Čech nerve. Their six orders are connected by adjacent swaps:

$$123 \longrightarrow 132 \longrightarrow 312 \longrightarrow 321 \longrightarrow 231 \longrightarrow 213 \longrightarrow 123. \quad (28)$$

Assume first that the lateral and seam terms vanish. At a vertex of Y , let x_a be the comparison vertex with cube coordinate $a \in \{0, 1\}^3$. Let $F_{r,\epsilon}$ denote the square at coordinate $r = \epsilon$, triangulated and oriented by the two remaining coordinates in increasing order. Use the pairwise square prisms of Section 3.2. Their oriented sum around (28) is

$$\sum_{\text{hexagon}} K_{P,Q} = F_{1,1} - F_{1,0} - F_{2,1} + F_{2,0} + F_{3,1} - F_{3,0} = \partial L, \quad (29)$$

where the cube's three-chain is explicitly

$$L = \sum_{\pi \in S_3} \text{sgn}(\pi) [x_{000}, x_{e_{\pi(1)}}, x_{e_{\pi(1)} + e_{\pi(2)}}, x_{111}].$$

The six tetrahedra cancel on interior faces. Evaluating their boundary with the closed two-cocycle gives

$$\sum_{\text{hexagon}} \kappa_{P,Q} = -\theta(\partial L) = 0, \quad \prod_{\text{hexagon}} k_{P,Q} = 1. \quad (30)$$

This is coherence of the supplied adjacent-swap comparisons, proved directly from $\delta\theta = 0$. With lateral terms present, the product instead records $\exp(2\pi i(\sum c_{P,Q} - \theta T))$; the corresponding seam identity makes it 1.

The three-parameter prism therefore supplies a composition law for the phase comparisons already constructed. Ordinary adjoint Wilson descent cannot detect the constant defect in (25); represented coefficient descent records the specified phases and their coherence. This calculation concerns the transgressed cocycle θ . It does not identify its evaluated phases with the native relative primitive, the original coefficient-band associator, or a physical operator correction. No additional analytic Wilson estimate is imposed by this coherence calculation.

4 Scale bordisms retain the physical passage

At the source scope of [3], Section III.B, a scale bordism over $a : s \rightarrow t$ contains the finite geometric refinement, physical comparison maps, vacuum comparison, and relative filling. Its invariant coefficient-assisted passage is written

$$J_a : \mathcal{E}_s \widehat{\otimes} L_a \longrightarrow \mathcal{E}_t. \quad (31)$$

Here the notation allows a Hilbert carrier or an admitted Hilbert-module carrier. The coefficient line L_a , its multiplication, and its reassociation are part of the datum. They are not replaced by a global selection of phases. A general passage is not invertible.

Proposition 4.1 (Local frame independence of supported observable transport). *Suppose (31) is an adjointable isometry of Hilbert modules over a fixed coefficient algebra, with L_a a complex Hermitian line. A unit vector $e \in L_a$ defines $V_e \xi = J_a(\xi \otimes e)$. Then*

$$\Phi_a(x) = V_e x V_e^* \quad (32)$$

is independent of e and is a $*$ -homomorphism into the supported corner with unit $V_e V_e^*$.

Proof. Every other unit vector is $e' = \lambda e$, $|\lambda| = 1$, so $V_{e'} = \lambda V_e$, and the phases cancel in (32). Since $V_e^* V_e = 1$, multiplication and adjoints are preserved. The image of the source identity is $V_e V_e^*$. $\square$

This proposition explains one way observable transport can forget a central phase while the represented coefficient diagram retains it. It neither trivializes that diagram nor makes Φ_a unital on an entire larger target algebra. If the passage is only a bounded comparison rather than an isometry, the same conjugation formula need not be multiplicative. For general correspondences, the actual observable action and its supported domain must be supplied separately.

The source's analytic composition defect is

$$D_{b,a}^{(v)} = U_{\chi_{b,a}}^{(v)} T_b^{(v)} T_a^{(v)} - T_{ba}^{(v)}. \quad (33)$$

The compositor $\chi_{b,a}$ identifies the relevant parenthesized targets before subtraction. Its higher coherence and a bound on $D_{b,a}^{(v)}$ are distinct statements. An actual bordism includes both kinds of

information, so it is a natural domain on which to formulate their relationship. The relation must nevertheless be checked, rather than inferred from the existence of the bordism label.

In a twisted realization, global scalar operators with an imposed nontrivial scalar associator are not an adequate replacement for coefficient-assisted composition. The native tensor order and reassociation must remain present. Local charts and their comparisons can be used for calculation without identifying all chart choices or filling histories globally.

5 The two Čech nerves and normal observable descent

The comparison of scales and the assembly of local presentations meet on a Čech nerve. The nerve records where local packets must be compared; a scale bordism supplies a physical passage between such packets. A filling must act compatibly with those comparisons before its boundary evaluation defines a descended observable.

5.1 The overlap diagram and its two uses

Fix an admitted finite cover $\mathcal{U} = \{U_i \rightarrow O\}_{i \in I}$ and retain the actual nonempty intersections. Its Čech object is

$$\check{C}_p(\mathcal{U}) = \coprod_{i_0, \dots, i_p} U_{i_0 \dots i_p}. \quad (34)$$

Face maps omit an index and use the corresponding intersection inclusion; degeneracies repeat an index. The physical diagrams over these intersections retain their own variance and restriction comparisons. The degree p counts overlaps, not spacetime dimensions. A triple overlap is not a geometric plaque, and an overlap simplex is not automatically a scale bordism.

Gauge-chart descent compares local gauge or represented presentations. Spacetime-cover descent compares physical realizations on admitted regions. These are the two Čech directions of the source [11], Section 8.1. They are distinct from the regional and scale composition directions. If both descent directions are organized into a diagram with two simplicial indices, the mixed comparison maps must be supplied; a Cartesian product of unspecified nerves is not a construction of them.

At degree zero one has local decorated physical diagrams $\mathcal{P}_i$. At degree one one has marking-preserving overlap equivalences

$$u_{ij} : \mathcal{P}_j|_{U_{ij}} \simeq \mathcal{P}_i|_{U_{ij}}.$$

Triple overlaps carry invertible comparisons $u_{ij}u_{jk} \Rightarrow u_{ik}$; quadruple overlaps carry their coherence. Units are retained through the degeneracies. Ordinary effective descent reconstructs an object from compatible local data [10]; here the object includes the source's higher cells and analytic decorations. All assertions concern the admitted cover and branch.

5.2 Local modules, coefficient phases, and actual Wilson actions

Suppose a local realization uses a self-dual Hilbert module E_i over a von Neumann algebra B_i . Its adjointable endomorphism algebra $\mathbb{L}_{B_i}(E_i)$ is a von Neumann algebra [9]. The Wilson actions must be actual elements of it. An arbitrary Hilbert C^* -module does not guarantee this normal endomorphism structure, and an abstract self-dual completion does not by itself identify the intended physical realization.

On an overlap chart in which the relevant coefficient algebras have been identified, a unitary module comparison $u_{ij} : E_j \rightarrow E_i$ induces the normal isomorphism

$$\alpha_{ij}(x) = u_{ij} x u_{ij}^*. \quad (35)$$

Restrictions to the overlap are understood in this formula. In the elementary local projective model,

$$u_{ij} u_{jk} = \zeta_{ijk} u_{ik}, \quad \zeta_{ijk} \in U(1), \quad (36)$$

ordinary associativity gives $\zeta_{ijk} \zeta_{ikl} = \zeta_{jkl} \zeta_{ijl}$ on quadruple overlaps. Nevertheless,

$$\alpha_{ij} \alpha_{jk}(x) = \zeta_{ijk} u_{ik} x u_{ik}^* \overline{\zeta_{ijk}} = \alpha_{ik}(x).$$

Thus adjoint observable comparisons can satisfy a strict cocycle while the represented module retains nontrivial coefficient descent. This is a local illustration of the ordinary observable shadow; it is not a global choice of Hilbert-space lifts for the native banded system. Its Čech two-cocycle ζ is not identified here with the source's categorical three-cocycle Ω .

A Wilson action must satisfy its ordinary overlap equation:

$$W_i|_{U_{ij}} = \alpha_{ij}(W_j|_{U_{ij}}). \quad (37)$$

For a Wilson tube crossing several patches, the local data must include the boundary and seam information that defines these evaluations. Traces of arbitrary open path segments do not constitute such a descent datum. Across the physical spacetime cover, the same insertion must also carry the Gauss–Ward and collar comparison evaluated in Section 5.4.

5.3 The overlap calculation for a one-sided bordism

Let $a : s \rightarrow t$ be an admitted scale passage. In a common overlap chart write

$$J_{a,i} : E_{s,i} \widehat{\otimes} L_{a,i} \longrightarrow E_{t,i}, \quad \ell_{a,ij} : L_{a,j} \longrightarrow L_{a,i}.$$

For the following concrete calculation the $J_{a,i}$ are adjointable isometries and $\ell_{a,ij}$ are unitary line identifications. Put $v_{a,ij} = u_{ij}^s \otimes \ell_{a,ij}$. The represented mixed comparison, after the given comparison cell is used to identify its targets, is

$$u_{ij}^t J_{a,j} = J_{a,i} v_{a,ij}. \quad (38)$$

In the native correspondence category this is a typed invertible cell between the two composites. Displaying it in a chart neither inverts J_a nor removes the higher comparisons.

Proposition 5.1 (Overlap covariance of supported transport and the Wilson residual). *Under (38), define*

$$\Phi_{a,i}(x) = J_{a,i}(x \otimes 1) J_{a,i}^*, \quad p_{a,i} = J_{a,i} J_{a,i}^*.$$

Then $\Phi_{a,i}$ is a normal $*$ -homomorphism into the supported endomorphism corner, and

$$\alpha_{ij}^t \Phi_{a,j} = \Phi_{a,i} \alpha_{ij}^s, \quad \alpha_{ij}^t(p_{a,j}) = p_{a,i}. \quad (39)$$

If source and target Wilson actions satisfy (37), their rectangular residual

$$K_{a,i} = W_{t,i} J_{a,i} - J_{a,i} (W_{s,i} \otimes 1) \quad (40)$$

obeys

$$u_{ij}^t K_{a,j} = K_{a,i} v_{a,ij}. \quad (41)$$

Proof. Multiplicativity follows from $J_{a,i}^* J_{a,i} = 1$, and adjoints are preserved. Normality follows from normality of amplification by a finite-dimensional line and ultraweak continuity of multiplication by fixed adjointable operators. On the overlap,

$$\begin{aligned} u_{ij}^t \Phi_{a,j}(x) (u_{ij}^t)^* &= J_{a,i} v_{a,ij} (x \otimes 1) v_{a,ij}^* J_{a,i}^* \\ &= J_{a,i} (u_{ij}^s x (u_{ij}^s)^* \otimes 1) J_{a,i}^*. \end{aligned}$$

Setting $x = 1$ gives the projection identity. For the residual, use Wilson covariance and the mixed comparison in each term:

$$\begin{aligned} u_{ij}^t K_{a,j} &= W_{t,i} J_{a,i} v_{a,ij} - J_{a,i} v_{a,ij} (W_{s,i} \otimes 1) \\ &= [W_{t,i} J_{a,i} - J_{a,i} (W_{s,i} \otimes 1)] v_{a,ij}. \end{aligned}$$

□

Equation (41) makes the actual Wilson residual a compatible morphism of the descended module diagrams whenever their morphism descent is effective. Its vanishing can then be checked locally, by faithfulness of descent. This is a statement about the same Wilson action and the same bordism, rather than separate knowledge of their existence.

The supported transport must remain distinct from a full receiving Wilson operator. If both $K_{a,i} = 0$ and the analogous residual for W_s^*, W_t^* vanish, then $p_{a,i}$ reduces $W_{t,i}$ and

$$\Phi_{a,i}(W_{s,i}) = p_{a,i} W_{t,i} p_{a,i} = W_{t,i} p_{a,i}. \quad (42)$$

Indeed the first intertwining equation gives $W_t p = J(W_s \otimes 1) J^*$, and the adjoint equation gives $p W_t$ equal to the same expression. Its action on $(1 - p) E_t$ is not determined. In particular, replacing every receiving Wilson action by $J(W_s \otimes 1) J^*$ is an additional choice. Range membership in the source-supplied physical observable subalgebra must also be verified; membership in the full endomorphism algebra alone does not establish it.

5.4 Ward identities across the physical spacetime cover

The physical spacetime overlap comparison uses the Ward identities. Coefficient rephasing and an ordinary algebra equalizer do not supply this physical step. The object crossing the cover includes the represented Wilson insertion core, its prepared Gauss–Ward graph domain, the collar-parameter quotient, and their native transports. Original Paper III [4], Section VI.C, equations (VI.18)–(VI.24), supplies the relevant charge and Ward laws. The spacetime-assembly draft [5], Sections 3.2–3.4, 7.4–7.6 and H.5–H.7, expands the splitting comparison on the full prepared domain.

Fix an admitted overlap U_{ij} , retaining gauge-chart indices separately from the spacetime indices i, j . Use the source’s actual overlap equivalence to transport the two collar presentations to the same graph domain and parameter quotient. A Wilson insertion uses its native path, port, and seam identifications to enter the common allowed insertion core. This specifies the comparison on which the Ward identity is evaluated.

Physical overlap contract. The following calculation assumes: (i) the actual overlap transports identify a common prepared graph domain and collar-parameter quotient; (ii) the insertion and its adjoint act on the domains needed by both charge pairings; (iii) transported splittings differ by

the stated interior parameter; (iv) their charge forms and insertion derivations obey (44), including any scalar zero point; and (v) the full interior Ward identity holds on these tests. The abstract calculation proves agreement of the boundary pairings under these assumptions. Identification of the operator insertion and effectiveness of module descent remain the separately specified source data. No global self-adjoint boundary charge is assumed where only forms have been supplied.

For an interior parameter η , the full Ward functional is

$$w^{\text{int}}(\eta; A)[\Psi, \Phi] = ig^{\text{int}}(\eta)[\pi(A)\Psi, \Phi] - ig^{\text{int}}(\eta)[\Psi, \pi(A^*)\Phi] - \langle \pi(\delta_\eta^{\text{int}} A)\Psi, \Phi \rangle. \quad (43)$$

The source's exact identity is $w^{\text{int}} = 0$ on its prepared graph and insertion domains. Those domains and their transport are part of the physical datum. The reduced relation involving only the last term applies on the appropriate Gauss-null face; it does not replace the three-term identity on general prepared tests.

Let s_i, s_j be the two transported splittings of the same collar quotient. Their difference is interior: $\eta_{ij}(\lambda) = s_i(\lambda) - s_j(\lambda)$. The native splitting and zero-point laws, on the common overlap domain, read

$$q_i^\partial(\lambda) - q_j^\partial(\lambda) = g^{\text{int}}(\eta_{ij}(\lambda)) + c_{ij}(\lambda)\langle \cdot, \cdot \rangle, \quad \delta_{i,\lambda}^\partial A - \delta_{j,\lambda}^\partial A = \delta_{\eta_{ij}(\lambda)}^{\text{int}} A. \quad (44)$$

Here q^∂ denotes a charge form, not an asserted global self-adjoint operator. The boundary Ward functional b_i^∂ is the three-term expression (43) with the boundary charge form and boundary insertion derivation.

Proposition 5.2 (Physical Ward overlap cancellation). *Under the native domain, insertion, and splitting transports above,*

$$b_i^\partial(\lambda; A) - b_j^\partial(\lambda; A) = w^{\text{int}}(\eta_{ij}(\lambda); A). \quad (45)$$

Consequently the boundary Ward pairing agrees across the physical overlap on prepared tests. This comparison is independent of the allowed scalar charge zero points.

Proof. Substitute (44) into the difference of the two boundary functionals. The two Gauss-form differences and the insertion-derivation difference give exactly (43). The scalar contribution is

$$ic_{ij}(\lambda)(\langle \pi(A)\Psi, \Phi \rangle - \langle \Psi, \pi(A^*)\Phi \rangle) = 0.$$

The prepared interior Ward identity makes the remaining difference zero. The native graph-projective continuity gives the corresponding normal-functional statement on its supplied domain. $\square$

In particular, with opposite orientations on a shared internal collar, the transported terms are $b_i^\partial(\lambda; A)$ and $b_j^\partial(-\lambda; A)$; their sum is zero. This is the internal-seam Ward cancellation used when crossing the spacetime cover. External boundary terms and the source's retained boundary defects remain. On triple overlaps the differences telescope after the prescribed parameter and graph transports. Scalar coefficient comparisons cancel in the adjoint action, and allowed residual interior splitting changes vanish by the same full Ward identity. The higher comparison cells and seam data are retained throughout.

Where the source supplies self-adjoint charge representatives, its affine overlap law makes the same mechanism transparent:

$$Q_i^\partial(\lambda)U_{ij} = U_{ij}(Q_j^\partial(\lambda) \otimes I) + \chi_{ij}(\lambda)U_{ij}.$$

The scalar character drops out of the adjoint commutator. Thus a global Ward action can exist while charge zero points retain a nontrivial comparison class. The form-level calculation above also accounts for interior splitting changes. It neither requires collar neutrality nor sets the current-algebra or boundary-Ward defects to zero.

This supplies the physical Ward compatibility accompanying the Wilson overlap equation (37). The native Wilson quotient and path/port comparison identify the insertion; Section 5.5 evaluates these endpoint contractions explicitly on a source seam. The Ward identities make its boundary implementation independent of the admitted local splitting. The coefficient filling supplies coherent line comparisons. These statements are used together on the physical cover. Ward identities do not identify arbitrary unrelated local operators or construct the region and overlap maps without their native source data.

The normal completion retains the appropriate domain distinction. In the self-adjoint Gauss branch, Original Paper III, equation (VI.14), places the regional Wilson bicommutant in the strong commutant of the interior Gauss system. That spectral commutant is a von Neumann algebra, so gauge invariance extends without taking ultraweak limits of unbounded commutators. Boundary Ward derivations and (45) retain their admitted insertion and graph domains. The source's fixed-window expectation-derivation identity and augmented rooted recovery transport the prepared Ward structure through the supplied scale and regional comparisons [5]; scalar prism identities alone do not establish that graph passage.

5.5 A Wilson insertion across one native seam

The source's open-edge seam gives an explicit Wilson insertion for Section 5.4. Use the graph calculation of [6], *Open Wilson operations and a native seam*. Subdivide an edge into two banks, retaining its exposed ports. With coarse holonomy g and bank holonomies x, y , the comparison is

$$p(x, y) = yx, \quad (If)(x, y) = f(yx), \quad h \cdot (x, y) = (hx, yh^{-1}). \quad (46)$$

The new seam gauge action leaves p invariant. Haar measure and $(x, y) \mapsto (x, g = yx)$ give the unitary

$$J : L^2(G) \longrightarrow L^2(G^2)^{G_{\text{seam}}}.$$

Other edge variables and retained factors are carried along by the supplied identity comparisons. All differentiations below first take place on $C^\infty(G^2)$, an invariant common core for the compact-group seam action.

For $X \in \mathfrak{g}$, define

$$L_x(X)F = \left. \frac{d}{dt} F(e^{tX}x, y) \right|_{t=0}, \quad R_y(X)F = \left. \frac{d}{dt} F(x, ye^{tX}) \right|_{t=0}.$$

The infinitesimal seam action is $D_X = L_x(X) - R_y(X)$. For a finite-dimensional unitary representation ρ , put $F_\rho(x, y) = \rho(y)\rho(x)$, a matrix of bounded smooth insertions. Differentiating the two banks gives

$$L_x(X)F_\rho = \rho(y)\rho_*(X)\rho(x) = R_y(X)F_\rho, \quad D_X F_\rho = 0. \quad (47)$$

The intermediate representation index is contracted: $(F_\rho)^a{}_c = \sum_b \rho(y)^a{}_b \rho(x)^b{}_c$. Thus (47) is cancellation of the two endpoint insertions at the same retained port. Closing the remaining endpoints with the native return word gives the coarse closed Wilson function $w_\gamma(g, \mathbf{u})$. Its cut presentation $w_\gamma^{\text{cut}} = w_\gamma \circ (p, \text{id})$ satisfies

$$\|w_\gamma^{\text{cut}}\|_\infty \leq 1, \quad D_X w_\gamma^{\text{cut}} = 0.$$

The return variables $\mathbf{u}$ are unaffected by the seam action. The open matrix indices and their contraction are retained until the loop is closed.

This also evaluates the full Ward identity. Let $G_X = -iD_X$ be the self-adjoint seam generator, with its smooth core, and put $g_X[\Psi, \Phi] = \langle G_X \Psi, \Phi \rangle$. For each scalar matrix entry F , define $\pi(F) = M_F$ and $\delta_X F = D_X F$. The Leibniz rule gives

$$[G_X, M_F] = -iM_{D_X F}.$$

Consequently, on the common core,

$$\begin{aligned} ig_X[M_F \Psi, \Phi] - ig_X[\Psi, M_F \Phi] - \langle M_{D_X F} \Psi, \Phi \rangle \\ = \langle (i[G_X, M_F] - M_{D_X F}) \Psi, \Phi \rangle = 0. \end{aligned} \quad (48)$$

This is (43) evaluated on the source seam. For the contracted Wilson insertion, (47) makes the insertion derivative vanish as well. The equality is proved before restricting to seam-invariant vectors.

The finite group action gives the normal completion directly. Let P_{seam} be Haar averaging over the seam group. Since w_γ^{cut} is invariant, $M_{w_\gamma^{\text{cut}}}$ commutes with every seam unitary and with P_{seam} . Thus it preserves the full seam-invariant carrier, and

$$M_{w_\gamma^{\text{cut}}} J = J M_{w_\gamma}, \quad J^* M_{w_\gamma^{\text{cut}}} J = M_{w_\gamma}. \quad (49)$$

Here J acts as the identity on the unchanged return variables. The adjoint equation holds too. This is the zero Wilson residual of Section 6, now with its actual seam Ward calculation and port contraction displayed.

Two successive cuts give an explicit comparison for three banks. On G^3 , the seam derivations are

$$D_{1,X} = L_{x_1}(X) - R_{x_2}(X), \quad D_{2,Y} = L_{x_2}(Y) - R_{x_3}(Y).$$

Left and right multiplication on x_2 commute, hence $[D_{1,X}, D_{2,Y}] = 0$. Both derivations annihilate $\rho(x_3)\rho(x_2)\rho(x_1)$. Their compact-group actions commute, so the two Haar projections satisfy

$$P_1 P_2 = P_2 P_1 = P_{G_{\text{seam},1} \times G_{\text{seam},2}}. \quad (50)$$

Fubini's theorem proves the projection identity; associativity of the matrix contraction identifies the two orders of closing the internal ports. The two cuts therefore give the same contracted Wilson action and the same full seam-invariant carrier. This evaluates the elementary comparison used in a physical cover admitting these cut and subdivision presentations.

The passage to a source-selected physical carrier retains its complete selection, insertion-domain, and graph-intertwining identities. In a charged presentation, the source's coefficient intertwiner replaces the elementary index contraction and must obey its stated Ward transport law. Section 5.4 supplies the native collar-splitting comparison on those prepared domains. Scalar coefficient rephasing preserves the ordinary adjoint overlap action, while the actual line comparisons and their higher coherence remain part of the represented descent. The evaluated neutral seam supplies a local physical crossing without imposing vanishing of an external charge or of the coefficient-band class.

5.6 A normal algebra on the admitted nerve

At setting s , suppose the actual ordinary observable diagram supplies von Neumann algebras $\mathcal{M}_{s,i}$, overlap algebras $\mathcal{M}_{s,ij}$, and normal unital $*$ -homomorphisms

$$r_{i,ij}^s : \mathcal{M}_{s,i} \rightarrow \mathcal{M}_{s,ij}, \quad r_{j,ij}^s : \mathcal{M}_{s,j} \rightarrow \mathcal{M}_{s,ij}.$$

Chart identifications such as α_{ij} are included in these maps so both sides have the same codomain. Higher faces and units are those of the source's coherent observable diagram. These maps are supplied data; arbitrary AQFT inclusions do not provide canonical restriction homomorphisms in the opposite direction.

Theorem 5.3 (Normal Čech algebra and supported scale maps). *The compatible-family algebra*

$$\mathcal{D}_s(\mathcal{U}) = \left\{ (x_i) \in \prod_{i \in I} \mathcal{M}_{s,i} : r_{i,ij}^s(x_i) = r_{j,ij}^s(x_j) \text{ on every overlap} \right\} \quad (51)$$

is a von Neumann algebra. Its coordinate maps are normal and jointly faithful.

Suppose an arrow $a : s \rightarrow t$ has normal $*$ -homomorphisms $\Phi_{a,i}$ and $\Phi_{a,ij}$ on its vertex and overlap algebras, with

$$r_{i,ij}^t \Phi_{a,i} = \Phi_{a,ij} r_{i,ij}^s, \quad r_{j,ij}^t \Phi_{a,j} = \Phi_{a,ij} r_{j,ij}^s. \quad (52)$$

They induce a normal $*$ -homomorphism

$$\Phi_a : \mathcal{D}_s(\mathcal{U}) \longrightarrow \mathcal{D}_t(\mathcal{U}), \quad (x_i) \longmapsto (\Phi_{a,i}(x_i)).$$

It is unital into $p_a \mathcal{D}_t(\mathcal{U}) p_a$, where $p_a = (\Phi_{a,i}(1))_i$. No surjectivity of the physical passage is required.

Proof. The finite product is a von Neumann algebra. Each overlap equation is the kernel condition for an ultraweakly continuous linear difference map. Their intersection is ultraweakly closed. Since both face maps are unital $*$ -homomorphisms, the intersection contains the unit and is closed under multiplication and adjoints. Coordinate evaluations have the asserted normality and separate elements.

For a compatible family, (52) gives

$$r_{i,ij}^t \Phi_{a,i}(x_i) = \Phi_{a,ij} r_{i,ij}^s(x_i) = \Phi_{a,ij} r_{j,ij}^s(x_j) = r_{j,ij}^t \Phi_{a,j}(x_j).$$

Thus the componentwise map preserves compatibility. It is normal and multiplicative because its finitely many components are. The image of the unit is a compatible projection p_a , and every image element is supported on that projection. $\square$

For a specified bounded compatible Wilson family $W^\lambda = (W_i^\lambda)_i$, set

$$\mathcal{M}_s^W(\mathcal{U}) = \overline{\text{alg}^* \{1, W^\lambda : \lambda \text{ admitted}\}}^{\text{uw}} \subseteq \mathcal{D}_s(\mathcal{U}). \quad (53)$$

This is a W^* -algebra in the specified normal realization. The generating family must first satisfy the overlap equations. Equality with the whole physical algebra is a separate generation statement. In particular, simultaneous equal-time Wilson character multipliers commute, so their generated algebra is abelian even for a non-Abelian gauge group. A larger physical algebra may retain electric or dynamical actions.

Proposition 5.4 (Restriction to the specified Wilson algebra). *Let $\Phi_a : \mathcal{D}_s \rightarrow \mathcal{D}_t$ be the normal *-homomorphism above, and put $p_a = \Phi_a(1)$. For the unital Wilson-generated algebras of (53), the inclusion*

$$\Phi_a(\mathcal{M}_s^W) \subseteq \mathcal{M}_t^W$$

holds if and only if

$$p_a \in \mathcal{M}_t^W, \quad \Phi_a(W_s^\lambda) \in \mathcal{M}_t^W \quad \text{for every admitted source generator } \lambda.$$

*When these conditions hold, the restriction is a normal *-homomorphism unital into $p_a \mathcal{M}_t^W p_a$. It need not be unital into all of $\mathcal{M}_t^W$.*

Proof. Necessity follows by applying the inclusion to the unit and each source generator. Conversely, the inverse image $\Phi_a^{-1}(\mathcal{M}_t^W)$ is an ultraweakly closed *-subalgebra of $\mathcal{D}_s$, by normality. The hypotheses put the source unit and all source generators in it. It therefore contains their generated von Neumann algebra. Normality and the supported-unit assertion follow by restriction. $\square$

For an isometric implementation, the intertwining equations in (42) give $\Phi_a(W_s^\lambda) = W_t^\lambda p_a$ when the receiving Wilson generator has been identified. Thus $p_a \in \mathcal{M}_t^W$ is sufficient in that case and is necessary for the full unital source algebra to have image in $\mathcal{M}_t^W$. Without it, the supported image still belongs to $\mathcal{D}_t$ but need not belong to the chosen Wilson subalgebra. The full word-pullback homomorphism in Proposition 6.1 has a different target action from supported conjugation; its generator transport is verified directly.

Remark 5.5 (Overlap covariance does not ensure Wilson range membership). On a one-patch cover take $\mathcal{D}_s = \mathbb{C}$, $\mathcal{D}_t = M_2(\mathbb{C})$ and

$$p = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Phi(\lambda) = \lambda p.$$

All overlap conditions are vacuous and Φ is a normal *-homomorphism unital into its supported corner. But for $\mathcal{M}_s^W = \mathbb{C}$ and $\mathcal{M}_t^W = W^*(Z)$ the image of the unit is not in the target Wilson algebra: $[p, Z] \neq 0$. This excludes the stronger conclusion without the range hypothesis and does not contradict exact native word transport.

For an infinite set-indexed cover, use the bounded product $\prod_i^\infty \mathcal{M}_{s,i}$; the same kernel proof applies. The present physical application uses the selected finite covers. Normal completely positive comparisons alone need not preserve products and do not satisfy this theorem's algebraic hypothesis. General normal correspondences are treated through the module descent statement, rather than silently being replaced by homomorphisms.

5.7 What the native source already supplies

There are concrete source results behind the overlap diagram. On the selected stationary pre-precision branch, Original Paper III [4], Theorem II.112, reconstructs the decorated fixed-scale object on carrier-saturated covers. Each persistent label occurs whole in a cover member; overlap comparisons identify repeated appearances, and ordered tensor product puts one copy of each label in the assembled packet. Definition II.113 and Theorem II.114 assemble the actual normal correspondence-valued scale section and its compositors. Their proof retains the noninvertible scale arrows and the finite analytic correction cells.

For the separately selected physical nonzero-flux branch, Lemmas II.255–II.257 identify the local physical Wilson quotient and regional W^* -factor, preserve the simplicial and mixed comparisons, and prove trivial ordinary observable isotropy. Theorem II.259 gives the nervewise realization equivalence. Theorem II.260 then defines the compatible-family algebra in its equation (II.337) and proves its presentation independence on that chosen nerve. Remark II.261 explicitly retains nonneutral represented coefficient descent. These statements appear on printed pp. 178–183 of the inspected source.

Thus the selected fixed-nerve normal realization is an imported construction, not an object that this note must invent by taking an arbitrary bicommutant. Theorem 5.3 exposes the elementary closure step when the actual ordinary overlap maps have its stated normal $*$ -homomorphism structure. Proposition 5.1 explains the supported-isometry calculation; it is applied to a native arrow only after the source’s mixed cell and observable action have been identified with its typed inputs. Neither the stationary and physical branches nor the gauge and spacetime nerves are identified by this account.

The inspected fixed-demand results supply neither a common receiving realization for every scale lineage nor an estimate uniform over thinning demands. Section 8 states the separate completion hypotheses. Optional operator repairs are confined to Appendix E.

5.8 Normal limits preserve verified overlap compatibility

Lemma 5.6 (Bounded normal completion respects the admitted nerve). *For the finite cover above, let $x^{(n)} \in \mathcal{D}_s(\mathcal{U})$ be uniformly bounded. Suppose each component converges in σ -strong * to $x_i \in \mathcal{M}_{s,i}$. Then $x = (x_i)_i$ belongs to $\mathcal{D}_s(\mathcal{U})$, and $x^{(n)} \rightarrow x$ in its σ -strong * topology.*

Proof. A normal $*$ -homomorphism is σ -strong * -continuous: the defining normal positive functionals pull back to normal positive functionals. Passing to the limit in each overlap equation gives compatibility of x . For a finite product, componentwise σ -strong * convergence is product σ -strong * convergence. On a bounded set its restriction to a von Neumann subalgebra is the subalgebra’s σ -strong * topology, as seen in a faithful normal representation. This gives the conclusion. $\square$

The lemma preserves descent after the analytic estimates have supplied the limits. It does not produce a common physical realization across scales or prove those estimates. When the hypotheses of the later completion theorem are verified within the assembled normal algebra, they concern its entire normal folium. A nonneutral coefficient object may still require a compatible system of local modules rather than one global untwisted module.

5.9 Dependence on choices

Fix the physical tube, representation, normalization, thickness, and sector. A change of these data changes the physical question. For a change of presentation, specified unitary carrier maps $v_i : E_i \rightarrow E'_i$ identify the same ordinary observable exactly when

$$W'_i v_i = v_i W_i, \quad W_i^{I*} v_i = v_i W_i^*. \quad (54)$$

The second equation follows from the first for a unitary, but must be retained separately for a one-sided isometry. In the latter case they identify only the reducing supported action; the complement remains unspecified.

If $u'_{ij} = z_{ij}v_i u_{ij} v_j^*$, scalar z_{ij} cancels in conjugation. The multiplier changes by $\zeta' = (\delta z)\zeta$, while the transported ordinary overlap equations agree. This does not trivialize the coefficient band or its algebra bundle.

Cover changes. For admitted covers $\mathcal{U}, \mathcal{V}$ with common refinement $\mathcal{R}$, suppose the physical restrictions induce normal unital $*$ -homomorphisms

$$r_{\mathcal{U}} : \mathcal{D}(\mathcal{U}) \rightarrow \mathcal{D}(\mathcal{R}), \quad r_{\mathcal{V}} : \mathcal{D}(\mathcal{V}) \rightarrow \mathcal{D}(\mathcal{R}). \quad (55)$$

Proposition 5.7 (Cover comparison). *Faithfulness makes an agreeing pair of images determine either Wilson family uniquely. If both maps are also onto, $r_{\mathcal{V}}^{-1}r_{\mathcal{U}}$ is a normal $*$ -isomorphism between the compatible-family algebras. With faithful compatible further-refinement maps and admitted common further refinements, this identification is independent of the common refinement.*

Proof. Injectivity gives uniqueness. Surjectivity gives the normal inverse isomorphism. For two common refinements, their commuting comparison squares give equal images in a common further refinement; faithfulness reflects that equality. $\square$

Thus equalizer closure alone is not effective descent. In the physical application, the actual insertion comparisons and full Ward identity supply the overlap compatibility, while effective descent retains its source hypotheses.

Prisms and retained holonomy. For fixed endpoint cocycles, two admitted fillings obey

$$\delta\beta = \delta\beta' = \theta_1 - \theta_0, \quad \eta = \beta' - \beta = \theta(H' - H) + (a' - a), \quad \delta\eta = 0.$$

They induce identical adjoint transition maps when the local Wilson actions are held fixed. Their coefficient transport can nevertheless differ by

$$\text{hol}_{\eta}(c) = \exp(2\pi i \eta(c)) \quad (56)$$

on a closed comparison path c . An exact $\eta = \delta\kappa$ is removed by a vertex rephasing; a nonzero H^1 class is retained. These are cohomology groups of the specified comparison complex. A physical flat-bundle interpretation requires its actual realization.

If the lateral operator and seam cochain are fixed, $Z = (H' - H)c$ is a two-cycle, and

$$\text{hol}_{\eta}(c) = \exp(2\pi i \theta(Z)). \quad (57)$$

It is one for an admitted boundary $Z = \partial B$. Changed lateral data must instead be included in η . The differential gerbe model of Appendix E.2 retains curvature: its bounding-surface holonomy can be $\exp(2\pi i \int_B H_G) \neq 1$. The flat simplicial and differential statements therefore have different hypotheses.

Operationally, the intertwining equations test equality of the Wilson action; closed-path and closed-surface evaluations test retained coefficient information. A phase multiplying a comparison cancels in its adjoint action. A phase multiplying a nonzero Wilson operator changes that operator unless the phase is one. These finite identifications persist in a common normal limit when the represented differences tend to zero in the topology of Section 8.

6 Finite Wilson algebras and refinement checks

The following calculations distinguish inherited Wilson transport, rebuilt finite compressions, and electric enlargement. They are evaluated on their specified carriers. Appendix D retains the complete separated-shell example; Appendix C gives the selected-arrow and first-cutoff certificates.

6.1 An exact Wilson comparison on the native frustum

The same-support cap-8 to cap-16 root-star refinement is recorded in [7], Section 14.61.1, with provenance to its geometric source's Development Notes V–AJ. It uses actual $PSU(3)$ edge variables, the scalar Gauss/copy selection, and unchanged spectator multiplicities. Its full coarse-carrier map, positive vacuum overlap, and finite polar vacuum comparison are supplied source data. We evaluate an inherited closed Wilson word on that map. This is the named scalar branch; it does not identify the charged linked-flux diagram with it.

For an oriented coarse edge $e = (v, w)$, let z_e be its native two-child word and k_v its collar holonomy at a promoted endpoint, or the identity at an unchanged endpoint. The source's geometric map is

$$p(g_f)_e = k_v^{-1} z_e k_w, \quad (J\psi)(g_f) = \psi(p(g_f)). \quad (58)$$

The child and collar coordinates have the disjointness specified by that finite construction. Its Haar and selection identities are

$$p_* \mu_f = \mu_c, \quad J^* J = I, \quad P_f^{\text{sel}} J P_c^{\text{sel}} = J P_c^{\text{sel}}.$$

Let J^{sel} denote the restriction to the complete selected coarse carrier. The inert spectators are retained, and both J and the Wilson operators below act as identities on them. The ordinary scalar comparison uses the trivial coefficient line; an already specified field-independent history line can be tensored with these identities without changing the residual.

The rebuilt target word. Fix a closed coarse path $\gamma = (e_1, \dots, e_N)$ in the retained positive-thickness window, with vertices $v_0, \dots, v_N = v_0$. Use the source's path-order convention; reversed edges use the inverse word and exchanged endpoints. Define the coarse observable and its geometrically rebuilt fine version by

$$w_c = \frac{1}{8} \chi_{\text{ad}}(g_{e_1} \cdots g_{e_N}), \quad w_f = \frac{1}{8} \chi_{\text{ad}}(z_{e_1} \cdots z_{e_N}).$$

The adjoint character is well defined on $PSU(3)$, so no fundamental lift is chosen. The fine word is specified by the child paths, before comparison with w_c . Direct cancellation gives

$$\prod_{a=1}^N p(g_f)_{e_a} = k_{v_0}^{-1} \left(\prod_{a=1}^N z_{e_a} \right) k_{v_0}, \quad w_c \circ p = w_f.$$

Every internal collar cancels. The remaining basepoint collar contributes a conjugation, removed by the character. Both functions are real contractions and commute with the native Gauss/copy selection.

Proposition 6.1 (Selected native Wilson transport). *On the complete selected carriers of this finite source, let $W_c = M_{w_c}$ and $W_f = M_{w_f}$. Then*

$$W_f J^{\text{sel}} = J^{\text{sel}} W_c, \quad W_f^* J^{\text{sel}} = J^{\text{sel}} W_c^*, \quad K_\gamma = 0. \quad (59)$$

The range projection $P_J = J^{\text{sel}}(J^{\text{sel}})^*$ reduces W_f , and

$$(J^{\text{sel}})^* W_f J^{\text{sel}} = W_c, \quad J^{\text{sel}} W_c (J^{\text{sel}})^* = W_f P_J.$$

Pullback is a faithful normal unital $*$ -homomorphism on the inherited gauge-invariant multiplication algebra, with image the functions of $p(g_f)$. In particular, it identifies the coarse Wilson-generated algebra with its inherited fine image.

Proof. Pointwise multiplication gives $M_{w_f} J \psi = (w_c \psi) \circ p = J M_{w_c} \psi$. Apply the same identity to the conjugate word, then use the selection square. The two intertwining equations imply reduction and the corner identities.

Haar pushforward proves that pullback on L^∞ preserves the essential norm and null sets; it preserves products and monotone bounded limits. These facts give faithfulness and normality. On the Gauss-selected carrier the invariant multiplication representation is still faithful, since the constant bulk function belongs to that carrier and is cyclic for testing nonzero invariant multipliers. A nonzero matching/spectator vector supplies the remaining factors. $\square$

This distinguishes two target operators explicitly. The rebuilt full multiplier is W_f , whereas supported conjugation gives $W_f P_J$. They agree on the receiving range; supported transport does not prescribe the complementary action. Newly created fine loops are not in this inherited algebra merely because they lie in the same window.

Seam and Ward compatibility. At an internal vertex, the common endpoint frame appears on adjacent words with opposite orientations. On the open representation matrices, the infinitesimal terms are $(-T\rho_*(X))S + T(\rho_*(X)S) = 0$. The same cancellation applies to the collar factors in (58); at the closed basepoint it is a commutator with zero trace. Thus this is the same contracted insertion on both sides, with its endpoint data retained before closure.

For the native endpoint frame changes, the word map is equivariant: $p \circ \tau_f = \tau_c \circ p$. The corresponding Haar unitaries therefore satisfy $U_f J = J U_c$. Together with the selection identity, this is the mixed square for this comparison. On the common smooth insertion core the seam generators satisfy the full three-term Ward identity of (48), not merely its ground restriction. For physical collar splittings transported to that prepared overlap, Proposition 5.2 cancels their interior difference. Hence the zero residual above remains zero on these admitted word/frame and cut-seam presentations. This calculation does not construct additional region maps or higher cells for arbitrary covers.

The native vacuum comparison is separate and already supplied. The source records positive normalized scalar grounds ϕ_c, ϕ_f with

$$\|J\phi_c - \phi_f\| < \delta_*, \quad \delta_* = 0.003089852, \quad \langle \phi_f, J\phi_c \rangle > 0.999995226.$$

These numerical certificates are imported from the named finite construction. The complete ground compression has polar map

$$V_{fc} = |\phi_f\rangle\langle\phi_c| \otimes I_{\text{spectator}}.$$

The carrier map J and ground polar map V_{f_c} have different domains and roles. The exact Wilson comparison does not require $J\phi_c = \phi_f$.

For every unit spectator vector, exact intertwining and the elementary vector-state estimate give

$$|\langle W_f \phi_f, \phi_f \rangle - \langle W_c \phi_c, \phi_c \rangle| < 2\delta_* = 0.006179704. \quad (60)$$

Indeed, compare ϕ_f with $J\phi_c$ in the fine carrier and use $|\langle A\xi, \xi \rangle - \langle A\eta, \eta \rangle| \leq 2\|A\|\|\xi - \eta\|$ for unit vectors. The same bound holds for every contraction in the inherited multiplication algebra, including a finite product of normalized Wilson functions. It is uniform over the retained spectator multiplicity.

There is no implied Hamiltonian intertwining. In the source's normalized units the complementary magnetic block is explicitly

$$\widehat{K}_H^* \widehat{K}_H = \frac{1}{64} \sum_F (h\beta_F)^2 I, \quad \widehat{K}_H = (1 - P_f) \widehat{H}_f J, \quad h = 1/32.$$

It can be nonzero while $K_\gamma = 0$. Thus this finite native comparison supplies exact inherited Wilson action, selected-carrier survival, and a certified vacuum-state error through the source's positive-overlap/polar interface. It supplies neither an unbounded refinement-family estimate nor an identification of all fine observables with inherited ones. In particular, simultaneous exact ground preservation and full multiplication transport is not added as a condition of that interface.

6.2 An exact SU(3) cutoff-boundary residual

The retained carrier in Original Paper III [4], printed p. 25, is

$$K_L = \text{span}\{\chi_{p,q} : p, q \geq 0, p + q \leq L\} \subset H_{\text{cl}} := L^2(SU(3), dg)^{\text{Ad}}.$$

Let Π_L be its orthogonal projection. Irreducible characters are orthonormal for normalized Haar measure. Put $w = \chi_{1,0}/3$, $W = M_w$, and $W_L = \Pi_L W|_{K_L}$, so $\|W_L\| \leq 1$. First evaluate the *raw character inclusion* $I_{L'/L} : K_L \hookrightarrow K_{L'}$, for $L' \geq L + 1$.

The one-box Pieri rule [8], applied to the partition $(p + q, q, 0)$, gives

$$\chi_{1,0} \chi_{p,q} = \chi_{p+1,q} + \chi_{p-1,q+1} + \chi_{p,q-1}, \quad (61)$$

with negative indices omitted. Indeed, the three permitted added boxes yield partitions $(p + q + 1, q, 0)$, $(p + q, q + 1, 0)$, and $(p + q, q, 1)$; subtracting a full determinant column in the last case gives $(p + q - 1, q - 1, 0)$. Their Dynkin labels are precisely those displayed. The second term exists only for $p \geq 1$, and the third only for $q \geq 1$.

Proposition 6.2 (Computed raw cutoff residual). *For $L' \geq L + 1$, the raw rectangular residual is*

$$K_{L'/L}^I := W_{L'} I_{L'/L} - I_{L'/L} W_L = \frac{1}{3} \sum_{p+q=L} |\chi_{p+1,q}\rangle \langle \chi_{p,q}|. \quad (62)$$

Writing $P_{\partial L}$ for the projection onto the characters with $p + q = L$, one has

$$(K_{L'/L}^I)^* K_{L'/L}^I = \frac{1}{9} P_{\partial L}, \quad \|K_{L'/L}^I\| = \frac{1}{3}. \quad (63)$$

The residual for the adjoint Wilson operator has the same norm, with output $\chi_{p,q+1}$ in place of $\chi_{p+1,q}$.

Proof. The residual is $(\Pi_{L'} - \Pi_L)W|_{K_L}$. In (61), only the first term raises $p + q$, and it crosses the boundary exactly when $p + q = L$. The output characters are distinct and orthonormal, giving both identities in (63). Complex conjugation exchanges (p, q) with (q, p) , proving the adjoint statement. $\square$

This nonvanishing operator norm does not obstruct strong observable existence in the raw Haar chart. Extend W_L to H_{cl} by $X_L = \Pi_L W \Pi_L$. Since $\Pi_L \rightarrow I$ strongly and $\|X_L\| \leq 1$,

$$X_L \longrightarrow W \quad \text{strongly-}^* \text{ on } H_{cl}. \quad (64)$$

For example, $\|(X_L - W)\psi\| \leq \|W\| \|(\Pi_L - I)\psi\| + \|(\Pi_L - I)W\psi\|$, and the same estimate applies to the adjoints. On every fixed finite character span the boundary residual is eventually zero. Thus an operator-norm summability condition on (62) would be unnecessarily strong for this realization.

There is an explicit W^* -algebra here. Let $Y = SU(3)/\text{Ad}$ and let μ be Haar measure pushed to Y . The fundamental trace determines a conjugacy class: the eigenvalues are the roots of

$$z^3 - \chi_{1,0}(g)z^2 + \overline{\chi_{1,0}(g)}z - 1.$$

Consequently w , its conjugate, and constants separate points of the compact space Y . Stone–Weierstrass and multiplication on $L^2(Y, \mu)$ give

$$W^*(M_w) = \{M_f : f \in L^\infty(Y, \mu)\}. \quad (65)$$

This is an abelian Wilson multiplication algebra in the Haar realization. The finite compressed matrices need not retain all multiplication relations, and adjoining electric operators gives a different observable algebra. Equation (65) makes no assertion about the factor type of a reconstructed interacting regional algebra.

6.3 The Wilson algebra and the electric extension on one cell

The triangular $SU(3)$ cutoff permits a more precise answer to the W^* -algebra question. Write $N_L = 3W_L$, $P = \Pi_L$, $Q = I - P$, $M = M_{\chi_{1,0}}$, and $d_L = (L + 1)(L + 2)/2$. The two outgoing boundary maps are

$$B_+ = QMP|_{K_L} = \sum_{p+q=L} |\chi_{p+1,q}\rangle \langle \chi_{p,q}|, \quad B_- = QM^*P|_{K_L} = \sum_{p+q=L} |\chi_{p,q+1}\rangle \langle \chi_{p,q}|. \quad (66)$$

Both have initial projection $P_{\partial L}$. Although a general compression of a normal operator need not be normal, these two boundary defects cancel exactly.

Proposition 6.3 (The finite Wilson algebra). *For every $L \geq 0$, the operator W_L is normal and has d_L distinct eigenvalues. The vector $\chi_{0,0}$ is cyclic for its unital * -algebra, and*

$$W^*(W_L) \cong \mathbb{C}^{d_L} \quad \text{is a maximal abelian subalgebra of } B(K_L). \quad (67)$$

Proof. Since $MM^* = M^*M$, expansion through $P + Q = I$ gives

$$N_L N_L^* - N_L^* N_L = -B_-^* B_- + B_+^* B_+ = -P_{\partial L} + P_{\partial L} = 0.$$

To prove cyclicity, induct on $p + q$. If $p \geq 1$ and $p + q \leq L$, the Pieri rule gives the vector identity

$$\chi_{p,q} = N_L \chi_{p-1,q} - \chi_{p-2,q+1} - \chi_{p-1,q-1}, \quad (68)$$

with negative indices omitted. The subtracted terms have lower total degree, and the multiplication produces no term outside K_L . For $p = 0, q \geq 1$, the conjugate rule gives $\chi_{0,q} = N_L^* \chi_{0,q-1} - \chi_{1,q-2}$. Starting with $\chi_{0,0}$, these identities produce every character in K_L . A cyclic vector for a finite-dimensional normal operator spans at most one direction in each eigenspace. Cyclicity therefore forces every eigenspace to be one-dimensional. The spectral theorem now gives (67). $\square$

In particular, the cutoffs at levels four and five have $W^*(W_4) \cong \mathbb{C}^{15}$ and $W^*(W_5) \cong \mathbb{C}^{21}$. This is an exact property of the rebuilt raw Wilson matrices, independent of the package-adapted physical arrows. It does not assert that compression preserves arbitrary products of classical Wilson functions.

A specified electric operator changes the generated algebra. Use the standard quadratic-Casimir normalization [18], equation (2.16),

$$D_L \chi_{p,q} = \frac{p^2 + q^2 + pq + 3p + 3q}{3} \chi_{p,q}. \quad (69)$$

Its zero eigenspace is exactly $\mathbb{C}\chi_{0,0}$. This is the electric Casimir on the declared raw character carrier; no identification with a transported regional Hamiltonian is being made.

Proposition 6.4 (The one-cell electric extension). *For the operators just specified,*

$$W^*(W_L, D_L) = B(K_L) \cong M_{d_L}(\mathbb{C}). \quad (70)$$

On the full Haar class-function carrier, let D be the positive diagonal self-adjoint operator with the eigenvalues in (69), with its spectral domain, and let $R = (1 + D)^{-1}$. Then

$$W^*(W, R) = B(H_{cl}). \quad (71)$$

Proof. The rank-one projection $P_0 = |\chi_{0,0}\rangle\langle\chi_{0,0}|$ belongs to $W^*(D_L)$. For polynomials p, q in W_L, W_L^* , the operators pP_0q^* are the rank-one operators $|p\chi_{0,0}\rangle\langle q\chi_{0,0}|$. Cyclicity from Proposition 6.3 makes their span all of $B(K_L)$.

For the infinite carrier, D has domain $\{\sum a_{p,q} \chi_{p,q} : \sum C_2(p,q)^2 |a_{p,q}|^2 < \infty\}$. The eigenvalue 1 of R has projection P_0 , so $P_0 \in W^*(R)$. The same Pieri recursion, with $3W$ in place of N_L , produces every character from $\chi_{0,0}$; these characters span a dense subspace of H_{cl} . The resulting rank-one operators have strong closure $B(H_{cl})$, proving (71). $\square$

The cutoff algebras also cannot be linked by declaring every rebuilt generator to be the image of its predecessor. Proposition C.1 proves that W_4 and W_5 have disjoint spectra. A unital $*$ -homomorphism sending W_4 to W_5 would force the latter spectrum to lie in the former, which is impossible. The strong- $*$ convergence (64) is therefore a different construction from such a generator-preserving inductive system.

Thus the Wilson multiplication algebra, its finite compressed version, and the algebra obtained by adjoining the electric Casimir are explicitly different algebras on the specified carriers. The infinite electric extension here is a type I algebra in the raw one-cell Haar representation. A reconstructed interacting regional algebra, its selected normal folium, and its factor type still require their actual physical realization. Coefficient rephasing preserves the ordinary adjoint overlap equations for whichever of these observable systems is transported; it does not choose the system or enlarge its generators.

7 Retained boundary modules and marked coefficient transport

The finite scalar comparison and the coefficient examples answer different questions. This section puts their roles together without identifying their source data. The native geometric record [1], Sections 4.1 and 6.5–6.8 and Development Notes X–AW, supplies the retained-carrier and marked-reference statements used below. Its finite certificates are used at their recorded scope; no uniform family is inferred from them.

7.1 The exterior factors are part of the comparison

For the selected regional square, the completed carriers are

$$\mathcal{K}_c = \mathcal{H}_C \hat{\otimes} L^2(G^{5542}), \quad \mathcal{K}_f = \mathcal{H}_D \hat{\otimes} L^2(G^{47324}).$$

Let $\mathcal{G}_c : \mathcal{K}_c \rightarrow \mathcal{H}_A$ and $\mathcal{G}_f : \mathcal{K}_f \rightarrow \mathcal{H}_B$ be the native gluing unitaries. Here $G = PSU(3)$ on this scalar branch. The integers 5542 and 47324 count the residual exterior group coordinates after the native open-root forest trivializations; they are not numbers of charge sectors or proposed continuum dimensions. Their incidence and forest provenance is [1], Development Notes X–Y; the completed corrected comparison is Note AR. The unitaries $\mathcal{G}_c, \mathcal{G}_f$, rather than the equality of abstract separable Hilbert-space dimensions, identify these coordinates with the physical carriers. The exterior factors retain the conjugate charge partners used in diagonal boundary matching. Replacing them by Haar lines is a regional readout with a kernel; it is not an identification of these full carriers. In particular, such a readout cannot preserve the complete excitation form whenever its kernel contains a vector of positive excitation energy.

For the raw native refinement J_U and its established vacuum-corrected isometry $\tilde{J}_U$, write

$$T = \mathcal{G}_f^* J_U \mathcal{G}_c, \quad \tilde{T} = \mathcal{G}_f^* \tilde{J}_U \mathcal{G}_c.$$

These are maps on the retained carriers. The identity $\mathcal{G}_f \tilde{T} = \tilde{J}_U \mathcal{G}_c$ follows from this construction; it does not prove equality with a second independently prescribed regional route.

The finite record gives $\|\tilde{T} - T\| \leq \delta$, with the outward bound $\delta = 0.003089852$. If the matched inherited multiplication actions satisfy $a_f T = T a_c$, then

$$a_f \tilde{T} - \tilde{T} a_c = a_f (\tilde{T} - T) - (\tilde{T} - T) a_c, \quad \|a_f \tilde{T} - \tilde{T} a_c\| \leq 2\delta \|a\|.$$

The full inherited scalar multiplication algebra can change boundary charge. This statement does not realize every separately prescribed charged insertion or intertwiner. Nor does an operator-norm estimate alone control electric derivatives. The native source here is uncut; additional representation cutoffs are optional changes of source, not prerequisites for this calculation.

7.2 What a comparison line repairs

The marked native flat reference extends over the transverse disk and has meridional transport I . The linked fundamental reference has transport $\omega^m I$, with $\omega = e^{2\pi i/3}$. A map a preserving these marked transports would satisfy

$$D_\gamma(a) = \omega^m a - a = (\omega^m - 1)a, \quad \|D_\gamma(a)\| = \sqrt{3}\|a\| \quad (m = 1, 2).$$

Thus a nonzero undecorated marked fiber identification is excluded. This statement concerns the specified flat references; it does not exclude curved connections on a disk.

A flat comparison line can change the domain transport. The precise test is elementary and useful.

Proposition 7.1 (Parallel line-decorated comparison). *Let Y be connected with a universal cover, and let two finite-rank flat unitary bundles have monodromies ρ_c, ρ_f . Let L_χ be a flat unitary line with character χ . A parallel map from the first bundle tensored with L_χ to the second is determined by a fiber map a_0 satisfying*

$$\rho_f(\eta)a_0 = a_0\rho_c(\eta)\chi(\eta) \quad \text{for every } \eta \in \pi_1(Y).$$

It is unitary precisely when a_0 is unitary.

Proof. Use the convention $s(\eta x) = \rho(\eta)s(x)$ for equivariant sections on the universal cover. The constant map $s(x) \mapsto a_0s(x)$ is equivariant exactly under the displayed identity, and hence descends. Conversely, a parallel map is constant in parallel frames upstairs and must obey that identity. Unitarity is a fiberwise condition. $\square$

Choosing $\chi(\gamma) = \omega^m$ repairs the meridional equation. On a transverse annulus this checks its generator. On a full tubular collar, the longitude must also be checked. The construction is conditional on all these identities; a matched meridional phase alone does not establish a physical bundle comparison. Furthermore, a center character used to label triality is not thereby a one-dimensional representation of $SU(3)$.

7.3 An exact seam calculation and its boundary of applicability

The native external comparator has the specified longitude identities

$$CSC^{-1} = \omega S, \quad \ell_h \mapsto \omega^{-1}\ell_h.$$

Its line-valued implementation $B = S \otimes \ell_h$ therefore satisfies

$$B' = (CSC^{-1}) \otimes (\omega^{-1}\ell_h) = B.$$

The coefficient seam defect is zero. If the line is forgotten, the remaining matrix has defect $(\omega - 1)S$, of norm $\sqrt{3}$ for unitary S , although $\text{Ad}_{CSC^{-1}} = \text{Ad}_S$. The adjoint action descends while this implementer requires its line.

The external comparator acts on the full charged ground factor, not on a selected entangled vector alone. In the source's chartwise notation,

$$P_0 = P_0^{\text{scalar}} \otimes I_V \otimes I_{\bar{V}_{\text{ref}}} \otimes P_+.$$

Consequently the displayed seam cancellation survives complete-ground compression. A local product $\tilde{T} \otimes B$ inherits the graph comparison bounds where the graph and coefficient actions actually separate. For graph-only forms a unitary coefficient map adds no energy cost. Spatial covariant derivatives, when part of the selected physical form, require their own comparison calculation.

For independently prescribed seam maps, the remaining test is

$$\mathfrak{S} = S_f A_{\text{before}} - A_{\text{after}} S_c,$$

with the comparison-line transport included in S_c . The calculation above evaluates its specified coefficient component. It does not set an independently prescribed graph or charged fusion component to zero.

7.4 Local cancellation does not trivialize composition

The linked source separately supplies selected graded lines and carry products

$$L_{m,a} \otimes L_{m,b} \longrightarrow K_m^{\epsilon_3(a,b)} \otimes L_{m,[a+b]_3}, \quad \epsilon_3(a,b) = \left\lfloor \frac{a+b}{3} \right\rfloor.$$

Moving its carry past the first triality line gives the source convention

$$\Omega_m(a, b, c) = \omega^{ma\epsilon_3(b,c)}, \quad I_3(\Omega_m) = \prod_{b=0}^2 \Omega_m(1, b, 1) = \omega^m.$$

The carry identity proves the pentagon. For an allowed two-cochain rephasing, the cyclic product of its coboundary is one, so this detector is nontrivial for $m = 1, 2$. This is compatible with exact cancellation of an individual seam. The two statements concern different levels of comparison.

The native longitude line, the line correcting a marked meridian, and the linked carry line have different prescribed roles. Their identification requires actual endpoint, line, and seam maps. In particular the fixed native charged fiber has exposed central character ω ; a center-equivariant map from grade t tensored with a character of grade a to grade s vanishes unless $s = t + a \pmod{3}$. This statement concerns $\mathcal{V}^{\text{fix}}$ alone. The graded package $\mathcal{V}^{\text{gr}}$ already exists in Section 2.2, with its nonzero coisometric insertions. What remains unprovided by that fact is an identification with the linked source's marked graded lines and carry products; it is not the existence of graded endpoints in the composed source.

The paper therefore retains the native finite comparison and the linked marked example as distinct results. The canonical associator of an ambient correspondence category does not supply this source-specific carry action. Likewise an added flat bundle alone does not construct the physical associator. A physical identification must preserve the declared source data and verify the actual overlap and composition maps.

8 Analytic realization throughout a specified normal folium

Let $\mathcal{M}$ be a von Neumann algebra in a fixed faithful normal representation. For $\varphi \in \mathcal{M}_*^+$, set

$$p_\varphi(x) = (\varphi(x^*x) + \varphi(xx^*))^{1/2}. \quad (72)$$

These seminorms generate the σ -strong* topology. In a module realization, $\mathcal{M}$ may be the adjointable endomorphism algebra of a specified self-dual Hilbert module over a von Neumann algebra. The actual Wilson actions must belong to that algebra; the choice of a module alone does not provide them.

Theorem 8.1 (Bounded observable completion from summable represented defects). *Let $x_n \in \mathcal{M}$ be the represented observables along a selected countable cofinal lineage. Assume*

$$\sup_n \|x_n\| \leq K < \infty, \quad (73)$$

and suppose there is a nonnegative summable sequence (e_n) such that, for every $\varphi \in \mathcal{M}_*^+$, a finite constant C_φ satisfies

$$p_\varphi(x_{n+1} - x_n) \leq C_\varphi e_n. \quad (74)$$

Then there is a unique $x \in \mathcal{M}$ with $x_n \rightarrow x$ in σ -strong*, $\|x\| \leq K$, and

$$p_\varphi(x - x_n) \leq C_\varphi \sum_{j=n}^{\infty} e_j. \quad (75)$$

If all x_n belong to a fixed von Neumann subalgebra $\mathcal{M}(O)$, then $x \in \mathcal{M}(O)$. If a specified normal automorphism fixes every x_n , it fixes x . For finitely many families satisfying these hypotheses, every finite $*$ -word in their limits is the σ -strong* limit of the corresponding words, and its expectations converge in every normal state.

Proof. The triangle inequality gives, for $m > n$,

$$p_\varphi(x_m - x_n) \leq C_\varphi \sum_{j=n}^{m-1} e_j.$$

Thus (x_n) is Cauchy in every displayed seminorm. A norm-bounded σ -strong*-Cauchy net in a von Neumann algebra has a σ -strong* limit in the same algebra: in a faithful normal representation its action and its adjoint action converge strongly, the uniform norm bound gives bounded limits, and strong closedness places them in the algebra. This also gives $\|x\| \leq K$. The seminorms separate points, proving uniqueness, and passage to the limit gives (75).

A von Neumann subalgebra is strongly closed. A normal automorphism is σ -strong*-continuous, as follows by pulling the normal positive functionals in (72) back through it. Finally, multiplication is jointly σ -strong*-continuous on bounded sets. Iteration proves convergence of finite words. Bounded σ -strong* convergence implies ultraweak convergence, hence convergence against every normal state. $\square$

The theorem addresses the full normal folium of the specified realization. It is not restricted to a chosen vacuum vector. It does not assert a limit in every disjoint representation or construct the common physical realization across scales. Section 5 constructs a normal compatible-family algebra on a specified nerve and records the native fixed-nerve realization; identifying a cross-scale receiving algebra remains a separate step. When the estimates are first proved in local coefficient charts, Lemma 5.6 preserves their verified ordinary overlap compatibility. Effective descent of the represented modules retains its own source hypotheses.

Proposition 8.2 (Comparison of two lineages). *Let (x_n) and (y_n) satisfy Theorem 8.1 after their admitted comparisons into the same $\mathcal{M}$. Suppose that along a cofinal family of common comparisons,*

$$p_\varphi(x_{n_j} - y_{m_j}) \longrightarrow 0 \quad \text{for every } \varphi \in \mathcal{M}_*^+, \quad n_j, m_j \longrightarrow \infty.$$

Then their limits coincide.

Proof. Subtract the two limits, insert x_{n_j} and y_{m_j} , and apply the triangle inequality and the assumed convergence. The separating seminorms give equality. $\square$

The comparison condition is essential. Coherence of finite cells alone does not prove it for every cofinal history. It must retain the same support, normalization, coefficient marking, and represented observable.

There is also a useful representation-independent sufficient condition. If $\mathcal{A}_n$ form a sequential system of C^* -algebras with injective $*$ -homomorphisms T_n , then

$$\sum_n \|w_{n+1} - T_n w_n\| < \infty \quad (76)$$

gives a norm limit in the C^* -inductive limit by the same telescoping argument. This condition can be stronger than the physically available topology. A general coefficient-assisted or one-sided correspondence system must first be shown to admit this particular C^* -algebraic realization before (76) is used.

8.1 The complete represented residual

Let w_n be the actual Wilson expressions on a selected lineage, with specified linear comparisons T_n and linear receiving realizations ι_n into $\mathcal{M}$. Put $x_n = \iota_n(w_n)$ and $r_n = w_{n+1} - T_n w_n$. Then the exact identity is

$$\begin{aligned} x_{n+1} - x_n &= \iota_{n+1}(r_n) + d_n, \\ d_n &= \iota_{n+1}(T_n w_n) - \iota_n(w_n). \end{aligned} \quad (77)$$

The realization-square defect d_n vanishes only when that square actually commutes on the observable. Omitting it does not establish the completion theorem. No additive repair of w_n is assumed here. If a relative filling is additionally represented as an observable correction, Appendix E gives its separate compatibility test.

8.2 Why coherence does not supply the estimate

Let $X = \mathbb{T}^{\mathbb{N}}$ with product Haar probability measure, and let $\mathcal{M} = L^\infty(X)$. Embed $\mathcal{A}_n = C(\mathbb{T}^{n+1})$ by the usual cylindrical inclusions and define

$$w_n(z_0, \dots, z_n) = z_n.$$

Each w_n is a unitary bounded observable. It is also the holonomy character of the newest independent $U(1)$ loop in a finite graph presentation. The cylindrical comparisons are strictly associative, and their coefficient band and higher reassociators may all be trivial.

For distinct n, m , independence and zero Haar mean give

$$\varphi((w_n - w_m)^*(w_n - w_m)) = 2, \quad p_\varphi(w_n - w_m) = 2, \quad (78)$$

where φ is integration against product Haar measure. There is no σ -strong* limit. The sequence can even have convergent individual expectations while failing to define a limiting operator. Perfect coherence of the comparison category therefore does not control an observable's refinement residual.

This is a kinematical example, not the canonical Yang–Mills measure. Its purpose is to disprove an implication based solely on coherence, boundedness, and a bordism indexing structure. It also shows why the physical identity of the observable across stages is part of the required data: choosing a different independent loop at each stage does not describe one transported loop.

Even successive errors tending to zero are insufficient. In $\mathcal{M} = \mathbb{C}$, the bounded sequence $x_n = e^{i \log(n+1)}$ obeys $|x_{n+1} - x_n| \rightarrow 0$ but does not converge. Subsequences with $n+1$ close to $e^{2\pi k}$ and $e^{(2k+1)\pi}$ approach opposite phases. A genuine Cauchy estimate, such as the summable bound in Theorem 8.1, supplies the missing control.

9 Established scope and remaining physical comparison

The native finite calculation starts from the frustum’s open-bank actions, retains the complete charged ground and its noninjective insertion maps, and realizes the selector slant by clock–shift operations with the inherited inverse line. The coefficient prism aligns endpoint multipliers; the double and third prisms give the specified higher comparisons. The full Ward identity crosses physical spacetime overlaps on the prepared source domains. Normal *-homomorphic overlap maps then give the ordinary compatible-family W^* -algebra and its Wilson-generated subalgebra.

These statements have different origins. The finite source’s Hamiltonian, complete selection, and admissible chart maps are imported from [7]; their displayed coefficient calculations are evaluated here. The selected fixed-nerve physical realization is imported from [4] at its stated hypotheses. The prism, normal-closure, and bounded-completion arguments are proved for their explicit inputs. The source’s interaction and selector prescriptions remain inputs, not consequences of the scalar formulas. Appendices retain the charged Haar tests, cyclic carry calculation, and finite cutoff certificates needed to distinguish these constructions.

For a further physical refinement limit at fixed thickness, the concrete task is to identify the actual receiving realization and control the full residual (77):

$$p_\varphi(\iota_{n+1}(r_n) + d_n) \leq C_\varphi e_n, \quad \sum_n e_n < \infty. \quad (79)$$

One must also verify the uniform operator bound, retained Wilson relations, support when asserted, and the common-lineage comparison needed for uniqueness. Proposition 6.1 now evaluates exact inherited Wilson transport on the named selected native frustum, with the vacuum-state bound (60). The raw Haar strong limit and separated-shell construction provide additional specific realizations. These do not identify every selected physical receiving system or establish a uniform refinement-family bound. Nonzero rebuilt-cutoff residuals likewise do not refute the source’s differently typed exact observable transport.

The earlier linked-flux class and the product selector class remain distinct under the computed cyclic test. Their equality is unnecessary for the native selector construction proved here. Any proposed identification between the full coefficient diagrams requires its actual line, intertwiner, and lateral-seam maps. Optional operator-valued realizations of a relative primitive are also independent of the coefficient-sewing result.

Sending the retained thickness ε to zero is a further question. Sharp quantum loops can require perimeter and cusp renormalization and intersection mixing [19]; the bound for normalized finite characters need not survive a divergent multiplicative renormalization. The abstract categorical construction is not specific to dimension four. Its $3 + 1$ -dimensional application depends on the native dynamics, regularity, and receiving estimates. Observable descent and completion do not by themselves assert a mass gap or a buffered spectral no-leakage theorem.

A Charged Haar seams and exact selection tests

A.1 An actual charged seam map for coefficient intertwiners

The remaining comparison concerns maps between specified carriers. The neutral Haar seam in Section 5.5 extends to a charged seam and transports actual intertwiners without choosing a scalar basis for their lines. We give this construction on its full equivariant Haar carrier, then state exactly how it applies to the native lines (102).

Let $R : G \rightarrow U(V)$ be a strongly continuous unitary representation of the compact seam group. Use normalized Haar measure and put

$$\mathcal{H}_R = L^2(G; V), \quad \mathcal{K}_R = \{F \in L^2(G^2; V) : F(hx, yh^{-1}) = R(h)F(x, y)\}.$$

The equality defining $\mathcal{K}_R$ is understood in L^2 , for every h . Define

$$(J_R f)(x, y) = R(x)f(yx). \quad (80)$$

The seam acts at its actual open port. In particular the charge factor is part of the equivariance condition rather than an inert extra tensor factor.

Proposition A.1 (Charged Haar seam and naturality). *The map $J_R : \mathcal{H}_R \rightarrow \mathcal{K}_R$ is unitary. For a bounded intertwiner $T : V \rightarrow V'$ with $TR(h) = R'(h)T$, let T_c, T_f denote its pointwise actions on the coarse and cut function spaces. Then*

$$T_f J_R = J_{R'} T_c. \quad (81)$$

This is an equality of bounded maps on the full coarse carrier; T need not be injective or unitary. For every bounded scalar Wilson function w ,

$$M_{w \circ p} J_R = J_R M_w, \quad p(x, y) = yx, \quad (82)$$

and the analogous adjoint identity holds.

Proof. Equivariance follows from $R(hx)f(yh^{-1}hx) = R(h)R(x)f(yx)$. The change of variables $(x, y) \mapsto (x, g = yx)$ and unitarity of $R(x)$ show that J_R is isometric. Under the same change of variables, $R(x)^*F(x, gx^{-1})$ is invariant under left translation in x for $F \in \mathcal{K}_R$, and hence is independent of x in L^2 . This proves surjectivity. Its inverse on that subspace is the Haar integral

$$(J_R^* F)(g) = \int_G R(x)^* F(x, gx^{-1}) dx.$$

No evaluation of an L^2 function at a point is used. Pointwise,

$$(T_f J_R f)(x, y) = TR(x)f(yx) = R'(x)Tf(yx) = (J_{R'} T_c f)(x, y).$$

Finally $w(yx)$ commutes with $R(x)$, which proves the Wilson identity and its conjugate. $\square$

The Ward identity includes the port charge. On a common smooth-vector core, define

$$D_X = L_x(X) - R_y(X), \quad \mathcal{D}_X = D_X - R_*(X).$$

Differentiating the equivariance equation gives

$$\mathcal{D}_X J_R f = 0. \quad (83)$$

The derivative of $R(x)$ is exactly the term that cancels the seam-port charge. The self-adjoint generator $G_X = -iD_X$ is taken with its unitary-group domain; the smooth-vector core is used for the following computation. For a smooth scalar insertion F ,

$$i[G_X, M_F] = M_{D_X F}.$$

With $g_X[\Psi, \Phi] = \langle G_X \Psi, \Phi \rangle$, the full three-term Ward functional is therefore

$$ig_X[M_F \Psi, \Phi] - ig_X[\Psi, M_{\bar{F}} \Phi] - \langle M_{D_X F} \Psi, \Phi \rangle = 0.$$

For $F = w \circ p$, $D_X F = 0$, and the Wilson multiplier preserves the charged seam carrier. This is the charged extension of the earlier neutral seam calculation. On infinite-dimensional V , vectors and insertions are kept in the declared common smooth domain; the bounded intertwining equations themselves require no differentiation.

Transport of a native coefficient line. For fixed representations R, R' , the assignment

$$T \mapsto \hat{T} := J_{R'}(I_{L^2(G)} \otimes T)J_R^* \quad \text{from } \mathcal{K}_R \text{ to } \mathcal{K}_{R'} \quad (84)$$

is linear, isometric, and faithful. It preserves adjoints, identities, and composition whenever the endpoints match. Each assertion follows by cancelling the intervening seam unitaries; the norm equality follows from $\|I \otimes T\| = \|T\|$. Thus a declared one-dimensional intertwiner space L is carried to an actual one-dimensional space $\hat{L}$ of maps on charged seam carriers, with its scalar action retained. This is a line identification, not merely equality of the multipliers of two unknown lines.

Apply this construction to a native

$$\ell_b : V_{j,m}^\Omega \hat{\otimes} \mathbb{C}_b \longrightarrow V_{j+b,m}^\Omega$$

when the seam action is the corresponding subgroup of the native Gauss action and the endpoint representations are its actual restrictions. Writing these as $R_{\text{in}} = R_j^\Omega \otimes \rho_b$ and $R_{\text{out}} = R_{j+b}^\Omega$, Gauss intertwining gives

$$(I_{L^2(G^2)} \otimes \ell_b)J_{R_{\text{in}}} = J_{R_{\text{out}}}(I_{L^2(G)} \otimes \ell_b). \quad (85)$$

This is the requested elementary lateral square on the charged Haar extension of those endpoint modules. It retains the norm and kernel of ℓ_b . In particular a contraction is never replaced by an inverse.

The center-naturality equation (103) also survives this faithful transport:

$$\widehat{C_{m,j+b}^\Omega \ell_b} = \ell_b(\widehat{C_{m,j}^\Omega \otimes \rho_b(\omega^m I_3)}).$$

The hat here means transport of each displayed composite by its endpoint seam unitaries. Since $\rho_b(\omega^m I_3) = \omega^{m\tilde{b}} I$, the native carry character remains $\omega^{m\tilde{b}}$. Any supplied coefficient multiplication diagram whose maps have these compatible Gauss endpoints is transported by the same functor, so its reassociation equality and detected phase are preserved as equalities of maps.

The precise boundary of this realization. Equations (80)–(85) construct the charged Haar carriers and the intertwiner lift explicitly. They establish the elementary seam for the native line whenever the indicated port action is its actual Gauss action. They do not identify an unspecified transgression selector with this construction.

If the physical seam imposes additional complete projections P_c, P_f , its restriction requires

$$P_f J_R = J_R P_c.$$

For a coefficient map between two such selections, it also requires $P'_c T_c = T_c P_c$ and $P'_f T_f = T_f P_f$. Under these exact identities the preceding square restricts to the selected carriers by multiplication with the projections. The complete-ground character of the endpoint line alone does not prove these identities for a new cut or a different complete constraint family.

The source [6], *Open Wilson operations and a native seam*, explicitly retains the charged intertwiner and complete selection as extra seam data. Its transgression paragraph supplies no particular identification of the resulting selector with (84). Accordingly the full physical application still needs that selector identification, the actual additional-selection equations where present, and the declared lateral closing cells. The new result removes the ambiguity about how a bounded Gauss intertwiner crosses the elementary charged Haar seam: its map, line transport, Ward cancellation, and Wilson square have all been evaluated.

A.2 Gauss projection and nonzero charged seam carriers

For the charged Haar realization just constructed, the Gauss selection can be evaluated rather than left as an unspecified projection. The endpoint actions are part of this calculation. In this subsection J_R is regarded as an isometry into the full cut space $L^2(G^2; V)$, with range $\mathcal{K}_R$.

On the coarse space $\mathcal{H}_R = L^2(G; V)$, define the source and target endpoint representations

$$(S_c(k)f)(g) = R(k)f(gk), \quad (T_c(k)f)(g) = f(k^{-1}g). \quad (86)$$

The $R(k)$ in the source action is essential: it is the endpoint charge action induced by the chosen charged seam coordinate J_R . On the full cut space put

$$\begin{aligned} (S_f(k)F)(x, y) &= F(xk, y), \\ (T_f(k)F)(x, y) &= F(x, k^{-1}y), \\ (A_{\text{seam}}(h)F)(x, y) &= R(h)F(h^{-1}x, yh). \end{aligned} \quad (87)$$

These are unitary representations, and the source, target, and seam actions commute with one another. Direct substitution gives

$$S_f(k)J_R = J_R S_c(k), \quad T_f(k)J_R = J_R T_c(k), \quad A_{\text{seam}}(h)J_R = J_R. \quad (88)$$

For instance, both sides of the first identity send f to $R(x)R(k)f(yxk)$. An endpoint convention without the displayed charge factor would fail this equality.

Let $P_s^c, P_t^c, P_s^f, P_t^f, P_{\text{seam}}$ be the corresponding Haar averages. They are orthogonal projections, with commuting factors at the same stage. Define the full vertex Gauss projections for this isolated-edge model by

$$P_G^c = P_s^c P_t^c, \quad P_G^f = P_s^f P_t^f P_{\text{seam}}.$$

Averaging (88) proves

$$P_G^f J_R = J_R P_G^c, \quad P_G^f = J_R P_G^c J_R^*. \quad (89)$$

For the second identity, use $P_{\text{seam}} = J_R J_R^*$, move the endpoint projections through J_R , and retain the ambient adjoint $J_R^* : L^2(G^2; V) \rightarrow L^2(G; V)$. Thus this complete finite vertex-Gauss selection is exactly compatible with the charged seam. No projection is defined by transporting a desired answer; every projection above is the Haar average of its displayed action.

Exact projection compatibility can still select zero. The selected coarse space can also be computed. Target averaging makes f constant:

$$(P_t^c f)(g) = \int_G f(u) du.$$

Source averaging then gives

$$(P_G^c f)(g) = P_{\text{inv}}^R \int_G f(u) du, \quad P_{\text{inv}}^R = \int_G R(k) dk. \quad (90)$$

Therefore

$$\text{Ran } P_G^c \cong V^G, \quad \text{Ran } P_G^f = J_R(V^G).$$

If R has nonzero $\text{SU}(3)$ triality b , then $R(\omega I_3) = \omega^b I$ and $V^G = 0$. Indeed, an invariant vector would satisfy $v = \omega^b v$. Both selected spaces are then zero, even though (89) holds exactly.

This tests a concrete failure mode for a proposed physical realization: imposing full gauge invariance at both endpoints of an isolated charged edge erases its nonzero-triality sector. Such a projection cannot serve as a faithful realization of that charged coefficient data. It does not imply that the native charged line is zero. It shows that this isolated, fully averaged carrier is the wrong boundary prescription for retaining it.

Retaining or compensating the charge. An exposed source port retains its charge action. If only the target endpoint and the seam are averaged, then

$$P_t^f P_{\text{seam}} J_R = J_R P_t^c, \quad \text{Ran } P_t^c \cong V.$$

The constants carry the residual source action R ; they have not been declared gauge singlets. With both endpoints exposed, the complete internal seam selection is simply P_{seam} , and its range is the nonzero $\mathcal{K}_R$.

Alternatively, introduce an actual compensating charge representation before imposing endpoint invariance. For a finite-dimensional unitary R on V , the representation $R \otimes \bar{R}$ has the invariant vector

$$\xi_R = \frac{1}{\sqrt{\dim V}} \sum_a e_a \otimes \bar{e}_a.$$

Thus the same fully averaged construction with coefficient representation $R \otimes \bar{R}$ has a nonzero selected space. For an irreducible R , Schur's lemma gives a one-dimensional invariant space. For the $\text{SU}(3)$ fundamental representation, this is the singlet in $\mathbf{3} \otimes \bar{\mathbf{3}}$. The pairing changes the declared charged configuration; it cannot be silently inserted to repair a different physical endpoint.

More generally, zero total triality is necessary for a tensor product of charged factors to contain a singlet, and is not sufficient. The actual invariant intertwiner must be retained. For example, the $\text{SU}(3)$ adjoint representation has zero triality but no invariant vector. A scalar phase calculation cannot determine whether a given complete projection retains the desired intertwiner.

Consequences for the physical seam identification. A bounded intertwiner between the displayed coarse representations commutes with the corresponding Haar averages, by integrating its intertwining equations. Together with (81), this restricts its lateral square to the selected spaces above. Whether the restricted line remains nonzero must then be checked; restriction to the zero carrier certainly destroys it.

For Wilson observables, the earlier identity $M_{w \circ p} J_R = J_R M_w$ remains exact. Preservation by endpoint averaging additionally uses an actual closed Wilson insertion, including its return edges and their endpoint actions. A generic function of one open edge is not independently invariant at both endpoints. For a specified larger graph, the incident-edge actions and port representations must be included in the endpoint Gauss average before asserting reduction by the Wilson multiplier.

This calculation discharges $P_f J_R = J_R P_c$ for the explicitly displayed finite Gauss family and determines when that family is nonzero. A physical complete Klauder selection can contain further constraints; its additional projection identities are not consequences of this Haar calculation. The native selector must identify which port is exposed, which compensating intertwiners are present, and which complete constraints are imposed. These choices are necessary to bind the evaluated charged seam to the intended nonneutral physical packet.

A.3 A closed Wilson carrier with retained endpoint charges

A return edge and its endpoint actions must be included before testing a closed Wilson observable. We now give a nonzero, fully vertex-Gauss-selected Haar realization with explicit charged endpoints. It supplies an exact finite graph comparison; identification with an additional physical selection remains a separate question.

Let $g : s \rightarrow t$ and $r : t \rightarrow s$ be two oriented edges, and let $R : G \rightarrow U(V)$ be finite dimensional, with $d = \dim V > 0$. The coefficient space is $\text{Hom}(V_s, V_t) \cong \text{End}(V)$, with its Hilbert–Schmidt norm. Thus both endpoint factors, $V_t \otimes V_s^*$, are retained. Define

$$\mathcal{H}_R^{\text{loop}} = \left\{ F \in L^2(G^2; \text{End}(V)) : \begin{array}{l} F(h_t g h_s^{-1}, h_s r h_t^{-1}) \\ = R(h_t) F(g, r) R(h_s)^* \end{array} \right\}.$$

This is the invariant space of the corresponding unitary endpoint gauge action, and its orthogonal projection is its two-vertex Haar average. The coefficient covariance supplies the endpoint charge contraction in that invariant space.

A nonzero carrier and a faithful scalar subspace. Put $q = rg$. The map

$$(\mathcal{I}f)(g, r) = R(g)f(rg) \tag{91}$$

is unitary from

$$\left\{ f \in L^2(G; \text{End}(V)) : f(hqh^{-1}) = R(h)f(q)R(h)^* \right\}$$

onto $\mathcal{H}_R^{\text{loop}}$. For covariance, substitute the transformed g, r into (91). Unitarity follows from the Haar change of variables $(g, r) \mapsto (g, q = rg)$ and the fact that left multiplication by $R(g)$ preserves the Hilbert–Schmidt norm. Surjectivity follows by taking invariants in the g variable after this change of variables. Equivalently, the inverse on the invariant subspace is

$$(\mathcal{I}^*F)(q) = \int_G R(g)^* F(g, qg^{-1}) dg.$$

This expresses the gauge fixing in L^2 , without evaluating a representative on a measure-zero slice.

In particular, for every square-integrable class function ψ , define

$$(C_R \psi)(g, r) = \frac{1}{\sqrt{d}} R(g) \psi(rg). \quad (92)$$

Then $C_R : L^2(G)^G \rightarrow \mathcal{H}_R^{\text{loop}}$ is an isometry. Indeed, $\|R(g)\|_{\text{HS}}^2 = d$, so $\|C_R \psi\|^2 = \int_G |\psi(q)|^2 dq$. This proves nonvanishing after the actual endpoint Haar projection, including when R has nonzero triality. The configuration contains both dual endpoint factors; it is different from the isolated single-charge configuration annihilated in Section A.2.

Wilson action and its exact norm. For any bounded measurable class function a , multiplication by $a(rg)$ preserves the selected carrier. Its action $\pi_R(a)$ satisfies

$$\pi_R(a) C_R = C_R M_a, \quad \|\pi_R(a)\| = \|a\|_{L^\infty(G)}. \quad (93)$$

The upper bound is the multiplication-operator bound. For the reverse inequality, use C_R and normalized characteristic functions of the conjugation-invariant sets $\{q : |a(q)| > \|a\|_\infty - \delta\}$. Thus the representation of $L^\infty(G)^G$ is faithful and normal. Its image is an ordinary W^* -algebra on the same nonzero selected carrier.

For the normalized Wilson character $w_\rho(q) = d_\rho^{-1} \text{Tr } \rho(q)$, this gives

$$\|\pi_R(w_\rho)\| = 1.$$

The upper bound follows from unitarity. The lower bound follows from continuity, $w_\rho(e) = 1$, and positive Haar measure of every neighborhood of the identity. Hence full vertex-Gauss selection in this declared charged configuration preserves the Wilson multiplication norm exactly.

For $G = SU(3)$, the fundamental character and its conjugate distinguish conjugacy classes as in Section 6.3. Therefore the von Neumann algebra generated by its represented Wilson operator is the faithful image of $L^\infty(SU(3))^{SU(3)}$. This is the multiplication algebra of this finite graph realization; it is not a claim that Wilson multiplication alone generates every physical regional observable.

Subdivision, selection, and the full Wilson word. Subdivide $g = yx$, retain r , and use

$$(JF)(x, y, r) = F(yx, r).$$

At the new seam, $x \mapsto hx$, $y \mapsto yh^{-1}$. The endpoint coefficient factors remain at s, t , so the new internal seam is neutral and contracts the two open matrix factors:

$$R(yx) = R(y)R(x).$$

The Haar quotient identifies the full seam-invariant space with the coarse two-edge space. The endpoint actions are also intertwined, since their products act on yx exactly as on g . If P_{loop}^c is the coarse endpoint Haar projection and P_{loop}^f averages the two endpoints and the new seam, then

$$P_{\text{loop}}^f J = J P_{\text{loop}}^c, \quad P_{\text{loop}}^f = J P_{\text{loop}}^c J^*. \quad (94)$$

The isometry restricts to a unitary between the full selected coarse and fine carriers. On these carriers,

$$M_{w_\rho(ryx)}J = JM_{w_\rho(rg)}, \quad JC_R\psi = \frac{1}{\sqrt{d}}R(y)R(x)\psi(ryx). \quad (95)$$

Both sides have Wilson norm one. The full closed word is invariant at each vertex. Its infinitesimal seam cancellation and the full three-term Ward identity are therefore the evaluated identities of Section 5.5, now with the return edge and endpoint factors present. Repeated subdivisions retain the same conclusion by Haar integration and matrix-product associativity.

Coefficient carry remains distinguishable. If R has triality b , a declared center carry on the target coefficient factor acts by

$$(B_m F)(g, r) = R(\omega^m I_3)F(g, r) = \omega^{mb} \tilde{F}(g, r).$$

It preserves $\mathcal{H}_R^{\text{loop}}$, commutes with its Wilson multiplication, and obeys

$$\text{Ad}(B_m)(\pi_R(a)) = \pi_R(a), \quad C_R^* B_m C_R = \omega^{mb} I.$$

Thus the same selected carrier can retain the coefficient character in a transported amplitude while its ordinary Wilson action is unchanged. Applying opposite carries to the two dual factors would instead cancel their characters; that is a different specified comparison. The single-factor action above matches the center character in the native carry law, without identifying the full associator or selector from this one equality.

We have now evaluated a closed Wilson word, its exact vertex-Gauss projection, a nonzero charged endpoint realization, and norm-preserving subdivision on one Haar graph. The native application must identify these endpoint factors and their carry/intertwiner maps with its declared physical packet. Any further constraint or cutoff can change the selected carrier and must be tested on that packet. The elementary graph result establishes the faithful kinematic realization required for that comparison.

A.4 Additional selection: products, support, and the actual intertwiner line

The physical-constraint paragraph of [6] requires both reduction of Wilson multiplication and faithfulness after complete selection. These are independent requirements. The closed carrier of Section A.3 makes their precise tests available before any identification with the native selector.

Write $Y = G/\text{Ad}$, let μ be the pushforward of normalized Haar measure, and put $\mathcal{A} = L^\infty(Y, \mu)$. On the closed carrier $\mathcal{H} = \mathcal{H}_R^{\text{loop}}$, retain the faithful normal representation $\pi = \pi_R$. Let P be an additional orthogonal selection projection on $\mathcal{H}$, with $P\mathcal{H} \neq 0$. Define the normal unital completely positive compression

$$\Phi_P(a) = P\pi(a)|_{P\mathcal{H}}.$$

The unit here is the identity of $P\mathcal{H}$.

Proposition A.2 (Exact selection test). *The compression Φ_P is a *-representation of $\mathcal{A}$ if and only if $P \in \pi(\mathcal{A})'$. In that case there is a measurable set $E \subseteq Y$, unique modulo null sets, such that*

$$\ker \Phi_P = \mathcal{A} 1_{Y \setminus E}, \quad \|\Phi_P(a)\| = \text{ess sup}_E |a|, \quad \Phi_P(\mathcal{A}) \cong L^\infty(E, \mu|_E). \quad (96)$$

In particular, the selected representation is faithful precisely when

$$P\pi(1_B) \neq 0 \quad \text{for every measurable } B \subseteq Y \text{ with } \mu(B) > 0. \quad (97)$$

Proof. For $a, b \in \mathcal{A}$, direct multiplication gives

$$\Phi_P(ab) - \Phi_P(a)\Phi_P(b) = P\pi(a)(1 - P)\pi(b)|_{P\mathcal{H}}.$$

Putting $a = b^*$ makes the right side

$$((1 - P)\pi(b)P)^*((1 - P)\pi(b)P)|_{P\mathcal{H}}.$$

Thus multiplicativity forces $(1 - P)\pi(b)P = 0$ for every b ; applying this to b^* gives reduction. Conversely reduction makes every displayed defect zero. A normal representation has an ultra-weakly closed ideal as kernel. An ideal of this kind in $L^\infty(Y, \mu)$ is multiplication by a characteristic projection, giving E . The induced faithful representation of $L^\infty(E, \mu|_E)$ is isometric, which proves the norm formula. Finally, $\Phi_P(1_B) = 0$ exactly when $P\pi(1_B) = 0$. These projections detect whether the kernel is zero. $\square$

For $G = SU(3)$, the fundamental Wilson function w generates $\mathcal{A}$. Hence checking $[P, \pi(w)] = 0$ suffices for reduction: since $P = P^*$, the adjoint also commutes, and commutation extends to $W^*(\pi(w))$. This check concerns the original multiplier, not a rebuilt cutoff matrix.

The norm-one conclusion after further selection is now exact:

$$\|\Phi_P(w_\rho)\| = 1 \iff \mu(E \cap \{|w_\rho| > 1 - \delta\}) > 0 \quad \text{for every } \delta > 0. \quad (98)$$

Faithfulness implies this condition, but it is stronger than necessary. Nonvanishing of $P\mathcal{H}$ alone implies neither faithfulness nor norm one. If reduction fails, the compressed Wilson operators still generate a concrete W^* -algebra in $B(P\mathcal{H})$, but compression does not represent the original multiplication relations. These are two different answers to the W^* -algebra question.

A sufficient witness with no global coefficient trivialization. The isometry C_R of (92) gives a direct sufficient condition:

$$PC_R = C_R, \quad [P, \pi(\mathcal{A})] = 0.$$

Under these assumptions, $\Phi_P(a)C_R = C_RM_a$, so Φ_P is faithful and isometric. Containing this entire scalar subspace is sufficient, not necessary; (97) is the exact condition. This scalar test uses one faithful ordinary subrepresentation. It does not identify or trivialize the coefficient band.

For the declared single-endpoint carry $B_m = \omega^{m\tilde{b}}I$, every P commutes with B_m , and on any nonzero selected carrier

$$B_m|_{P\mathcal{H}} = \omega^{m\tilde{b}}I_{P\mathcal{H}}, \quad \text{Ad}(B_m|_{P\mathcal{H}})(\Phi_P(a)) = \Phi_P(a).$$

The coefficient character survives this further selection even when the ordinary representation has lost faithfulness. Its survival does not prove that the native line or its fusion maps survive.

The line must survive as a map. Let $T : \mathcal{H}_{\text{in}} \rightarrow \mathcal{H}_{\text{out}}$ be an actual bounded native intertwiner, and let $P_{\text{in}}, P_{\text{out}}$ be the complete selection projections. The required restriction identity is

$$P_{\text{out}}T = TP_{\text{in}}. \quad (99)$$

It implies the adjoint identity and defines $T^K = T|_{P_{\text{in}}\mathcal{H}_{\text{in}}}$ with values in $P_{\text{out}}\mathcal{H}_{\text{out}}$. For a composable S satisfying its corresponding identity, $(ST)^K = S^K T^K$. Thus genuine identities between the retained maps descend, including a specified center-naturality identity. A one-dimensional native line $L = CT$, with $T \neq 0$, embeds faithfully into the selected Hom space if and only if

$$TP_{\text{in}} \neq 0. \quad (100)$$

This is only faithfulness on that line: it does not say that T^K is injective or isometric.

Arbitrary compression cannot replace (99). Its composition error is

$$P_{\text{out}}STP_{\text{in}} - P_{\text{out}}SP_{\text{mid}}TP_{\text{in}} = P_{\text{out}}S(1 - P_{\text{mid}})TP_{\text{in}}.$$

Consequently an equality of scalar carry factors cannot establish the selected fusion diagram. One must retain the actual tensor identifications, verify the selection square for each line map, and check nonvanishing of the selected line products used by that diagram. These are precisely the data missing from a comparison of cocycle formulas alone.

Subdivision and physical overlaps. Let $J : \mathcal{H}_c \rightarrow \mathcal{H}_f$ be the closed-loop Haar unitary between the already vertex-Gauss-selected carriers. If additional selections satisfy

$$P_f J = J P_c,$$

then J restricts to a unitary between their selected ranges. Since $\pi_f(a)J = J\pi_c(a)$, their compressed maps intertwine. When the projections reduce these representations, they have the same set E in (96). Subdivision then preserves the selected algebra and its norm exactly, even when $E \neq Y$. For a physical arrow that is only an isometry, this conclusion applies to its reducing supported image; surjectivity onto the complete target selection must be supplied separately.

Across a physical spacetime overlap, the corresponding requirement is $P_i u_{ij} = u_{ij} P_j$, together with the actual Wilson insertion identity. The full prepared Ward identity and collar cancellation of Section 5.4 establish the stated physical passage on their supplied domains; they do not by themselves prove this additional projection identity or (100). Once the selected overlap maps and line maps are identified, their commuting squares pass to the selected objects, and scalar coefficient multipliers continue to cancel in the adjoint observable descent.

The native packet in Section 2.2 supplies actual charged selections and surviving maps. The Haar construction here is a reference realization: identifying its full Wilson representation and lateral comparisons with another selected packet requires those actual maps.

B The cyclic linked-flux class and its fusion

B.1 The native SU(3) coefficient realization

The physical realization is already specified in Original Paper III [4], Section VIII.B and Appendix I.3, printed pp. 239–240 and 280–282. It uses meridional carry acting on complete-ground, center-natural coefficient lines. The normal dictionary on an admitted physical nerve is supplied separately

by Lemmas II.255–II.257 and Theorems II.259–II.260, printed pp. 178–183. The current spacetime assembly [5], Sections 3.2–3.4, retains precisely this dictionary. The following calculation evaluates its coefficient law and ordinary Wilson comparison at that source scope.

The retained associator lies in $H^3(B\mathbb{Z}_3; U(1))$. Section 2.3 compares it with the product selector class; the calculation here supplies its actual carry lines and fusion data.

The physical carry and its action on coefficient lines. Fix the source’s selected positive thickness and flux branch $m \in \{1, 2\}$, and write $\omega = e^{2\pi i/3}$. On the universal angular cover of $X_{\text{inc}} = \mathbb{R}^3 \setminus \ell$, the positive deck transformation is $T : \tilde{\theta} \mapsto \tilde{\theta} + 2\pi$. The source’s flat bundle has positive meridional holonomy $\omega^m I_3$. Strict lifted corridors obey

$$\Gamma_n \Gamma_k = \Gamma_{n+k}, \quad \Gamma_3 = T.$$

On the coefficient-preserving skeleton, use representatives $\tilde{\alpha} \in \{0, 1, 2\}$ and the carry

$$\epsilon_3(\alpha, \beta) = \left\lfloor \frac{\tilde{\alpha} + \tilde{\beta}}{3} \right\rfloor, \quad \tilde{U}_{m,\alpha} \tilde{U}_{m,\beta} = C_m^{\epsilon_3(\alpha,\beta)} \tilde{U}_{m,\alpha+\beta}. \quad (101)$$

Here sums in labels are reduced modulo three. The source retains declared rank-one lines

$$L_{m,\alpha} \subset \text{Hom}_{\text{Gauss}, \blacksquare, \text{cen}}(V_{j,m}^\Omega \hat{\otimes} C_\alpha, V_{j+\alpha,m}^\Omega), \quad \rho_\alpha(\omega I_3) = \omega^{\tilde{\alpha}} I. \quad (102)$$

Rank one belongs to the selected endpoint package. For $\ell_\alpha \in L_{m,\alpha}$, the actual center-naturality identity is

$$C_{m,j+\alpha}^\Omega \ell_\alpha = \ell_\alpha (C_{m,j}^\Omega \otimes \rho_\alpha(\omega^m I_3)). \quad (103)$$

Thus the induced carry action on this intertwiner line is $C_m \cdot \ell_\alpha = \omega^{m\tilde{\alpha}} \ell_\alpha$. This is the source’s represented action, rather than a phase assigned to an auxiliary tensor factor.

Pairwise multiplication retains its carry line:

$$\mu_{m,\alpha,\beta}^{\text{fib}} : L_{m,\alpha} \hat{\otimes} L_{m,\beta} \longrightarrow K_m^{\epsilon_3(\alpha,\beta)} \hat{\otimes} L_{m,\alpha+\beta}. \quad (104)$$

Putting all carry lines to the left, the right-associated product must move the inner carry across $L_{m,\alpha}$. Equation (103) supplies

$$\Omega_m^{\text{fib}}(\alpha, \beta, \gamma) = \omega^{m\tilde{\alpha}} \epsilon_3(\beta, \gamma). \quad (105)$$

The number of carry lines agrees in the two bracketings because

$$\epsilon_3(\alpha, \beta) + \epsilon_3(\alpha + \beta, \gamma) = \epsilon_3(\beta, \gamma) + \epsilon_3(\alpha, \beta + \gamma).$$

For the marked triple $(1, 2, 1)$, each bracketing has one carry line. The right inner product of labels 2, 1 creates it to the right of the triality-one line. Moving it left gives exactly ω^m .

A detector invariant under changes of line multiplication. The single marked triple depends on the normalized representative. The full detector

$$I_3(\Omega_m^{\text{fib}}) = \prod_{b=0}^2 \Omega_m^{\text{fib}}(1, b, 1) = 1 \cdot 1 \cdot \omega^m = \omega^m \quad (106)$$

is invariant under normalized two-cochain changes. Indeed, for

$$(\delta q)(a, b, c) = \frac{q(b, c)q(a, b + c)}{q(a + b, c)q(a, b)},$$

both products in

$$\prod_{b=0}^2 (\delta q)(1, b, 1) = \prod_{b=0}^2 \frac{q(b, 1)}{q(1 + b, 1)} \prod_{b=0}^2 \frac{q(1, b + 1)}{q(1, b)}$$

telescope to one. Therefore the two nonzero flux branches have a nontrivial coefficient class; their detectors are ω and ω^2 . Since the cocycle itself takes values in μ_3 , each nonzero class has order exactly three.

The calculation already takes place on the selected complete-ground lines. Compressing to those lines cannot erase its detected phase. This is a statement about the coefficient multiplication, not a requirement that the ordinary Wilson operator preserve a rank-one ground space.

The same physical comparison on ordinary Wilson operators. On the marked reassociation cell, let V_A, V_B denote the represented comparisons after the specified carry-line identifications, oriented so that $V_A = \omega^m V_B$. For an admitted ordinary operator A , their supported adjoint transports agree:

$$V_A A V_A^* = V_B A V_B^*. \quad (107)$$

For endpoint Wilson actions and a line-assisted isometry, write the source action with its coefficient identity understood. Their actual residuals satisfy

$$W_t V_A - V_A W_s = \omega^m (W_t V_B - V_B W_s). \quad (108)$$

The adjoint residual has the same scalar transformation. Thus changing this coefficient bracketing preserves exact Wilson intertwining when it holds, and preserves the norm of any retained residual. The equation does not set a nonzero physical residual to zero.

Original III's Lemma II.257 supplies the corresponding statement for the selected presentation groupoid: a closed linked presentation loop can retain $\omega^q I$ while inducing the identity on the ordinary regional algebra. Its Theorem II.259 identifies the source and retained physical diagrams on the same admitted nerve, preserving the coefficient lines, their reassociator, the complete-ground packet, and the interior Gauss–Ward data. Hence, for the selected Wilson generator,

$$\Phi_i(W_i^{\text{src}}) = W_i^{\text{ret}}, \quad \Phi_{ij} d_{ij,i}^{\text{src}} = d_{ij,i}^{\text{ret}} \Phi_i.$$

These are the source's normal comparison maps. Theorem II.260 then identifies their compatible-family algebras on that fixed nerve.

This binds the ordinary Wilson action and the nonneutral coefficient object to the same native physical packet. Crossing a physical overlap still uses the full Ward and collar identities of Section 5.4 on the transported prepared domains. The source dictionary preserves those data; scalar reassociation leaves their ordinary adjoint action unchanged. The elementary Haar calculation of Section 5.5 is retained at its own carrier scope.

An explicit transgression test and its lost information. The native three-class does not imply that every selected transgressed two-class is nonzero. To make this precise, define a cyclic slant comparison on the abelian skeleton, with fixed loop label $\ell \in \mathbb{Z}_3$:

$$t_{m,\ell}(b,c) := \frac{\Omega_m^{\text{fib}}(\ell, b, c) \Omega_m^{\text{fib}}(b, c, \ell)}{\Omega_m^{\text{fib}}(b, \ell, c)} = \omega^{m\tilde{\ell}\epsilon_3(b,c)}. \quad (109)$$

The last two factors cancel because the carry is symmetric. This formula explicitly specifies the comparison being tested. It gives a normalized two-cocycle. In additive notation it has the primitive

$$\theta_{m,\ell}(b,c) = \frac{m\tilde{\ell}}{3}\epsilon_3(b,c), \quad \beta_{m,\ell}(b) = \frac{m\tilde{\ell}\tilde{b}}{9}, \quad \delta\beta_{m,\ell} = \theta_{m,\ell} \quad \text{in } \mathbb{R}/\mathbb{Z}. \quad (110)$$

Indeed, $\tilde{b} + \tilde{c} - \widetilde{b+c} = 3\epsilon_3(b,c)$. For $m\ell \neq 0$ modulo three, the primitive uses ninth roots of unity after exponentiation. It is unavailable if the allowed primitive is required to be μ_3 -valued: for a normalized μ_3 -valued one-cochain r ,

$$\prod_{b=0}^2 (\delta r)(b,1) = r(1)^3 = 1, \quad \prod_{b=0}^2 t_{m,\ell}(b,1) = \omega^{m\tilde{\ell}} \neq 1.$$

Thus this particular two-cocycle is exact with unrestricted $U(1)$ coefficients while its μ_3 -valued version need not be exact. Other physical admissibility conditions on a primitive must likewise be retained.

Two unrestricted primitives differ by a character of $\mathbb{Z}_3$:

$$\beta'_{m,\ell}(b) - \beta_{m,\ell}(b) = \frac{q\tilde{b}}{3} \pmod{\mathbb{Z}}, \quad q \in \{0,1,2\}.$$

Their generator-loop phases differ by ω^q , with the same adjoint comparison. Nevertheless $I_3(\Omega_m^{\text{fib}}) = \omega^m$ remains nontrivial. This explicitly exhibits a filled transgression layer together with retained flat information and a surviving native band.

Equation (109) is an evaluated cyclic comparison, not an identification of an unspecified selector $\text{trg}_{\ell_\lambda}$ on the full physical nerve. Using it in a particular scale bordism requires the actual selector to restrict to this formula and the primitive to satisfy its marking, support, and seam conditions. The source's physical associator realization and fixed-nerve ordinary descent are already identified above; extending this explicit primitive to that full selector and its bordism is the remaining comparison.

B.2 Fusion of the cyclic fillings retains the native carry

The selector in [6], immediately before Section 1 and in its transgression proposition, includes the actual lines, modules, and comparison maps. Those statements treat the selector as supplied input; they do not specify that it equals the cyclic insertion (109). A further consequence of the evaluated insertion can nevertheless be computed completely: its primitives over all three loop labels retain exactly the native carry character.

Fix $m = 1$ or 2 , and abbreviate the primitives in (110) by $\beta_\ell(b)$, with $\ell, b \in \mathbb{Z}_3$. Their exponentials are

$$r_\ell(b) = \exp\left(2\pi i \frac{m\tilde{\ell}\tilde{b}}{9}\right), \quad \delta r_\ell = t_{m,\ell}.$$

Each r_ℓ trivializes its own multiplier with unrestricted $U(1)$ coefficients. The family has an additional comparison under addition of the loop label:

$$F_{\ell,k}(b) := \beta_\ell(b) + \beta_k(b) - \beta_{\ell+k}(b) = \frac{m\tilde{b}}{3}\epsilon_3(\ell,k) \pmod{\mathbb{Z}}. \quad (111)$$

This follows by substituting $\tilde{\ell} + \tilde{k} - \widetilde{\ell+k} = 3\epsilon_3(\ell,k)$. For fixed ℓ, k , $F_{\ell,k}$ is a character in the b variable: $\delta_b F_{\ell,k} = 0$. Equivalently,

$$\frac{r_\ell(b)r_k(b)}{r_{\ell+k}(b)} = \omega^{m\tilde{b}\epsilon_3(\ell,k)} = \Omega_m^{\text{fib}}(b, \ell, k). \quad (112)$$

The permutation of arguments is explicit: b is the triality label across which the carry from fusing ℓ, k passes.

Equation (112) evaluates the same scalar as the actual native operation

$$K_m^{\epsilon_3(\ell,k)} \text{ crossing } L_{m,b} \longmapsto \omega^{m\tilde{b}\epsilon_3(\ell,k)}$$

in (103). Thus the cyclic primitive has a definite fusion comparison that can be matched to the source's carry action. Equality of these evaluated scalars is established here; identification of the full selector's coefficient lines and intertwiners still uses their actual realization maps.

Proposition B.1 (No strict simultaneous filling on the cyclic branch). *For $m = 1, 2$, there is no family of normalized $U(1)$ -valued primitives r'_ℓ satisfying both*

$$\delta_b r'_\ell = t_{m,\ell} \quad \text{and} \quad r'_\ell(b)r'_k(b) = r'_{\ell+k}(b) \quad (\ell, k, b \in \mathbb{Z}_3).$$

The primitives r_ℓ above do give coherent comparisons when the character in (112) is retained.

Proof. The ratio r'_ℓ/r_ℓ must be a character of $\mathbb{Z}_3$, so it has the form $\omega^{q_\ell \tilde{b}}$, with $q_\ell \in \mathbb{Z}_3$. The fusion character therefore changes from $m\epsilon_3(\ell, k)$ to

$$m\epsilon_3(\ell, k) + q_\ell + q_k - q_{\ell+k} \quad \text{in } \mathbb{Z}_3.$$

If all these characters vanished, summing the equations for $(\ell, k) = (0, 1), (1, 1), (2, 1)$ would give $m + 3q_1 = 0$ in $\mathbb{Z}_3$, which is impossible for $m = 1, 2$. For coherence of the retained characters, the carry identity gives

$$F_{\ell,k}(b) + F_{\ell+k,h}(b) = F_{k,h}(b) + F_{\ell,k+h}(b).$$

This is precisely the equality of the two successive fusion comparisons. □

There is also a direct multiplicative detector:

$$\prod_{\ell=0}^2 \frac{r_\ell(1)r_1(1)}{r_{\ell+1}(1)} = r_1(1)^3 = \omega^m. \quad (113)$$

Multiplying any r_ℓ by a character leaves this detector unchanged. Its value equals $I_3(\Omega_m^{\text{fib}})$. For example, $r_1^2 = r_2$, while $r_2 r_1 = \omega^{m\tilde{b}} r_0$. The failure appears exactly at the wrap from the lifted label three to the reduced label zero. It is the retained meridional carry of the native geometry.

What this adds to admissibility. If an admitted comparison fixes a single loop label, the primitive r_ℓ is available at the scalar $U(1)$ -cochain level. If the demand also compares fused labels, its coefficient package must retain (112), or an equivalent rephased comparison. Requiring all those comparisons to become identities would contradict (113). A scale-bordism filling therefore cannot be declared admissible solely because its separate endpoint two-cocycles are exact.

The retained comparison has identity adjoint action on ordinary operators. At any fixed labels, multiplying a physical implementer by $\omega^{m\tilde{b}\epsilon_3(\ell,k)}$ leaves its supported adjoint transport unchanged. It multiplies its actual Wilson residual by that scalar, as in (108). The character remains visible on the coefficient transport. Ward cancellation across physical spacetime overlaps is still evaluated on the native prepared domains; it does not remove this character or select a strict simultaneous filling.

This completes the fusion test of the cyclic primitives. It identifies their residual character with the source's evaluated meridional carry and proves that the character cannot be removed by any other sectorwise primitive. To apply this family to a particular physical bordism, its selector must still supply the corresponding line/intertwiner comparison and lateral seam data. On the cyclic branch, the required higher comparison is now explicit, and its detector equals the native one.

C Selected cutoff comparisons and finite certificates

C.1 The native line seam and the bare cutoff comparison

We now attempt the comparison against named source data. There are three distinct entries: the retained one-cell packet of [4], Definition II.11; the linked endpoint line of its equation (I.23); and the chart-scale line map of its equation (V.21). Treating these as one unspecified selector conceals the comparison that must actually be made.

The native line-valued lateral comparison. At a fixed selected chart overlap $\alpha\beta$ and a selected scale arrow $a : r_0 \rightarrow r_1$, the source's line branch retains

$$U_{\alpha\beta,r} : H_{\beta,r} \hat{\otimes} L_{\alpha\beta,r} \longrightarrow H_{\alpha,r}, \quad V_{\alpha;a} : H_{\alpha,r_0} \hat{\otimes} L_{\alpha;a} \longrightarrow H_{\alpha,r_1}.$$

Section V.B states the latter as a unitary only in its strong representation branch; the alternative is a normal intertwiner correspondence. We do not impose that unitary branch on the finite packets of changing dimension. The mixed line identification has the actual type

$$\lambda_{\alpha\beta;a} : L_{\beta;a} \hat{\otimes} L_{\alpha\beta,r_1} \longrightarrow L_{\alpha\beta,r_0} \hat{\otimes} L_{\alpha;a}. \quad (114)$$

In a line-implemented branch, the mixed cell identifies the routes

$$\begin{aligned} &U_{\alpha\beta,r_1}(V_{\beta;a} \otimes 1), \\ &V_{\alpha;a}(U_{\alpha\beta,r_0} \otimes 1)(1 \otimes \lambda_{\alpha\beta;a}), \end{aligned} \quad (115)$$

with the declared tensor associators understood. Both start at $H_{\beta,r_0} \otimes L_{\beta;a} \otimes L_{\alpha\beta,r_1}$. Any represented comparison cell required by the native convention is retained in this identification. No inverse of a one-sided physical scale arrow is introduced.

This is the native object with which a proposed lateral seam must be compared. It is a map of lines, together with its represented mixed cell. In the source's normalized scalar convention,

$$(\delta\lambda_a)_{\alpha\beta\gamma} = \vartheta_{\alpha\beta\gamma,r_0}^{-1} \vartheta_{\alpha\beta\gamma,r_1}. \quad (116)$$

Here ϑ is the chart multiplier of (V.9), not automatically the cyclic slant of the linked associator in Section B.1. Hypotheses III.8–III.10 explicitly supply the central line reduction and mixed coherence. Equations (V.21)–(V.24) record their consequences. Locating this map is therefore a source identification, not an independent construction of those hypotheses.

After identifying endpoint lines, markings, and comparison nerves, a prism phase $t_a = \exp(2\pi i\beta_a)$ with the same coboundary can be compared with the native scalar λ_a . Their ratio satisfies

$$z_a = t_a \lambda_a^{-1}, \quad \delta z_a = 1.$$

It can retain flat comparison information. This ratio is meaningful only after the line identifications have been made; it cannot create them. In particular, the cyclic formula for $t_{m,l}$ does not specify which native insertion produces ϑ , nor which physical mixed map it produces.

The explicit one-cell comparison. Choose the first retained cutoff $j = 2$, $L_j = 4$. Its class-function carrier is

$$K_4 = \text{span}\{\chi_{p,q} : p + q \leq 4\}, \quad \dim K_4 = 15.$$

Let $i_4 : K_4 \hookrightarrow L^2(SU(3))^{SU(3)}$ be the actual character inclusion, $\Pi_4 = i_4 i_4^*$, and $W_4 = i_4^* M_w i_4$, where $w = \chi_{1,0}/3$. Into the closed charged Haar carrier of Section A.3, the natural coordinate map is

$$I_{R,4} = C_R i_4, \quad Q_{R,4} = I_{R,4} I_{R,4}^*.$$

This is an explicit isometry. The projection $Q_{R,4}$ is the chosen coordinate cutoff; it is not being identified with an unspecified complete Klauder projection. Direct substitution gives

$$I_{R,4}^* \pi_R(w) I_{R,4} = W_4, \quad \pi_R(w) I_{R,4} - I_{R,4} W_4 = C_R (1 - \Pi_4) M_w i_4. \quad (117)$$

The Pieri calculation in Proposition 6.2 evaluates the last map exactly:

$$(1 - \Pi_4) M_w i_4 = \frac{1}{3} \sum_{p+q=4} |\chi_{p+1,q}\rangle \langle \chi_{p,q}|.$$

Its norm is $1/3$, and its rank is five. Thus the named native cutoff generator is recovered by compression, but this coordinate map does not intertwine it with the full Haar Wilson multiplier. In particular, $Q_{R,4}$ fails the reduction test of Proposition A.2.

There is also a strict norm distinction, independent of numerical diagonalization. For every nonzero $\psi \in K_4$,

$$\|W_4 \psi\| \leq \|M_w \psi\| < \|\psi\|.$$

Indeed, $|w(g)| = 1$ only when all three eigenvalues of g agree, hence only at the finite center of $SU(3)$, a Haar-null set. Compactness of the unit sphere of K_4 then gives $\|W_4\| < 1$, whereas $\|\pi_R(w)\| = 1$. Diagonalizing the explicit fifteen-dimensional fusion matrix gives the supplementary numerical value $\|W_4\| \simeq 0.7489932012$. The exact norm inequality and the $1/3$ comparison defect do not depend on that numerical value.

For every $A \in B(K_4)$, the map $A \mapsto I_{R,A} A I_{R,A}^*$ is nevertheless a faithful normal $*$ -isomorphism onto $B(Q_{R,A} \mathcal{H}_R^{\text{loop}})$, viewed as a supported corner. It preserves the finite packet algebra in that sense. The image of W_4 is $Q_{R,A} \pi_R(w) Q_{R,A}$, not the original multiplier. Subdivision remains exact for this chosen cutoff: with the Haar seam unitary J , set $I_{R,A}^f = J I_{R,A}^c$. Then

$$Q_{R,A}^f J = J Q_{R,A}^c, \quad (I_{R,A}^f)^* M_{w(ryx)} I_{R,A}^f = W_4.$$

This computes the coordinate seam and its compressed observable; it does not identify the native scale arrow with a graph subdivision.

The charged line cannot be supplied by the bare class packet. Definition II.11 of [4] gives the retained characters the conjugation action. This action, including the center, is trivial. It explicitly contains no collar charge or boundary module. Therefore if A, B are any two of these bare carriers or their ground subspaces, and C_b has nonzero triality $b = 1, 2$, then

$$\text{Hom}_{Z_3}(A \hat{\otimes} C_b, B) = 0. \quad (118)$$

For a center-equivariant T , the generator $z = \omega I_3$ gives $T(1 \otimes \rho_b(z)) = T$, hence $(\omega^b - 1)T = 0$. This proves the displayed vanishing without a multiplicity assumption.

The linked line of (I.23),

$$L_{m,b} \subset \text{Hom}_{\text{Gauss}, \Omega, \text{cen}}(V_{j,m}^\Omega \hat{\otimes} C_b, V_{j+b,m}^\Omega),$$

is declared nonzero. Consequently its charged endpoint modules cannot both be replaced by the bare center-trivial character carriers with the action just tested. The source does not make that replacement: it introduces the linked endpoint package separately. Neither a scalar rephasing nor a genuinely neutral spectator repairs (118). One must identify the actual boundary action and the charged endpoint factors. Likewise the endpoint $R(g)$ in C_R is not by itself an identification of those native ground modules.

Scope of the comparison. The bare character cutoff cannot supply the charged line. The composed source in Section 2.2 supplies an actual charged packet, and Section 2.4 identifies its selector slant with the native lift multiplier and inverse line. These results do not identify it with the cyclic carry diagram (114); Section 2.3 excludes the ordinary cyclic-pullback identification.

C.2 Evaluation on the actual selected regional arrow

The physical scale arrow of Original Paper III is *not declared to be* the raw inclusion. Definitions II.11–II.12 use a package-adapted isometry class $U_{kj} : K_{L_j} \rightarrow K_{L_k}$, transporting complete-ground and physical-channel flags. Choose a representative on an admitted coefficient chart, retaining the source's compositor. For the compressed fundamental Wilson operator the one-cell matrix to evaluate is

$$K_{kj}^U = W_{L_k} U_{kj} - U_{kj} W_{L_j}. \quad (119)$$

The source also supplies an observable comparison c_{kj}^A . Its selected P3 construction, Definition II.53 and the DN4 proof of Theorem II.54 on printed pp. 63–65, asserts exact represented-action transport. Retain that assertion with its own observable map. If

$$\hat{W}_k = \pi_k(c_{kj}^A(W_{L_j})), \quad R_{kj}^{\text{act}} = \hat{W}_k U_{kj} - U_{kj} W_{L_j},$$

then the exact identity is

$$K_{kj}^U = (W_{L_k} - \widehat{W}_k)U_{kj} + R_{kj}^{\text{act}}. \quad (120)$$

On the source's asserted exact represented-action branch, $R_{kj}^{\text{act}} = 0$. The remaining expression compares the transported observable with the independently rebuilt cutoff operator on the physical comparison range. Exact action transport alone does not identify them.

Relative to the raw inclusion I_{kj} , put $B_{kj} = U_{kj} - I_{kj}$. Direct expansion gives the exact decomposition

$$K_{kj}^U = K_{kj}^I + W_{L_k}B_{kj} - B_{kj}W_{L_j}. \quad (121)$$

Thus the evaluated boundary term (62) is known, but cannot be substituted for the entire physical residual. Conversely, flag transport by itself does not cancel that term.

There is a precise regional reduction, with no new ground vector. Write $E_j(B)$ for the old labels and $G_{kj}(B) = \widehat{\bigotimes}_{\eta \in E_k(B) \setminus E_j(B)} \text{Ran } P_{\Omega, k, \eta}$. Up to the declared tensor ordering and coefficient line, the native arrow is

$$T_{kj}^B = \left(\widehat{\bigotimes}_{\lambda \in E_j(B)} U_{kj, \lambda} \right) \widehat{\otimes} \text{incl}_{G_{kj}(B)}.$$

For the inherited one-label Wilson action at λ , its rectangular residual is exactly

$$K_{kj, \lambda}^B = \left(\widehat{\bigotimes}_{\mu \neq \lambda} U_{kj, \mu} \right) \widehat{\otimes} K_{kj, \lambda}^U \widehat{\otimes} \text{incl}_{G_{kj}(B)}, \quad \|K_{kj, \lambda}^B\| = \|K_{kj, \lambda}^U\|. \quad (122)$$

The norm equality holds because every other factor is an isometry on a nonzero space. Added labels contribute no factor proportional to their number for this inherited single-label action. The physical issue has therefore reduced to the actual one-cell Wilson matrix comparison, rather than a new uncontrolled regional estimate.

More generally, for a word $W_{r,1} \cdots W_{r,d}$ of contractions at endpoint r , telescoping gives

$$W_{k,1} \cdots W_{k,d} T - T W_{j,1} \cdots W_{j,d} = \sum_{a=1}^d W_{k,1} \cdots W_{k,a-1} (W_{k,a} T - T W_{j,a}) W_{j,a+1} \cdots W_{j,d}. \quad (123)$$

Its norm is at most the sum of the generator residual norms. Adjoint letters require their own residuals, already covered in the raw computation above.

For any isometry U , set $p = UU^*$ and $A = U^*W_kU - W_j$. The same residual also has the orthogonal decomposition

$$K^U = UA + (1 - p)W_kU, \quad (K^U)^*K^U = A^*A + U^*W_k^*(1 - p)W_kU. \quad (124)$$

These are the compression discrepancy and the Wilson action escaping the comparison range. Neither term is the Hamiltonian ultraviolet–infrared mixed block. No buffered spectral no-leakage assertion has entered this calculation.

C.3 The first selected cutoffs cannot intertwine exactly

This distinction can be settled on the first selected arrow, without knowing the numerical entries of U_{32} . Definition II.9 of the source fixes $L_j = j + 2$, beginning at $j = 2$. Thus its first consecutive cutoffs are $L_2 = 4$ and $L_3 = 5$.

Proposition C.1 (A computed obstruction for the rebuilt Wilson generators). *Let W_4 and W_5 be the normalized compressed fundamental Wilson matrices defined above. Then*

$$\sigma(W_4) \cap \sigma(W_5) = \emptyset. \quad (125)$$

Consequently $W_5 U = U W_4$ forces $U = 0$ for every linear map $U : K_4 \rightarrow K_5$. In particular, the residual K_{32}^U is nonzero for every isometry, independently of which ground and channel flags it preserves.

Proof. Put $N_L = 3W_L$. In the orthonormal character basis its columns are given by (61), deleting indices outside the cutoff. Triality $p + 2q \pmod{3}$ increases by one under N_L . The three triality spaces have equal dimension five at $L = 4$, and seven at $L = 5$. Let C_L be N_L^3 restricted to triality zero. Cyclic block elimination gives

$$\det(xI - N_L) = \det(x^3 I - C_L).$$

Order the triality-zero basis by increasing p , then increasing q . At $L = 4$ it is $(0,0)$, $(0,3)$, $(1,1)$, $(2,2)$, $(3,0)$; at $L = 5$ it is $(0,0)$, $(0,3)$, $(1,1)$, $(1,4)$, $(2,2)$, $(3,0)$, $(4,1)$. Three applications of the displayed fusion rule give

$$C_4 = \begin{pmatrix} 1 & 1 & 2 & 0 & 0 \\ 0 & 3 & 2 & 3 & 1 \\ 2 & 3 & 6 & 3 & 2 \\ 0 & 2 & 3 & 3 & 3 \\ 1 & 0 & 3 & 2 & 3 \end{pmatrix}, \quad C_5 = \begin{pmatrix} 1 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 3 & 3 & 1 & 0 \\ 2 & 3 & 6 & 1 & 3 & 2 & 0 \\ 0 & 2 & 0 & 3 & 3 & 0 & 1 \\ 0 & 3 & 3 & 2 & 6 & 3 & 3 \\ 1 & 0 & 3 & 0 & 3 & 3 & 2 \\ 0 & 0 & 1 & 0 & 2 & 3 & 3 \end{pmatrix}. \quad (126)$$

Their characteristic polynomials are

$$\begin{aligned} P_4(y) &= y^5 - 16y^4 + 59y^3 - 67y^2 - 37y + 8, \\ P_5(y) &= y^7 - 25y^6 + 191y^5 - 559y^4 + 531y^3 - 507y^2 + 341y + 27. \end{aligned} \quad (127)$$

These determinant identities are finite integer calculations. For a short coprimality certificate, divide P_5 by P_4 over $\mathbb{F}_5$, then continue the Euclidean algorithm. The successive nonzero remainders are

$$2y^4 + 3y^3 + 3y^2 + 4y + 3, \quad y^2 + 4y + 3, \quad y + 2, \quad 4.$$

The monic integer polynomials are therefore coprime over $\mathbb{Q}$. A rational Bezout identity remains an identity after substituting $y = x^3$, so $P_4(x^3)$ and $P_5(x^3)$ are coprime as well. They are the characteristic polynomials of N_4 and N_5 . Their spectra, and hence those of W_4, W_5 , are disjoint.

If $N_5 U = U N_4$, then $P_4(N_5^3)U = U P_4(N_4^3) = 0$. The same Bezout identity and $P_5(N_5^3) = 0$ show that $P_4(N_5^3)$ is invertible. Thus $U = 0$. $\square$

There is even a positive lower bound for this single finite comparison. If v is a unit eigenvector of W_4 with eigenvalue λ , then every isometry satisfies

$$\|K_{32}^U\| \geq \|(W_5 - \lambda I)Uv\| \geq s_{\min}(W_5 - \lambda I) > 0.$$

Here $s_{\min}$ is the smallest singular value. This is a bound for the first arrow, with no assertion of a uniform bound along later cutoffs. It does not obstruct the strong limit (64).

The conclusion identifies the exact finite issue. The source's transported observable $c_{32}^A(W_4)$ can have exact represented action through its admitted correspondence. It cannot be identified with the independently rebuilt W_5 on that correspondence's range while preserving W_4 exactly. A proposed filling that changes the physical operator comparison must be evaluated on that comparison and its retained defects. Requiring every rebuilt finite-cutoff generator to intertwine strictly would impose a false condition already on the first selected arrow.

C.4 Where the filling and the two nerves enter this calculation

The actual mixed overlap equation of Proposition 5.1 transports (119) and (122) between admitted charts. Since its implementers are unitary, the residual norm is unchanged on overlaps. Rephasing U by a central unit scalar merely rephases K^U ; it cannot remove a nonzero residual. On a carrier-saturated spacetime cover, the native one-copy assembly identifies repeated old labels. The tensor expression (122) must then agree on those overlaps; it is not summed once for every appearance of a label.

The coefficient filling acts by (17) and (18). It aligns endpoint multipliers while preserving each endpoint's ordinary adjoint descent and the admitted mixed square. This coefficient comparison need not be an additive correction of the physical arrow.

If a further operator-valued realization of the relative primitive is proposed, it must act on the same physical comparison. The shell calculation supplies exact inherited transport with a trivial relative component; the raw cutoff calculation supplies an evaluated boundary residual and a strong limit in its Haar carrier. Neither identifies the native relative primitive with an operator correction of the package-adapted U_{kj} . Equation (120) retains the separate comparison between transported and rebuilt generators.

D The separated-shell realization

D.1 The source's separated-shell bordism

Original Paper II [3], Appendix C.9, gives

$$H_n = K^{\otimes n}, \quad V = \text{Ran } P, \quad J_{n;k} : H_n \widehat{\otimes} V^{\otimes k} \longrightarrow H_{n+k},$$

where $J_{n;k}$ is tensor reassociation followed by inclusion of the complete new-shell ground spaces. Let $W_n \in B(H_n)$ be an inherited Wilson operator and take its target extension to be $W_{n+k}^{\text{old}} = W_n \otimes I_{K^{\otimes k}}$. On elementary tensors,

$$W_{n+k}^{\text{old}} J_{n;k}(\psi \otimes v) = W_n \psi \otimes v \tag{128}$$

$$= J_{n;k}((W_n \otimes I_{V^{\otimes k}})(\psi \otimes v)). \tag{129}$$

Density gives $K_{n;k}^{\text{old}} = 0$, and applying the same calculation to W_n^* gives the adjoint residual zero. No unit vector in V was selected. In particular, the target range projection $p = I_{H_n} \otimes P^{\otimes k}$ reduces every inherited operator, while

$$J_{n;k}(W_n \otimes I) J_{n;k}^* = W_n \otimes P^{\otimes k}$$

is its supported corner action. The full target Wilson operator remains $W_n \otimes I$, including its action outside that range.

Theorem C.6 also proves the differently typed channel identities

$$J_{n;k}(F_n^W \otimes I) = F_{n+k}^W E_{n;k}, \quad J_{n;k}(F_n^{\text{phys}} \otimes I) = F_{n+k}^{\text{phys}} E_{n;k}. \quad (130)$$

Here $E_{n;k}$ embeds the old coefficient summands. These are identities for maps from coefficient spaces to physical carriers. Equation (129) is the separate operator identity. Newly added Wilson channels are additional observables, not an error in the transport of an inherited one.

On this depth branch, unital inclusions of the inherited finite observable algebras are $A \mapsto A \otimes I$. They give a C^* -inductive limit, and any specified representation π gives the W^* -algebra $\pi(\mathcal{A}_\infty)''$. Each fixed inherited observable is already stationary in that limit. This supplies an explicit existence mechanism at fixed shell data; it does not identify this representation with an arbitrary physical regional realization.

There is also a decisive source-side qualification about filling. Appendix E.6, equation (E10), chooses the relative component of this shell family trivial while retaining its nonzero marked coefficient-band class. Accordingly, this example demonstrates exact Wilson comparison and nontrivial coefficient descent together. It does not demonstrate that a nontrivial relative filling is necessary for the existence of the Wilson operators.

D.2 A native filled comparison with a nontrivial coefficient phase

This finite example puts the filling and physical Wilson action on one native diagram: the separated-shell $SU(2)$ family of Original Paper II [3], Appendices A, C, D, and I. Its relative refinement component is zero, while its marked parity-band class is nonzero. The coefficient filling compares two local lifting objects in that same twisted family.

The one-shell carrier is $K = \text{span}\{\chi_0, \chi_{1/2}, \chi_1, \chi_{3/2}\}$. In this ordered basis, the normalized fundamental Wilson operator and seed Hamiltonian are

$$W_\star = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad H_\star = \text{diag}(1, 4, 9, 16) - W_\star. \quad (131)$$

Thus $\|W_\star\| = \cos(\pi/5) < 1$. Let P be the source's complete rank-one ground projection of $H_\star$, and $V = \text{Ran } P$. For $1 \leq i < j \leq 4$, use the actual native objects

$$H_i = K^{\otimes i}, \quad L_{ij} = V^{\otimes(j-i)}, \quad J_{ij} : H_i \widehat{\otimes} L_{ij} \longrightarrow H_j, \quad W_i = W_\star \otimes I_{K^{\otimes(i-1)}}. \quad (132)$$

Here J_{ij} is the canonical reassociation and inclusion of the complete new-shell ground lines. The inherited observable is always the Wilson operator on the first shell.

The coefficient band is $\mathbb{Z}_2$, with grade $|i \rightarrow j| = (j - i) \bmod 2$ and $\Omega_1(g, h, k) = (-1)^{ghk}$. On the ambient depth simplex $0 < 1 < 2 < 3 < 4$, write its additive representative as

$$\omega[i, j, k, \ell] = \frac{1}{2}((j - i) \bmod 2)((k - j) \bmod 2)((\ell - k) \bmod 2) \quad \text{in } \mathbb{R}/\mathbb{Z}.$$

It satisfies $\delta\omega = 0$. On $E = \Delta\{1, 2, 3, 4\}$, define two local lifting objects by the source's rooted formulas:

$$\lambda_A[i, j, k] = \omega[0, i, j, k], \quad \lambda_B[i, j, k] = \begin{cases} 0, & i = 1, \\ \omega[1, i, j, k], & i > 1. \end{cases} \quad (133)$$

Both satisfy $\delta\lambda_A = \delta\lambda_B = \omega|_E$. The first is defined on E using the ambient apex; this does not assert a root-preserving chart morphism that changes the root.

Here the chosen degree-lowering coefficient comparison is explicitly $\text{trg}_{0,1}^{\text{root}} \omega = (h_0 - h_1)\omega|_E$, where h_r inserts the apex r . Its output is the native lift-transition cocycle $\theta = \lambda_A - \lambda_B$. The cone identities give $\delta\theta = 0$, and every value is evaluated in the following table:

$[i, j, k]$	$[1, 2, 3]$	$[1, 2, 4]$	$[1, 3, 4]$	$[2, 3, 4]$
λ_A	1/2	0	0	0
λ_B	0	0	0	1/2
θ	1/2	0	0	1/2

This specifies the comparison insertion and its target. It is a rooted lift comparison, rather than an unspecified identification with a loop-space transgression or the separate relative mapping-fiber primitive.

The unique native batch arrows $1 \rightarrow i$ give the comparison from the constant root diagram to the identity diagram; their physical components are J_{1i} , with the identity at $i = 1$. Retain their coefficient lines and the chosen lifting tensorators. On the depth nerve, this comparison gives the cone prism. For the filling, take $f_0 : E \rightarrow E$ constant at vertex 1, $f_1 = \text{id}_E$, and that normalized cone prism $H[v_0, \dots, v_n] = [1, v_0, \dots, v_n]$. Repeated vertices give zero. Then

$$f_{1\#} - f_{0\#} = \partial H + H\partial, \quad S = 0, \quad a = 0, \quad \beta = \theta H, \quad \delta\beta = \theta.$$

The complete list of edge values is

$$\beta_{23} = \frac{1}{2}, \quad \beta_{12} = \beta_{13} = \beta_{14} = \beta_{24} = \beta_{34} = 0. \quad (134)$$

Consequently $\kappa_{23} = -1$, with every other $\kappa_{ij} = e^{2\pi i \beta_{ij}}$ equal to 1. Writing $\Lambda_A = e^{2\pi i \lambda_A}$ and similarly for B , the source's transition equation is

$$\Lambda_A(i, j, k) = \frac{\kappa_{ik}}{\kappa_{ij}\kappa_{jk}} \Lambda_B(i, j, k). \quad (135)$$

Its inverse one-cochain differential convention agrees with the present calculation because all phases are signs. Equation (135) is exactly the one-morphism of the native lifting two-groupoid in Definition I.1 and Proposition I.3. On $[1, 2, 3]$ it reads $-1 = (-1) \cdot 1$; on $[2, 3, 4]$ it reads $1 = (-1) \cdot (-1)$.

The same transition acts on the actual coefficient line in (132). Put $q_{ij} = \kappa_{ij} I_{L_{ij}}$. The physical residual and its coefficient-transition comparison are

$$\begin{aligned} K_{ij} &= W_j J_{ij} - J_{ij}(W_i \otimes I_{L_{ij}}) = 0, \\ K_{ij}^q &= W_j J_{ij}(I \otimes q_{ij}) - J_{ij}(W_i \otimes q_{ij}) = K_{ij}(I \otimes q_{ij}) = 0. \end{aligned} \quad (136)$$

Indeed, the first-shell action commutes with adding the ordered ground-line factors. The same holds for adjoints. In particular $K_{23}^q = -K_{23} = 0$. The scalar belongs to the coefficient transition between lifting objects; its tensorators in (135) remain present. It is not a new physical refinement arrow or a global scalar encoding of the band associator.

There is a compatible physical state on this branch:

$$\omega_i(A) = \text{Tr}(P^{\otimes i} A), \quad \omega_j(A \otimes I) = \omega_i(A).$$

Thus every finite word in the inherited Wilson operator has the same expectation at all four depths and in both lifting charts. For example, $\omega_i(W_i) = \text{Tr}(PW_\star) \simeq 0.161205779266$; the equality across depths is exact. This evaluates coefficient filling, full-carrier Wilson transport, and state compatibility together.

The indices 1, 2, 3, 4 above are depth indices. Gauge-chart and spacetime-cover Čech indices remain separate and require their actual restriction and overlap functors. Original Paper II, Warning I.7, constructs a two-presheaf on rooted finite charts and expressly supplies no effective stack or continuum gerbe theorem. This example supplies an evaluated native local input to the mixed descent calculations of Section 5. The physical spacetime passage uses the full Ward overlap cancellation of Section 5.4 on its native prepared domain, together with the actual Wilson path and port identifications. Its scope is the fixed separated-shell branch. The phase filling relates local coefficient charts while the original band class remains nonzero; the physical Wilson residual vanishes because of the actual tensor action of J_{ij} .

E Relative fillings and optional operator realizations

E.1 Relative filling and its ambiguity

Let $Q : A^\bullet \rightarrow B^\bullet$ be a cochain map of additive complexes. Fix the allowed support, marking, and locality conditions in the definition of the complexes. If those conditions do not form a subcomplex, their satisfaction must be checked in addition to the cochain equations. Use the convention

$$\text{Fib}(Q)^k = A^k \oplus B^{k-1}, \quad D_Q(\alpha, \tau) = (d_A \alpha, Q\alpha - d_B \tau). \quad (137)$$

The identity $Qd_A = d_B Q$ gives $D_Q^2 = 0$.

Proposition E.1 (Filling criterion and its ambiguity). *For a fixed cocycle $\Delta \in Z^k(A)$, a filling $\tau \in B^{k-1}$ with*

$$Q\Delta = d_B \tau \quad (138)$$

exists in the specified complex exactly when $[Q\Delta] = 0$ in $H^k(B)$. If a filling exists, the set of fillings is a torsor over $Z^{k-1}(B)$. Modulo exact changes of the filling while Δ is fixed, the ambiguity is a torsor over $H^{k-1}(B)$.

Proof. The existence statement is the definition of exactness. Subtracting two instances of (138) shows that $\tau' - \tau$ is closed. Conversely, adding a closed cochain to one filling gives another. Taking the quotient by exact differences proves the last assertion. $\square$

The relative cocycle (Δ, τ) can have a nonzero class in $H^k(\text{Fib}(Q))$ even though $Q\Delta$ has been filled. Thus a filling does not assert that the entire relative class vanishes. When simultaneous changes of the raw representative are admitted, they have the form

$$(\Delta, \tau) \longmapsto (\Delta + d_A \lambda, \tau + Q\lambda - d_B v). \quad (139)$$

For a fixed raw cocycle, allowing closed λ further identifies filling classes by the image of $Q : H^{k-1}(A) \rightarrow H^{k-1}(B)$, whenever all these changes are permitted. One must use the equivalence relation of the physical source.

Original Paper II also carries the coefficient-line reassociator Ω , whose pentagon is a categorical cocycle equation. Neither (138) nor (139) identifies its class with the relative class. A proposed

transgression between them must supply a map, prove compatibility with differentials and permitted changes, and evaluate that map on the native data. The existence of τ is not itself that evaluation.

E.2 A geometric model of boundary-valued transgression

There is a standard setting in which filling surfaces have a precise boundary meaning. A smooth $U(1)$ -gerbe $\mathcal{G}$ with connection on X transgresses to a line bundle $L_{\mathcal{G}}$ with connection on LX , retaining a fusion product [15, 16]. On characteristic classes,

$$H^3(X; \mathbb{Z}) \longrightarrow H^2(LX; \mathbb{Z}), \quad h \longmapsto \int_{S^1} \text{ev}^* h. \quad (140)$$

A cylinder of loops supplies parallel transport between its two boundary lines. For two cylinders with the same oriented boundaries, the ratio of their transports is the gerbe holonomy on the closed surface obtained by gluing one to the reverse of the other. This follows from the surface-holonomy gluing law. Consequently, retaining the boundary line gives a coherent object without choosing a global scalar trivialization; numerical independence of a filling requires the appropriate closed-surface phases to cancel or be trivial on the admitted comparisons.

This model explains why coefficient-valued boundary data can be essential. It does not identify the Yang–Mills coefficient band with this gerbe, nor does it turn an ordinary non-Abelian Wilson loop into gerbe surface holonomy. Those identifications would require additional maps. The smooth transport-functor formulation provides a related geometric comparison [17]; its regularity assumptions do not establish quantum ultraviolet regularity.

For a differential gerbe with curvature $H_{\mathcal{G}}$, normalized in $\mathbb{R}/\mathbb{Z}$ units,

$$\text{Hol}_{\mathcal{G}}(\partial B) = \exp\left(2\pi i \int_B H_{\mathcal{G}}\right).$$

A bounding surface therefore need not have trivial differential holonomy. This curvature datum is additional to the flat simplicial cocycle calculations of the main text.

E.3 An additive model for an observable correction

The coefficient filling in Section 3.1 acts directly on overlap phases. A different proposal may use a relative primitive to correct observable expressions. The statement below is a sufficient additive model for that additional realization. It is not a universal requirement for coefficient sewing or Wilson descent.

Let $\mathcal{D}$ be a selected diagram of settings. At $s \in \mathcal{D}$, let $\mathcal{A}_s$ be a normed vector space of admitted observable expressions. An arrow $a : s \rightarrow t$ has a specified bounded linear comparison $T_a : \mathcal{A}_s \rightarrow \mathcal{A}_t$. Define

$$C_T^0 = \prod_s \mathcal{A}_s, \quad C_T^1 = \prod_{a:s \rightarrow t} \mathcal{A}_t, \quad (\delta_T w)_a = w_t - T_a w_s. \quad (141)$$

This two-term construction does not assume that arbitrary comparison maps form a strict functor. If a full higher cochain complex is used, its higher differential must incorporate the actual comparison cells.

Lemma E.2 (Represented filling and residual). *Fix $\Delta \in Z^k(A)$ and a filling τ satisfying (138). Suppose there are additive realization maps*

$$R_0 : B^{k-1} \longrightarrow C_T^0, \quad R_1 : B^k \longrightarrow C_T^1, \quad \delta_T R_0 = R_1 d_B. \quad (142)$$

Suppose the actual observable comparison has the identity

$$\delta_T w = R_1 Q \Delta + r. \quad (143)$$

Then the corrected section $\tilde{w} = w - R_0 \tau$ satisfies

$$\delta_T \tilde{w} = r. \quad (144)$$

If τ' fills the same Δ , the two corrected sections differ by a δ_T -closed section.

Proof. Use (142) and (138) to compute

$$\delta_T(w - R_0 \tau) = R_1 Q \Delta + r - R_1 d_B \tau = r.$$

For two fillings, $d_B(\tau' - \tau) = 0$, hence $\delta_T R_0(\tau' - \tau) = 0$. □

The content of an application lies in (142) and (143). The first requires the admissible filling to have a representation on the same observables being compared. The second identifies that representation with their actual defect. Merely knowing the relative complex and the Wilson action separately supplies neither identity. No cohomological degree has been silently reassigned: R_0, R_1 explicitly record which two degrees are being realized as vertex and arrow data.

Even when $r = 0$, the filling need not select a unique section. A nonzero closed section can be added. One therefore needs boundary normalization or an admitted equivalence that identifies precisely the allowed differences. If R_0 is continuous in chosen seminorms, an estimate of τ gives an estimate of the correction. Algebraic existence alone gives no such bound.

For Wilson observables, the additive model has one further limitation. The corrected expression need not be a normalized character of a unitary holonomy, and arbitrary additive repair need not preserve Wilson-word relations or the physical state. An application must prove that the prescribed boundary evaluation is the corrected object, with its required relations and normalization. Otherwise Lemma E.2 constructs a compatible observable expression, not the claimed Wilson observable. A nonlinear or correspondence-valued realization can replace the additive model, but then its actual comparison identity must be proved. In particular, no global logarithm or additive realization of a multiplicative coefficient band is presumed here.

E.4 Overlap test for an optional additive correction

Suppose one proposes the additive observable realization of Lemma E.2. Its overlap compatibility is a separate test. Let $w_{s,i}$ be local observable expressions and let $c_{s,i} = R_{0,i} \tau_i$ be the represented local filling correction of Lemma E.2. Define the two overlap mismatches

$$\begin{aligned} b_{ij}^s &= r_{i,ij}^s(w_{s,i}) - r_{j,ij}^s(w_{s,j}), \\ h_{ij}^s &= r_{i,ij}^s(c_{s,i}) - r_{j,ij}^s(c_{s,j}). \end{aligned} \quad (145)$$

These are elements of the same declared overlap algebra.

Proposition E.3 (Overlap test for the corrected observable). *The family $\tilde{w}_{s,i} = w_{s,i} - c_{s,i}$ belongs to $\mathcal{D}_s(\mathcal{U})$ exactly when $b_{ij}^s = h_{ij}^s$ for every overlap. If this holds at both endpoints of a , if (52) holds, and if local represented-filling identities give*

$$\tilde{w}_{t,i} - \Phi_{a,i}(\tilde{w}_{s,i}) = r_{a,i},$$

then $r_a = (r_{a,i})_i \in \mathcal{D}_t(\mathcal{U})$ and the same identity holds in the descended algebras.

Proof. The overlap difference of $\tilde{w}_s$ is $b_{ij}^s - h_{ij}^s$, proving the first assertion. Theorem 5.3 places both terms of the scale difference in $\mathcal{D}_t(\mathcal{U})$; their difference is precisely the stated residual family. $\square$

If the uncorrected Wilson family already descends, the test reduces to $h_{ij}^s = 0$. The local cochain identity $d_B \tau_i = Q \Delta_i$ does not by itself imply this equation. Even when the difference of two local fillings is closed after comparison, its represented value can be a nonzero closed observable section. The native realization must show that the overlap mismatch cancels or belongs to the declared equivalence. If its natural residual is the rectangular operator K_a in (40), the realization must take values in those morphism spaces; an algebra-valued subtraction is then not a substitute for the typed identity.

Changing the relative filling. For $d_B \tau = d_B \tau' = Q \Delta$, set $h = \tau' - \tau \in Z^{k-1}(B)$. The relative filling ambiguity is exactly that of Proposition E.1, with the source's admitted equivalences. It has no automatic action on the Wilson operator. If the optional additive realization of Section E.3 is supplied, it gives the precise comparison

$$\tilde{w}_{\tau'} - \tilde{w}_{\tau} = -R_0 h, \quad \delta_T(R_0 h) = 0. \quad (146)$$

The two represented observables are equal exactly when $R_0 h = 0$ in the declared faithful realization, or when a separately specified observable quotient identifies this difference with zero. Closedness alone gives a compatible difference, not a vanishing difference. Even an exact change $h = d_B v$ requires $R_0 d_B v = 0$, or its explicitly admitted equivalence, to be invisible in that realization.

E.5 Wilson evaluation of a physical composition correction

The source's composition defect also has a direct Wilson evaluation. Consider two composable admitted arrows $a : s \rightarrow t$, $b : t \rightarrow u$, their direct arrow ba , and their actual represented maps. Insert the declared coefficient identifications first, so every composite below has the same source coefficient domain. Suppress only the resulting tensor identities. Let V represent the target comparison $U_{\chi_{b,a}}$ in (33), and set

$$D_{b,a} = V J_b J_a - J_{ba}, \quad K_a = W_t J_a - J_a W_s, \quad (147)$$

with K_b and K_{ba} defined at their own endpoints in the same way. Expanding both products gives

$$K_{ba} = V(K_b J_a + J_b K_a) + [W_u, V] J_b J_a - (W_u D_{b,a} - D_{b,a} W_s). \quad (148)$$

Indeed, $K_b J_a + J_b K_a = W_u J_b J_a - J_b J_a W_s$; adding the commutator moves W_u past V , and subtracting the last parenthesis replaces $V J_b J_a$ by J_{ba} . No physical arrow is inverted.

Thus the represented correction F , when it is an actual operator between the same endpoint carriers, has the typed Wilson evaluation

$$\mathcal{T}_W(F) = W_u F - F W_s, \quad \|\mathcal{T}_W(F)\| \leq 2\|F\| \quad (\|W_s\|, \|W_u\| \leq 1). \quad (149)$$

Equation (148) computes the contribution of a physical composition correction to its Wilson boundary comparison. If V acts only by a central coefficient phase, the commutator term vanishes. For a general physical comparison it must be retained. Finite coherence does not justify discarding it.

This evaluation is compatible with the actual chart overlaps. If a correction has overlap law $u_{ij}^u F_j = F_i v_{ij}^s$, and the endpoint Wilson actions obey their overlap laws, direct multiplication gives

$$u_{ij}^u \mathcal{T}_{W,j}(F_j) = \mathcal{T}_{W,i}(F_i) v_{ij}^s. \quad (150)$$

It can therefore be checked on the gauge-chart nerve and then on the admitted spacetime-cover nerve in the same morphism spaces as the Wilson residual. The coefficient associator remains in the higher composition data.

An additional identification of a relative filling with a physical composition correction would have to reproduce these identities. It is not required by the coefficient prism filling of Section 3.1. Controlling a difference between refinement histories also does not require every rebuilt-generator residual to vanish, as Proposition C.1 shows.

E.6 Test of a proposed arrow-valued filling

The additive proposal can be tested precisely. Work on the underlying one-cell analytic carriers, retaining the coefficient decoration separately. For the strict raw inclusions, $I_{\ell k} I_{kj} = I_{\ell j}$. Put $C_{kj} = U_{kj} - I_{kj}$, and define

$$(\partial_I C)_{\ell kj} = I_{\ell k} C_{kj} - C_{\ell j} + C_{\ell k} I_{kj}. \quad (151)$$

Expansion gives the physical composition defect

$$D_{\ell kj}^U := U_{\ell k} U_{kj} - U_{\ell j} = (\partial_I C)_{\ell kj} + C_{\ell k} C_{kj}. \quad (152)$$

Identifying a degree-one relative primitive with C , its boundary with D^U , and its differential with ∂_I therefore requires $C \smile C = 0$, or an evaluation that accounts for the quadratic term. This is a test of that proposed identification. It is not a requirement on the prism primitive β , whose scalar phase target is additive.

For rebuilt Wilson generators, set $(\mathcal{T}_W C)_{kj} = W_k C_{kj} - C_{kj} W_j$ and $K_{kj}^I = W_k I_{kj} - I_{kj} W_j$. Direct expansion yields

$$\begin{aligned} (\partial_I \mathcal{T}_W C)_{\ell kj} - \mathcal{T}_W((\partial_I C)_{\ell kj}) \\ = -K_{\ell k}^I C_{kj} - C_{\ell k} K_{kj}^I. \end{aligned} \quad (153)$$

On the raw $SU(3)$ windows, $K_{kj}^I = S_{kj}/3$, with S_{kj} the norm-one boundary shift. The right side has norm at most

$$\frac{1}{3}(\|C_{kj}\| + \|C_{\ell k}\|). \quad (154)$$

No vanishing or cofinal decay of this expression is asserted.

Replacing I with U in the degree-one differential gives

$$(\partial_U K^U)_{\ell kj} = W_\ell D_{\ell kj}^U - D_{\ell kj}^U W_j, \quad (155)$$

where $K_{kj}^U = W_k U_{kj} - U_{kj} W_j$. For vertex operators, use $(\partial_U X)_{kj} = U_{kj} X_j - X_k U_{kj}$, so $K^U = -\partial_U W$. Then $(\partial_U^2 X)_{\ell kj} = D_{\ell kj}^U X_j - X_\ell D_{\ell kj}^U$. Cancellation of the intermediate products proves these identities. Thus arbitrary physical transport cannot silently replace a cochain differential. Overlap covariance follows by transporting both I and U with the endpoint unitaries. The native relative mapping-fiber datum remains at its source type until a specific additional realization is supplied.

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