Admissible Witness Descent for Higher Gauge Fields

J. Scott Little
Independent Researcher, Columbus, Ohio, USA

Original manuscript: August 23, 2026
Website Edition v1.0: September 8, 2026

Abstract

Higher gauge fields are naturally organized by higher stacks, and their usual local-to-global passage is governed by descent. We formulate a relative question internal to that established setting: when only a declared subdiagram of the local descent data is retained, what does that selected information determine? Given an n-truncated higher stack $\mathcal{F}$, a cover $\mathcal{U}$ of a spacetime or manifold M, and an admitted witness functor $j\colon \mathsf{W} \to \mathsf{C}_{\mathcal{U}}$ into the Čech indexing category, we define the witness reconstruction object

$$\mathrm{Rec}_{\mathsf{W}}(\mathcal{F}) := \operatorname*{holim}_{w \in \mathsf{W}} D_{\mathcal{U}}(j(w))$$

and the comparison map

$$\rho_{\mathsf{W}}\colon \mathcal{F}(M) \longrightarrow \mathrm{Rec}_{\mathsf{W}}(\mathcal{F}).$$

Its homotopy fiber is a common carrier for failure of globalization, ambiguity of globalization, and gauge symmetry invisible to the admitted witnesses. Truncated homotopy-initiality supplies a standard diagram-independent sufficient criterion; the paper's calculations begin when the diagram-specific relative fiber must instead be computed. For abelian bundle gerbes with connection, a shifted mapping cone calculates all three defect layers. A missing tetrahedral coherence gives a sharp $U(1)$ obstruction, deletion of a dominated patch star gives an explicit sparse reconstruction certificate, and barycentric subdivision distributes one invariant obstruction among 24 local coherence tests without multiplying it. For crossed-module gauge fields, where no single additive cone encodes the nonlinear cocycle equations, a transported filler tower organizes successive completion; a split crossed module with vanishing Postnikov class nevertheless has a nontrivial obstruction generated by the π_1-action on π_2. Witnesswise-complete refinement preserves the relative defect. Conditional BRST/BV and Lorentzian Yang–Mills applications then identify the differential, multiplicative, equation-bundle, and coherent arrow data that must themselves descend. An adversarial audit proves that raw witness counts are not invariants, universal finite cores need not exist, and nonempty connected fillers do not imply reconstruction. No quantization, anomaly-cancellation, mass-gap, or confinement claim is made.

Keywords. Higher stacks; descent; higher gauge theory; bundle gerbes; homotopy limits; mapping cones; crossed modules; local coefficients; BRST/BV; algebraic quantum field theory.

2020 Mathematics Subject Classification. Primary 18F20; secondary 18N60, 55N30, 55N25, 81T13, 81T05.

Publication notice. Independent research article; public website edition. This version preserves the frozen theorem spine of the August 23, 2026 release candidate.

Contents

1 Introduction

1.1 The question after descent

The modern geometric formulation of gauge theory makes the local-to-global problem precise. Principal bundles, connections, bundle gerbes, higher connections, gauge transformations, and gauge-of-gauge transformations are assembled into stacks or higher stacks. On a cover $\mathcal{U} = \{U_i \to M\}$, the global field object is recovered from a homotopy limit of its Čech descent diagram. In this form, descent does not merely glue coordinate representatives; it retains the full homotopy type of fields and their symmetries.

This paper begins one step later. Suppose that the full Čech diagram exists, but only some of its objects, overlaps, higher overlaps, and coherence morphisms are declared to be admitted reconstruction data. Such a selection may arise from a finite presentation, a restricted family of local probes, a computational reduction, a boundary or defect interface, or a deliberate comparison between two local descriptions. The question is then not whether the field theory is a stack. The question is:

> Which selected portions of a descent diagram determine the global higher gauge field, and what precisely remains undetermined when they do not?

The answer cannot depend only on the number of retained pieces. Removing a redundant patch may change nothing, while omitting a single higher coherence may destroy globalization. The relevant object is the homotopy type of the admitted diagram together with its coefficient transport and differential structure.

1.2 Relation to established frameworks

Higher principal bundles and their cocycles are naturally treated in smooth ∞-toposes and higher nonabelian cohomology; higher connections and differential characteristic maps are described by differential cohomology on higher moduli stacks [13, 8, 16]. These works supply the higher-geometric environment used here. The present paper does not propose a replacement for higher-stack descent. It studies the comparison between full descent and a declared subdiagram of that descent.

For ordinary Yang–Mills theory, Benini, Schenkel, and Schreiber constructed the stack of nonabelian Yang–Mills fields on globally hyperbolic Lorentzian manifolds and formulated a stacky Cauchy problem [4]. Homotopical and fibered formulations of algebraic quantum field theory provide a complementary setting for derived observables and theories on categories carrying extra geometric data [3, 5]. We use these results as the global field-theoretic input. Our additional question is which admitted local witnesses reconstruct their field and solution objects.

The quantum interface is BRST/BV rather than a coherent-state or projection quantization. Fredenhagen and Rejzner's perturbative algebraic formulation of the renormalized BV complex identifies the anomaly term with the anomalous Master Ward Identity, while Rejzner emphasizes locality, deformation, and homology as structural reasons for the BV description of gauge theories [9, 15]. In the present paper, these ideas enter only through a conditional Ward-compatible descent statement: local BV complexes must descend together with their BRST/BV differential and its coherent compatibility with restriction.

1.3 Contributions

The standard inputs are higher-stack descent, truncated cofinality, abelian mapping-cone obstruction theory, crossed-module obstruction theory, and derived BRST/BV totalization. The paper does not rename these as a new descent formalism. Its contribution is to place a declared witness diagram between full descent and the global field object, organize the resulting relative fiber uniformly, and compute it in finite examples where omitted coherence and transported coefficients are visible. More precisely, the paper makes six principal contributions.

(i) It formulates admitted witness reconstruction as a relative homotopy-limit problem inside an existing higher-stack descent diagram. Its reconstruction fiber simultaneously records nonexistence, inequivalent globalization, and witness-invisible higher symmetry. Standard homotopy-initiality technology then gives a diagram-independent sufficient criterion.

(ii) For $U(1)$-bundle gerbes with connection, it constructs the admitted derived Čech–Deligne totalization. The shifted cone of restriction computes the obstruction, ambiguity, and relative-symmetry layers of the same reconstruction fiber.

(iii) It provides complementary finite calculations. The boundary of a tetrahedron without its tetrahedral coherence carries a sharp $U(1)$ obstruction, while deletion of a dominated patch star produces an explicit chain contraction and sparse reconstruction certificate. Under subdivision, the 24 refined tetrahedral defects combine into one alternating product representing the original relative class.

(iv) For strict crossed-module gauge fields, it organizes reconstruction by a completion filtration and successive transported homotopy filler spaces rather than a single abelianized cone. The crossed module

$$U(1) \xrightarrow{0} \mathbb{Z}_2, \qquad t \triangleright z = z^{-1},$$

shows that nontrivial coefficient transport may obstruct reconstruction even when the Postnikov class vanishes.

(v) It proves stability of the reconstruction defect under witnesswise-complete refinement. Its Yang–Mills solution-stack and BRST/BV Ward results are conditional interfaces specifying the additional equation, differential, multiplicative, and coherent transport data required for those applications.

(vi) It stress-tests three tempting strengthenings: raw witness counts are not presentation invariants, universal finite reconstruction cores need not exist, and nonempty connected nonabelian filler spaces do not suffice for unique reconstruction.

The conceptual conclusion is concise:

The quantity of retained local data does not determine reconstructibility; its admitted homotopy type does.

1.4 Scope and nonclaims

The word *admitted* has a deliberately narrow meaning in this paper. It means included in the declared reconstruction diagram, together with the restriction morphisms and coherences required to make that diagram well typed. No independent physical ontology is attached to the term.

The higher-gerbe calculations are higher-gauge test cases. Ordinary Yang–Mills connections form a 1-stack and therefore have a different coskeletal bound. The BRST/BV discussion is conditional on the existence of the relevant local complexes, quantum master equation, renormalization scheme, and homotopy-coherent restriction maps. In particular, this paper does not:

- construct a nonperturbative quantum Yang–Mills theory;
- prove cancellation of a gauge anomaly;
- prove a mass gap, confinement, or Haag–Kastler realization;
- identify higher-gauge degree with BRST ghost degree; or
- replace full stack descent by a smaller universal framework.

1.5 Organization

Section 2 fixes the higher-stack, Čech, and witness conventions. Section 3 constructs the comparison and reconstruction fiber. Section 4 develops the abelian Čech–Deligne defect and its finite examples. Section 5 treats crossed modules by filtered fillers. Section 6 proves refinement stability. Section 7 introduces the separate BRST/BV grading and Ward-compatible descent. Section 8 applies the construction to the classical Lorentzian Yang–Mills solution stack. Section 9 gives the adversarial audit of minimality, finite cores, and existence-only completion. Section 10 separates the standard inputs from the paper's reconstruction content and records the current open frontier. The appendices collect orientation, sign, and cellular calculations.

2 Foundational setup

2.1 Site, cover, and truncated higher stack

Let $(\mathcal{C}, \tau)$ be a Grothendieck site and let $M \in \mathcal{C}$. Fix a cover $\mathcal{U} = \{U_i \to M\}_{i \in I}$. Its simplicial Čech object has degree-p term

$$U_p = \coprod_{(i_0,\dots,i_p)\in I^{p+1}} U_{i_0\cdots i_p}, \qquad U_{i_0\cdots i_p} := U_{i_0} \times_M \cdots \times_M U_{i_p}.$$

We write $\mathsf{C}_{\mathcal{U}}$ for a chosen small ∞-category presenting the associated descent diagram. An ordinary simplicial, semi-simplicial, cellular, or nondegenerate-intersection indexing category is always inserted through its simplicial nerve. Displayed 1-categorical diagrams are strict presentations of the corresponding homotopy-coherent diagrams. Different presentations are allowed only when their homotopy limits compute the same totalization compatibly with the descent comparison.

Let $\mathcal{F}$ be an n-truncated higher stack on $(\mathcal{C}, \tau)$, valued in an ambient ∞-category $\mathcal{X}$ admitting the required homotopy limits. Evaluation on the Čech object gives a diagram

$$D_{\mathcal{U}} \colon \mathsf{C}_{\mathcal{U}} \longrightarrow \mathcal{X}.$$

Descent supplies an equivalence

$$\mathcal{F}(M) \simeq \operatorname*{holim}_{c \in \mathsf{C}_{\mathcal{U}}} D_{\mathcal{U}}(c).$$

Warning 2.1 (Orientation)*.* Because reconstruction is expressed by a homotopy *limit*, the relevant cofinality notion is homotopy initiality. All truncated connectivity shifts and comma-category conventions will be audited explicitly in Appendix A; the paper will not alternate informally between "initial" and "final."

2.2 Admitted witness systems

Definition 2.2 (Admitted witness system)**.** An admitted witness system for $(\mathcal{F}, \mathcal{U})$ is a small ∞-category W together with a functor

$$j \colon \mathsf{W} \longrightarrow \mathsf{C}_{\mathcal{U}}.$$

The functor records not only which local intersections are retained, but also which restriction maps, gauge transformations, higher transformations, and coherence morphisms are part of the declared reconstruction diagram.

Definition 2.3 (Witness reconstruction)**.** The reconstruction object associated with W is

$$\mathrm{Rec}_{\mathsf{W}}(\mathcal{F}) := \operatorname*{holim}_{w \in \mathsf{W}} D_{\mathcal{U}}(j(w)).$$

The descent equivalence followed by restriction of the indexing diagram defines the comparison map

$$\rho_{\mathsf{W}} \colon \mathcal{F}(M) \longrightarrow \mathrm{Rec}_{\mathsf{W}}(\mathcal{F}).$$

Definition 2.4 (Reconstruction fiber)**.** For $y \in \mathrm{Rec}_{\mathsf{W}}(\mathcal{F})$, define

$$\mathfrak{R}_{\mathsf{W}}(y) := \mathrm{hofib}_y(\rho_{\mathsf{W}}).$$

The witness system is *complete for* y when this fiber is contractible, and *complete for* $\mathcal{F}$ when ρ_{W} is an equivalence.

The interpretation of the homotopy groups is intrinsic to the comparison:

$$\begin{aligned} \mathfrak{R}_{\mathsf{W}}(y) = \varnothing \quad &\iff \quad y \text{ has no globalization}, \\ \pi_0 \mathfrak{R}_{\mathsf{W}}(y) \quad &= \quad \text{globalization ambiguity}, \\ \pi_k \mathfrak{R}_{\mathsf{W}}(y),\ k \geq 1 \quad &= \quad \text{higher automorphisms invisible to } \mathsf{W}. \end{aligned}$$

When a globalization x and an identification $\alpha\colon \rho_{\mathsf{W}}(x) \simeq y$ have been chosen, the relative automorphism space is the based loop space

$$\mathrm{Aut}_{\mathrm{rel}}(x, \alpha; \mathsf{W}) \simeq \Omega \mathfrak{R}_{\mathsf{W}}(y).$$

2.3 Three gradings that must remain distinct

The applications use up to three independent degrees. We write

$$C^{p,q,r},$$

where p is Čech degree, q is internal higher-gauge or Deligne degree, and r is BRST/BV ghost degree. Under strict pairwise compatibility, a total differential has the schematic form

$$D_{\mathrm{tot}} = \delta_{\check{C}} + (-1)^p d_{\mathrm{hg}} + (-1)^{p+q} s_{\mathrm{BRST}}.$$

For homotopy-coherent compatibility, this expression is shorthand for the corresponding derived totalization with its higher coherence terms.

Warning 2.5. Higher-gauge degree records geometric gluing and gauge-of-gauge structure. BRST/BV degree resolves gauge redundancy and grades ghosts, antifields, and their differentials. They interact in a totalized complex but are not the same degree and will not be identified.

3 Admitted witness comparison and reconstruction fibers

3.1 The limit orientation

Let $j\colon \mathsf{W} \to \mathsf{C}$ be a functor of small ∞-categories. For $c \in \mathsf{C}$, define the comma ∞-category

$$(j \downarrow c) := \mathsf{W} \times_{\mathsf{C}} \mathsf{C}_{/c}.$$

Its objects are pairs (w, α) with $\alpha\colon j(w) \to c$. We write $|(j \downarrow c)|$ for its groupoid completion. This is the comma construction appropriate to limits. The opposite comma category, built from arrows $c \to j(w)$, is the one that controls colimits.

We use the connective convention standard in higher category theory: a space K is r-connective when $\tau_{\leq r-1} K \simeq *$. Thus 0-connective means nonempty, 1-connective means connected, and 2-connective means connected and simply connected.

Definition 3.1 (Truncated limit-initiality)**.** For $n \geq -1$, the functor j is *n-limit-initial* when, for every $c \in \mathsf{C}$, the space $|(j \downarrow c)|$ is $(n+1)$-connective. Equivalently, j is left $(n+1)$-cofinal in the truncated cofinality convention of [7]. For $n = \infty$, we require these comma spaces to be weakly contractible, recovering ordinary homotopy initiality [12].

The shift by one in Definition 3.1 is essential. A diagram of n-truncated objects takes values in an $(n+1,1)$-category because its mapping spaces are n-truncated. The low-degree cases are therefore

target values	required comma space	ordinary description
(-1)-truncated	0-connective	nonempty
0-truncated	1-connective	connected
1-truncated	2-connective	simply connected
2-truncated	3-connective	$\pi_0 = *,\ \pi_1 = \pi_2 = 0.$

Theorem 3.2 (Truncated witness comparison)**.** *Let $\mathcal{X}$ be an ∞-category admitting limits indexed by C and W. Let*

$$D\colon \mathsf{C} \longrightarrow \mathcal{X}$$

take values in n-truncated objects, and let $j\colon \mathsf{W} \to \mathsf{C}$ be n-limit-initial. Then restriction of cones induces an equivalence

$$\lim_{c\in\mathsf{C}} D(c) \xrightarrow{\simeq} \lim_{w\in\mathsf{W}} D(j(w)).$$

Proof. Let $T \in \mathcal{X}$. Since every $D(c)$ is n-truncated, each mapping space $\mathrm{Map}_{\mathcal{X}}(T, D(c))$ is n-truncated. Mapping out of T preserves limits, so there are natural equivalences

$$\begin{aligned}\mathrm{Map}_{\mathcal{X}}\Big(T, \lim_{\mathsf{C}} D\Big) &\simeq \lim_{c\in\mathsf{C}} \mathrm{Map}_{\mathcal{X}}(T, D(c)),\\ \mathrm{Map}_{\mathcal{X}}\Big(T, \lim_{\mathsf{W}} D\circ j\Big) &\simeq \lim_{w\in\mathsf{W}} \mathrm{Map}_{\mathcal{X}}(T, D(j(w))).\end{aligned}$$

The ∞-category of n-truncated spaces is an $(n+1,1)$-category. The truncated Quillen–A criterion and the n-cofinality theorem, applied in the limit orientation, identify the two limits of mapping spaces because every $|(j{\downarrow}c)|$ is $(n+1)$-connective [7]. The resulting equivalence is natural in T; the ∞-categorical Yoneda lemma gives the asserted equivalence in $\mathcal{X}$. ☐

Proposition 3.3 (Sharpness of the connectivity shift)**.** *For every $n \geq 0$, n-connectivity of the comma spaces is insufficient for preservation of all limits of n-truncated diagrams.*

Proof. Let $j\colon S^n \to *$ be the unique functor from an ∞-groupoid to the terminal category. Its sole comma space is S^n, which is n-connective but not $(n+1)$-connective. If $n=0$, take a set X with at least two elements; the comparison is the diagonal $X \to X \times X$ and is not an isomorphism. If $n \geq 1$, take a nonzero abelian group A and the constant diagram with value $K(A,n)$. The comparison map is

$$K(A,n) \longrightarrow \mathrm{Map}(S^n, K(A,n)).$$

The target has components indexed by $[S^n, K(A,n)] \cong A$, so the comparison is not an equivalence. Thus the $(n+1)$-connectivity in Definition 3.1 cannot be lowered uniformly. ☐

Theorem 3.4 (Witness comparison)**.** *Let $\mathcal{F}$ be an n-truncated higher stack, let $\mathcal{U}$ be a cover of M, and let $j\colon \mathsf{W} \to \mathsf{C}_{\mathcal{U}}$ be n-limit-initial. Then*

$$\rho_{\mathsf{W}}\colon \mathcal{F}(M) \xrightarrow{\simeq} \mathrm{Rec}_{\mathsf{W}}(\mathcal{F})$$

is an equivalence.

Proof. Descent identifies $\mathcal{F}(M)$ with $\lim_{\mathsf{C}_{\mathcal{U}}} D_{\mathcal{U}}$. The values of $D_{\mathcal{U}}$ are n-truncated. Apply Theorem 3.2 to the admitted witness functor j. ☐

Remark 3.5 (What is standard and what is being applied). The proof of Theorem 3.4 is truncated homotopy-cofinality, not a new descent theorem. Its role here is to turn selected higher-gauge descent data into a checkable reconstruction criterion. The new calculations begin when this sufficient criterion fails and the relative reconstruction defect is computed.

3.2 The reconstruction fiber

For the rest of this subsection, the higher stack is groupoid valued, so both $\mathcal{F}(M)$ and $\mathrm{Rec}_{\mathsf{W}}(\mathcal{F})$ are spaces. Given $y \in \mathrm{Rec}_{\mathsf{W}}(\mathcal{F})$, the reconstruction fiber

$$\mathfrak{R}_{\mathsf{W}}(y) = \mathrm{hofib}_y(\rho_{\mathsf{W}})$$

is the space of pairs (x, α) where $x \in \mathcal{F}(M)$ and $\alpha \colon \rho_{\mathsf{W}}(x) \simeq y$ is an identification of its admitted restriction with the declared witness datum.

Proposition 3.6 (Exact reconstruction sequence)**.** *Choose a point $(x, \alpha) \in \mathfrak{R}_{\mathsf{W}}(y)$. There is a natural long exact sequence*

$$\cdots \longrightarrow \pi_{k+1} \mathrm{Rec}_{\mathsf{W}}(\mathcal{F}, y) \longrightarrow \pi_k \mathfrak{R}_{\mathsf{W}}(y; x, \alpha) \longrightarrow \pi_k \mathcal{F}(M, x) \longrightarrow \pi_k \mathrm{Rec}_{\mathsf{W}}(\mathcal{F}, y) \longrightarrow \cdots .$$

In degrees zero and one it continues as the usual exact sequence of groups, pointed sets, and components associated with a homotopy fiber.

Proof. This is the homotopy long exact sequence of the fiber sequence based at (x, α),

$$\mathfrak{R}_{\mathsf{W}}(y) \longrightarrow \mathcal{F}(M) \xrightarrow{\rho_{\mathsf{W}}} \mathrm{Rec}_{\mathsf{W}}(\mathcal{F}).$$

□

Corollary 3.7 (Relative gauge symmetry)**.** *For a chosen globalization (x, α),*

$$\mathrm{Aut}_{\mathrm{rel}}(x, \alpha; \mathsf{W}) \simeq \Omega_{(x,\alpha)} \mathfrak{R}_{\mathsf{W}}(y).$$

Consequently,

$$\pi_k \mathrm{Aut}_{\mathrm{rel}}(x, \alpha; \mathsf{W}) \cong \pi_{k+1} \mathfrak{R}_{\mathsf{W}}(y; x, \alpha).$$

Proof. A loop in the fiber is an automorphism of x together with a coherent null-homotopy of its admitted restriction relative to α. This is precisely an automorphism invisible to the declared witness datum. □

Corollary 3.8 (Completeness criterion)**.** *The witness comparison ρ_{W} is an equivalence if and only if $\mathfrak{R}_{\mathsf{W}}(y)$ is contractible for every $y \in \mathrm{Rec}_{\mathsf{W}}(\mathcal{F})$. If $\mathcal{F}$ is n-truncated, every nonempty reconstruction fiber is also n-truncated.*

3.3 Functoriality under enlargement of witnesses

Suppose

$$\begin{tikzcd} \mathsf{W} \arrow[rr, "u"] \arrow[dr, "j"'] & & \mathsf{W}' \arrow[dl, "j'"] \\ & \mathsf{C}_{\mathcal{U}} & \end{tikzcd}$$

commutes up to a specified natural equivalence. Restriction along u gives

$$q_u \colon \operatorname{Rec}_{\mathsf{W}'}(\mathcal{F}) \longrightarrow \operatorname{Rec}_{\mathsf{W}}(\mathcal{F})$$

and a homotopy-commutative triangle

$$\begin{tikzcd} & \mathcal{F}(M) \arrow[dl, "\rho_{\mathsf{W}'}"'] \arrow[dr, "\rho_{\mathsf{W}}"] & \\ \operatorname{Rec}_{\mathsf{W}'}(\mathcal{F}) \arrow[rr, "q_u"'] & & \operatorname{Rec}_{\mathsf{W}}(\mathcal{F}). \end{tikzcd}$$

Proposition 3.9 (Comparison of reconstruction fibers)**.** *For $y' \in \operatorname{Rec}_{\mathsf{W}'}(\mathcal{F})$ and $y = q_u(y')$, there is a natural map*

$$\mathfrak{R}_{\mathsf{W}'}(y') \longrightarrow \mathfrak{R}_{\mathsf{W}}(y).$$

If q_u is an equivalence, then this map is an equivalence. More generally, the failure of this map to be an equivalence measures the additional completion problem introduced by the witnesses in W' but absent from W.

Proof. Homotopy fibers are functorial for homotopy-commutative squares. The second statement follows by invariance of homotopy pullbacks under equivalence. □

Warning 3.10 (No cardinal monotonicity)*.* Adding more witness objects does not by itself imply a monotone numerical change in the homotopy groups of the reconstruction fiber. The comparison depends on the map q_u, including its coherences. The filtered filler spaces of Section 5 retain this map-level information.

4 Abelian gerbe defects and finite certificates

4.1 The Čech–Deligne model

Fix a finite good cover $\mathcal{U} = \{U_i \to M\}$. We place the Deligne complex

$$\mathcal{D}(2): \quad C^\infty(-, U(1)) \xrightarrow{d\log} \Omega^1 \xrightarrow{d} \Omega^2$$

in internal degrees $q = 0, 1, 2$, where $d\log(g) = (2\pi i)^{-1} g^{-1} dg$. Its Čech bicomplex is

$$C^{p,q}_{\mathcal{U}} := \prod_{i_0 < \cdots < i_p} \mathcal{D}(2)^q(U_{i_0 \cdots i_p}), \qquad D = \delta + (-1)^p d_{\mathcal{D}(2)}.$$

Thus a total degree-two cocycle is represented by

$$B_i \in \Omega^2(U_i), \qquad A_{ij} \in \Omega^1(U_{ij}), \qquad g_{ijk} \in C^\infty(U_{ijk}, U(1)),$$

with equations

$$\delta B - dA = 0, \qquad \delta A + d\log g = 0, \qquad \delta g = 1.$$

The signs are fixed once and for all by the displayed total differential; degree-one and degree-zero cochains encode gauge and gauge-of-gauge data. This is the standard differential cocycle presentation of $\mathbf{B}^2 U(1)_{\mathrm{conn}}$ [8].

Write

$$A^\bullet_{\mathcal{U}} := \mathbb{R}\lim_{c \in \mathsf{C}_{\mathcal{U}}} \mathcal{D}(2)(U_c) \simeq \operatorname{Tot} C^{\bullet,\bullet}_{\mathcal{U}}$$

for the full derived totalization. For a witness functor $j\colon \mathsf{W} \to \mathsf{C}_{\mathcal{U}}$, define instead

$$A^\bullet_{\mathsf{W}} := \mathbb{R}\lim_{w\in\mathsf{W}} \mathcal{D}(2)(U_{j(w)}), \qquad r_{\mathsf{W}}\colon A^\bullet_{\mathcal{U}} \longrightarrow A^\bullet_{\mathsf{W}}.$$

Here and below complexes are regarded as objects of the derived ∞-category of abelian sheaves; a strict complex is a chosen model. This matters: a partial totalization is not, in general, obtained by naively deleting coordinates from the full total complex. The derived limit retains every compatibility morphism and higher coherence present in W.

We work in the category of complexes of abelian sheaves and use additive notation for cone calculations; $U(1)$ components remain multiplicative in the displayed cocycle formulas. Our homotopy-fiber complex is

$$K^m_{\mathsf{W}} = A^m_{\mathcal{U}} \oplus A^{m-1}_{\mathsf{W}}, \qquad d_K(a,b) = (da, r_{\mathsf{W}}(a) - db). \tag{1}$$

Thus $K^\bullet_{\mathsf{W}} = \mathrm{Cone}(r_{\mathsf{W}})[-1]$ in the convention audited in Appendix B; see [17] for mapping cones and derived categories.

Proposition 4.1 (Realization of the abelian reconstruction fiber)**.** *For a cochain complex $A^\bullet$, choose a fibrant complex model QA and define the nonnegative chain complex $N_2(QA)$ by*

$$N_2(QA)_0 = Z^2(QA), \qquad N_2(QA)_1 = (QA)^1, \qquad N_2(QA)_2 = (QA)^0,$$

with boundary induced by the cochain differential and zero in higher degrees. Write

$$\mathfrak{Z}^2(A^\bullet) := \mathrm{DK}(N_2(QA)) \simeq \mathrm{Map}_{D(\mathbf{Ab})}(\mathbb{Z}[-2], A^\bullet)$$

for its Dold–Kan realization: the simplicial abelian group of degree-two cocycles, gauge transformations, and gauge-of-gauge transformations. For $y \in Z^2(A^\bullet_{\mathsf{W}})$, the homotopy fiber of

$$\mathfrak{Z}^2(A^\bullet_{\mathcal{U}}) \longrightarrow \mathfrak{Z}^2(A^\bullet_{\mathsf{W}})$$

over y is empty exactly when the connecting class of y in $H^3(K^\bullet_{\mathsf{W}})$ is nonzero. When a lift is chosen, translation to the zero fiber gives a natural equivalence with the Dold–Kan realization of the truncated shifted fiber complex. Consequently

$$\pi_k \mathfrak{R}_{\mathsf{W}}(y) \cong H^{2-k}(K^\bullet_{\mathsf{W}}), \qquad 0 \le k \le 2.$$

Proof. Dold–Kan identifies the displayed good truncation with the derived mapping space from $\mathbb{Z}[-2]$. The functor $\mathrm{Map}_{D(\mathbf{Ab})}(\mathbb{Z}[-2], -)$ preserves limits. Apply it to the fiber sequence

$$K^\bullet_{\mathsf{W}} \longrightarrow A^\bullet_{\mathcal{U}} \xrightarrow{r_{\mathsf{W}}} A^\bullet_{\mathsf{W}}.$$

A point over y exists exactly when its connecting class vanishes. Translation by a chosen point identifies that fiber with the fiber over zero, whose homotopy groups are the homology groups of its Dold–Kan chain complex. Undoing the degree-two shift gives $H^{2-k}(K^\bullet_{\mathsf{W}})$. See [10, 17]. $\square$

4.2 The relative defect theorem

Theorem 4.2 (Abelian witness defect)**.** *Let $y \in Z^2(A^\bullet_{\mathsf{W}})$ be admitted bundle-gerbe data. Its primary globalization obstruction is*

$$\partial[y] = [(0,y)] \in H^3(K^\bullet_{\mathsf{W}}).$$

It vanishes if and only if there is a full degree-two cocycle whose admitted restriction is gauge-equivalent to y. If a globalization is chosen, then

$$\pi_k \mathfrak{R}_{\mathsf{W}}(y) \cong H^{2-k}(K^\bullet_{\mathsf{W}}), \qquad k = 0, 1, 2,$$

where π_0 is naturally a torsor under $H^2(K^\bullet_{\mathsf{W}})$ before the choice of basepoint.

Proof. The triangle

$$K^\bullet_{\mathsf{W}} \longrightarrow A^\bullet_{\mathcal{U}} \xrightarrow{r_{\mathsf{W}}} A^\bullet_{\mathsf{W}} \longrightarrow K^\bullet_{\mathsf{W}}[1]$$

sends the class of y to the class of $(0, y)$. The latter is a cocycle by (1). It is exact precisely when there is $(a, b) \in K^2_{\mathsf{W}}$ satisfying

$$da = 0, \qquad r_{\mathsf{W}}(a) - db = y.$$

Here a is the desired full gerbe cocycle and b is the admitted gauge equivalence to y. Once one such pair is fixed, every other lift is obtained by translating by the degree-two cocycle 2-groupoid of $K^\bullet_{\mathsf{W}}$. The asserted homotopy groups follow from Proposition 4.1. Without a chosen lift, the set of components has the corresponding torsor action. □

The theorem separates three logically different failures. The group $H^3(K_{\mathsf{W}})$ detects existence, $H^2(K_{\mathsf{W}})$ detects inequivalent globalizations, and $H^1(K_{\mathsf{W}})$ and $H^0(K_{\mathsf{W}})$ detect gauge and gauge-of-gauge symmetry invisible to the witnesses. The next two finite calculations show that the shape of the omission, rather than its cardinality, controls all four layers.

4.3 A sharp missing-coherence certificate

Proposition 4.3 (Sharp tetrahedral defect)**.** *Let the full indexing nerve be Δ^3 and retain its complete 2-skeleton $\mathrm{sk}_2\,\Delta^3 = \partial\Delta^3$. Restrict the smooth Deligne model to the flat sector in which all differential-form components vanish and the coefficient system is the constant discrete abelian group underlying $U(1)$. Then*

$$H^3(\Delta^3, \partial\Delta^3; U(1)) \cong U(1).$$

For triangular data with oriented tetrahedral defect $e^{i\theta}$,

$$\mathfrak{R}_{\le 2}(y_\theta) \simeq \begin{cases} \varnothing, & e^{i\theta} \neq 1, \\ *, & e^{i\theta} = 1. \end{cases}$$

Proof. In the stated constant-flat sector choose face values

$$g_{012} = g_{013} = g_{023} = 1, \qquad g_{123} = e^{i\theta}.$$

Every such 2-cochain is a cocycle on $\partial\Delta^3$, because the retained complex has no 3-simplex. The omitted equation on the tetrahedron is

$$(\delta g)_{0123} = g_{123} g_{023}^{-1} g_{013} g_{012}^{-1} = e^{i\theta}.$$

The restriction from constant cochains on Δ^3 to its boundary is degreewise surjective, so its shifted cone is quasi-isomorphic to the relative complex $C^\bullet(\Delta^3, \partial\Delta^3; U(1))$. Hence

$$H^q(K_{\le 2}) = 0 \quad (q = 0, 1, 2), \qquad H^3(K_{\le 2}) \cong U(1),$$

and the connecting morphism $H^2(\partial\Delta^3; U(1)) \to H^3(\Delta^3, \partial\Delta^3; U(1))$ is an isomorphism. The obstruction is therefore $e^{i\theta}$. If it is nontrivial the fiber is empty; if it is trivial the lift exists and the vanishing of the three lower relative groups makes its reconstruction fiber contractible. □

This example is sharp in exactly the relevant sense: all vertices, edges, and triangular gerbe data are present, but the first omitted tetrahedral coherence supports the obstruction. Merely retaining the full 2-skeleton cannot reconstruct a 2-stack-valued descent object.

4.4 A sparse positive certificate

Proposition 4.4 (Dominated-star reconstruction)**.** *Suppose $\mathcal{U} = \mathcal{V} \cup \{U_*\}$ is a good cover with $U_* \subset V_0$ and $\mathcal{V}$ still covers M. Retaining the complete gerbe descent diagram on $\mathcal{V}$ while omitting all data incident to U_* gives a contractible defect complex and reconstructs every $U(1)$-bundle gerbe with connection on M.*

Proof. Collapse the dominated vertex by $p(*) = 0$ and $p(i) = i$. Because $U_* \subset V_0$, every intersection U_I is contained in $V_{p(I)}$ after repeated indices are removed. Restriction therefore defines a cochain map

$$s \colon A^\bullet_{\mathcal{V}} \longrightarrow A^\bullet_{\mathcal{U}}$$

that fills every omitted component from its V_0 counterpart. If $r \colon A^\bullet_{\mathcal{U}} \to A^\bullet_{\mathcal{V}}$ is restriction, then $rs = 1$.

The inclusion and collapse of nerves are contiguous simplicial maps. The simplicial prism gives a degree -1 operator h that commutes with the Deligne differential and satisfies

$$1 - sr = Dh + hD, \qquad rh = 0.$$

The second identity is obtained by taking the normalized relative prism. On the fiber complex (1), define

$$H(a, b) = (ha + sb, 0).$$

A direct calculation gives

$$d_K H + H d_K = 1.$$

Indeed, the first component is $(Dh + hD)a + sr(a) = a$, and the second is b because $rh = 0$ and $rs = 1$. Thus $K^\bullet_{\mathcal{V}}$ is contractible. The conclusion follows from Theorem 4.2. □

Corollary 4.5 (Half-size certificate in the maximal five-patch case)**.** *If five patches have all nonempty intersections through tetrahedral degree and one patch is dominated as above, the full 3-skeletal presentation has*

$$\binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} = 30$$

nondegenerate entries. The complete four-patch subpresentation has

$$\binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 15$$

and still reconstructs with contractible choice.

The negative and positive certificates differ topologically. The tetrahedral omission retains a boundary 2-sphere while deleting its filler, thereby creating a relative degree-three class. The dominated-star omission deletes an entire redundant cone and is a deformation retract. Consequently no criterion based only on the number or maximum degree of retained witnesses can capture admissibility.

5 Crossed modules and successive fillers

5.1 Crossed-module cocycles

Let

$$\mathcal{G} = (H \xrightarrow{\partial} G, \triangleright)$$

be a strict crossed module. Thus G acts on H, the map ∂ is G-equivariant, and the Peiffer identity $\partial(h) \triangleright h' = hh'h^{-1}$ holds. Its strict 2-group has

$$\pi_1(B\mathcal{G}) = \operatorname{coker} \partial, \qquad \pi_2(B\mathcal{G}) = \ker \partial,$$

with the induced π_1-action on π_2. Crossed modules present connected homotopy 2-types, while the corresponding nonabelian Čech cocycles present principal 2-bundles and nonabelian gerbes [2, 14].

On a good cover, a normalized cocycle has maps

$$g_{ij} \colon U_{ij} \to G, \qquad h_{ijk} \colon U_{ijk} \to H,$$

satisfying, in one fixed convention,

$$g_{ij} g_{jk} = \partial(h_{ijk}) g_{ik}, \tag{2}$$

$$h_{ijk} h_{ikl} = (g_{ij} \triangleright h_{jkl}) h_{ijl}. \tag{3}$$

Gauge transformations and modifications supply the morphisms and 2-morphisms of the cocycle 2-groupoid. Unlike the abelian complex of Section 4, these equations do not linearize globally: the second one depends on transport by the first. Because they do not define a morphism of abelian complexes, there is no single additive mapping cone whose cohomology captures all stages. Reconstruction is instead organized by successive spaces of coherent fillers.

5.2 Completion towers

Let

$$\mathsf{W}^{(0)} \subset \mathsf{W}^{(1)} \subset \cdots \subset \mathsf{W}^{(r)} = \mathsf{C}_{\mathcal{U}}$$

be a finite completion filtration and put

$$X_a := \operatorname{Rec}_{\mathsf{W}^{(a)}}(B\mathcal{G}), \qquad p_a \colon X_{a+1} \longrightarrow X_a.$$

For $y_a \in X_a$, define the next-stage filler space

$$\operatorname{Fill}_a(y_a) := \operatorname{hofib}_{y_a}(p_a).$$

A point of this homotopy fiber includes both the new local data and the coherent identification of its restriction with y_a.

Definition 5.1 (Reachability in a completion tower)**.** A point $y_a \in X_a$ is *reachable from* $y_0 \in X_0$ when it lies in the essential image of the projection

$$\operatorname{hofib}_{y_0}(X_a \longrightarrow X_0) \longrightarrow X_a;$$

equivalently, y_a is equipped with a coherent path from its image in X_0 to y_0. Reachability is therefore a property of a completion branch, not merely of a component of the underlying set of objects.

Lemma 5.2 (Fiber of a composite)**.** *For maps of spaces $Z \xrightarrow{q} Y \xrightarrow{p} X$ and $x \in X$, there is a natural equivalence*

$$\mathrm{hofib}_x(pq) \simeq \int_{y \in \mathrm{hofib}_x(p)} \mathrm{hofib}_y(q),$$

where the right side is the dependent homotopy sum of the fibers of q over all lifts of x to Y.

Proof. Both sides are the homotopy pullback of $Z \to X$ along $\{x\} \to X$; the right side computes that pullback in two stages. This is the associativity or pasting law for homotopy pullbacks [1, 12]. ☐

Theorem 5.3 (Filtered nonabelian reconstruction)**.** *Fix $y_0 \in X_0$. Suppose that for every $a < r$ and every $y_a \in X_a$ reachable from y_0, the space $\mathrm{Fill}_a(y_a)$ is contractible. Then*

$$\mathrm{hofib}_{y_0}(X_r \longrightarrow X_0) \simeq *.$$

Hence y_0 determines a full crossed-module field uniquely up to contractible choice. If the hypothesis holds for every $y_0 \in X_0$, then $X_r \to X_0$ is an equivalence.

Proof. Apply Lemma 5.2 successively to the tower $X_r \to \cdots \to X_0$. At the first stage the completion space is $\mathrm{Fill}_0(y_0)$ and is contractible. Inductively, the completion space through stage $a+1$ is a dependent homotopy sum over the stage-a completion space, with fibers $\mathrm{Fill}_a(y_a)$. A dependent homotopy sum over a contractible base with contractible fibers is contractible. This proves the pointwise statement. A map of spaces is an equivalence exactly when all its homotopy fibers are contractible, which proves the uniform one. ☐

Remark 5.4 (Failure modes remain separated). An empty filler space kills the corresponding completion branch. A nonempty but disconnected filler space gives inequivalent choices, while higher homotopy records automorphisms of a filler invisible at the previous stage. Thus the filtration is the nonabelian analogue of the existence, ambiguity, and relative-symmetry layers in Theorem 4.2; it does not replace them by a yes/no extension test.

5.3 Transported action with trivial Postnikov class

The following elementary observation prevents two different sources of nonabelian behavior from being conflated.

Proposition 5.5 (Split action versus associator)**.** *Let A be an abelian group with an action of a group Q, and let*

$$Y = K(A,2)_{hQ} \longrightarrow BQ$$

be the split homotopy 2-type. Its Postnikov class is zero. For a pair $L \subset X$ and a fixed holonomy $\rho\colon X \to BQ$, a chosen lift over L can nevertheless have a nonzero relative extension obstruction in

$$H^3(X, L; A_\rho).$$

This obstruction is computed with the ρ-twisted differential and may be nonzero solely because the Q-action on A is nontrivial.

Proof. The split projection has a section, so the universal Postnikov obstruction and every pullback $\rho^* k$ vanish. After fixing ρ, its fiber is the local Eilenberg–Mac Lane system $K(A_\rho, 2)$. Relative obstruction theory places the first obstruction to extending a specified section from L in $H^3(X, L; A_\rho)$. The cellular or Čech coboundary in this group transports coefficients by ρ; therefore it can be nontrivial even though the associator class is zero [6]. ☐

Proposition 5.6 (Obstruction from transported action)**.** *For the split crossed module*

$$U(1) \xrightarrow{0} \mathbb{Z}_2, \qquad t \triangleright z = z^{-1},$$

fix the nontrivial $\mathbb{Z}_2$ holonomy on $\mathbb{RP}^3$ and retain $\mathbb{RP}^2$. Here $U(1)_\rho$ is a local system of abstract abelian groups with inversion monodromy, and the calculation is cellular cohomology with local coefficients, not smooth Deligne cohomology. If $u \in U(1)$ is the partial two-cell witness, then

$$\mathrm{Fill}_\rho(u) \simeq \begin{cases} \varnothing, & u^2 \neq 1, \\ *, & u^2 = 1. \end{cases}$$

The Postnikov class vanishes; the obstruction is generated by the nontrivial π_1-action on π_2.

Proof. The crossed module presents the split 2-type

$$K(U(1), 2)_{h\mathbb{Z}_2},$$

with $\pi_1 = \mathbb{Z}_2$, $\pi_2 = U(1)$, and inversion monodromy. The strict splitting and trivial associator imply $k = 0$. Give $\mathbb{RP}^3$ its standard CW structure with one cell in every degree $0 \leq q \leq 3$, and let t denote the nontrivial deck transformation. The cellular boundary over $\mathbb{Z}[\mathbb{Z}_2]$ is

$$\partial_q = 1 + (-1)^q t.$$

For the chosen nontrivial holonomy, the degree-two twisted coboundary is dual to $\partial_3 = 1 - t$. In multiplicative notation it sends

$$u \longmapsto u\,(t \triangleright u)^{-1} = u\,(u^{-1})^{-1} = u^2.$$

There is no degree-three equation on $\mathbb{RP}^2$, so every u is admitted partial two-cell data. It extends over the unique 3-cell precisely when $u^2 = 1$. When an extension exists, its filler space is a torsor for $\Omega^3 B\mathcal{G}$; this space is contractible because $B\mathcal{G}$ is 2-truncated. The asserted dichotomy follows. □

Corollary 5.7 (The surviving global sectors)**.** *For the fixed nontrivial holonomy, the components of global lifts on $\mathbb{RP}^3$ are*

$$H^2(\mathbb{RP}^3; U(1)_\rho) \cong \{u \in U(1) : u^2 = 1\} = \{1, -1\}.$$

Thus the completion condition removes generic partial witnesses but does not identify the two witnesses that survive it.

Proof. The preceding cellular complex has trivial degree-one coboundary because $1 + t$ acts on $U(1)$ as $u(t \triangleright u) = uu^{-1} = 1$. Its degree-two cocycles are the kernel of $u \mapsto u^2$, giving the displayed group. □

The example is deliberately relative. The identity witness $u = 1$ always extends, in agreement with $k = 0$. What fails for generic u is extension of a particular partial field through a particular admitted cell. The obstruction is the transported comparison between u and its inverse, not an unobserved associator. This is why nonabelian witness descent must retain both the completion filtration and the action carried along it.

6 Witnesswise-complete refinement

6.1 The refinement square

A refinement of the underlying cover is not by itself enough for witness invariance. The admitted subdiagram must also be transported. Abstractly, a refinement datum is a homotopy-commutative square

$$\begin{tikzcd} X_{\mathcal{U}} \arrow[r, "\rho_{\mathcal{U}}"] \arrow[d, "f"'] & Y_{\mathcal{U}} \arrow[d, "g"] \\ X_{\mathcal{V}} \arrow[r, "\rho_{\mathcal{V}}"'] & Y_{\mathcal{V}}, \end{tikzcd} \tag{4}$$

where $X_{\mathcal{U}}$ and $X_{\mathcal{V}}$ are full descent objects and $Y_{\mathcal{U}}$ and $Y_{\mathcal{V}}$ are their admitted witness reconstructions.

Definition 6.1 (Witnesswise-complete refinement)**.** The refinement datum (4) is *witnesswise complete* when both f and g are equivalences. Thus the refinement preserves not only the full field object but also the information retained by the declared witness diagram.

The condition on g is essential. A refinement may preserve ordinary descent while changing the admitted reconstruction problem by adding, deleting, or duplicating witness coherences.

Theorem 6.2 (Refinement stability of reconstruction fibers)**.** *For a witnesswise-complete refinement and $y \in Y_{\mathcal{U}}$, the square* (4) *induces a natural equivalence*

$$\mathrm{hofib}_y(\rho_{\mathcal{U}}) \xrightarrow{\simeq} \mathrm{hofib}_{g(y)}(\rho_{\mathcal{V}}).$$

Consequently globalization, ambiguity of globalization, and relative gauge symmetry are invariant under witnesswise-complete refinement.

Proof. The source fiber is the homotopy pullback $X_{\mathcal{U}} \times^h_{Y_{\mathcal{U}}} \{y\}$. Apply f, g, and the specified homotopy in (4). Homotopy pullbacks are invariant under equivalence in each input, giving

$$X_{\mathcal{U}} \times^h_{Y_{\mathcal{U}}} \{y\} \simeq X_{\mathcal{V}} \times^h_{Y_{\mathcal{V}}} \{g(y)\}.$$

The three listed layers are the emptiness, components, and higher homotopy of this one fiber, so each is preserved. □

6.2 Abelian defect complexes

For the Čech–Deligne presentation of Section 4, the refinement square is a square of derived complexes

$$\begin{tikzcd} A^\bullet_{\mathcal{U}} \arrow[r, "r_{\mathcal{U}}"] \arrow[d, "f \simeq"'] & A^\bullet_{\mathsf{W}_{\mathcal{U}}} \arrow[d, "g \simeq"] \\ A^\bullet_{\mathcal{V}} \arrow[r, "r_{\mathcal{V}}"'] & A^\bullet_{\mathsf{W}_{\mathcal{V}}}. \end{tikzcd} \tag{5}$$

Corollary 6.3 (Refinement stability of the defect complex)**.** *The square* (5) *induces a quasi-isomorphism*

$$\kappa\colon K^\bullet_{\mathsf{W}_{\mathcal{U}}} \xrightarrow{\simeq_{\mathrm{qis}}} K^\bullet_{\mathsf{W}_{\mathcal{V}}}.$$

For $y \in Z^2(A^\bullet_{\mathsf{W}_\mathcal{U}})$, its obstruction is natural:

$$H^3(\kappa)(\partial_\mathcal{U}[y]) = \partial_\mathcal{V}[g(y)].$$

The ambiguity and relative-symmetry groups $H^q(K)$ are likewise identified in every degree.

Proof. The rows of (5) extend to exact triangles

$$K \longrightarrow A_{\text{full}} \longrightarrow A_{\text{adm}} \longrightarrow K[1].$$

The coherent square supplies a morphism of triangles. Since the two displayed vertical maps are quasi-isomorphisms, so is the induced map on their homotopy fibers. Naturality of the connecting morphism sends the representative $(0, y)$ to the representative $(0, g(y))$ in the derived category. The cohomology statements follow. □

6.3 Barycentric subdivision of the missing tetrahedron

Let

$$(T, B) = (\Delta^3, \partial\Delta^3), \qquad (T', B') = (\operatorname{sd}\Delta^3, \operatorname{sd}\partial\Delta^3).$$

Subdivision gives a chain equivalence of pairs

$$s\colon C_\bullet(T, B) \xrightarrow{\simeq} C_\bullet(T', B')$$

and hence a quasi-isomorphism of the two relative defect complexes. In particular,

$$H^q(T', B'; U(1)) \cong \begin{cases} U(1), & q = 3, \\ 0, & q \neq 3. \end{cases}$$

To see the preserved class explicitly, write b_I for the barycenter of the face spanned by a nonempty set $I \subset \{0, 1, 2, 3\}$. For $\sigma \in S_4$, let $I_k(\sigma) = \{\sigma(0), \ldots, \sigma(k)\}$. The subdivided relative fundamental chain is

$$s[0123] = \sum_{\sigma \in S_4} \operatorname{sgn}(\sigma)[b_{I_0(\sigma)}, b_{I_1(\sigma)}, b_{I_2(\sigma)}, b_{I_3(\sigma)}]. \tag{6}$$

It contains $4! = 24$ oriented tetrahedra.

Proposition 6.4 (Aggregate refined defect)**.** *Let $d_\sigma \in U(1)$ be the degree-three coherence defect on the refined tetrahedron indexed by σ, evaluated with the vertex order in (6). Then the refinement-invariant obstruction is*

$$\operatorname{Def}_{\text{sd}}(d) := \prod_{\sigma \in S_4} d_\sigma^{\operatorname{sgn}(\sigma)}. \tag{7}$$

Individual d_σ depend on the chosen partial fillers, but their product depends only on the relative cohomology class. If the unrefined tetrahedral defect is $e^{i\theta}$ and the refinement data are witnesswise complete, then

$$\operatorname{Def}_{\text{sd}}(d) = e^{i\theta}.$$

Proof. The product in (7) is precisely the multiplicative evaluation $\langle d, s[0123]\rangle$. Changing a partial filler changes d by a relative coboundary, whose evaluation on the relative cycle $s[0123]$ is 1. Thus the product descends to $H^3(T', B'; U(1))$. Naturality of the cohomology–homology pairing under s, equivalently Corollary 6.3, identifies it with the evaluation of the unrefined class on $[0123]$, namely $e^{i\theta}$. □

Corollary 6.5 (The subdivided reconstruction fiber)**.** *Let y be boundary data on $B' = \operatorname{sd}\partial\Delta^3$, and let d be any relative degree-three defect representative obtained from a choice of partial fillers. Then*

$$\mathfrak{R}_{B'}(y) \simeq \begin{cases} \varnothing, & \mathrm{Def}_{\mathrm{sd}}(d) \neq 1, \\ *, & \mathrm{Def}_{\mathrm{sd}}(d) = 1. \end{cases}$$

Subdivision therefore distributes one obstruction among 24 *local coherence tests; it neither creates* 24 *independent obstruction classes nor erases the original one.*

Proof. The aggregate is the unique relative degree-three class. If it is nontrivial, globalization fails. If it vanishes, globalization exists, and the vanishing of $H^q(T', B'; U(1))$ for $q = 0, 1, 2$ makes the reconstruction fiber contractible by Theorem 4.2. □

7 BRST/BV and Ward-compatible descent

7.1 The local quantum BV diagram

Fix a cover $\mathcal{U}$ and let $\mathsf{C}_{\mathcal{U}}$ be its descent indexing category. For every intersection U_c, suppose perturbative renormalization has produced a complete filtered BV algebra $\mathfrak{A}(U_c)$ and a degree-one quantum BV differential $\widehat{s}_c$. The local quantum master equation is used here only through the consequence

$$\widehat{s}_c^{\,2} = 0.$$

For every arrow $\alpha \colon c \to c'$ in $\mathsf{C}_{\mathcal{U}}$, restriction must be represented by a chain map

$$R_\alpha \colon (\mathfrak{A}(U_c), \widehat{s}_c) \longrightarrow (\mathfrak{A}(U_{c'}), \widehat{s}_{c'}), \qquad R_\alpha \widehat{s}_c = \widehat{s}_{c'} R_\alpha,$$

strictly or by a specified chain homotopy, with coherent higher homotopies for composition. These data define a derived diagram

$$\mathfrak{A}_{\mathrm{BV}} \colon \mathsf{C}_{\mathcal{U}} \longrightarrow \mathrm{Ch}_{\mathrm{comp}}$$

in complete filtered complexes. This is the precise meaning of *Ward-compatible local descent* used below [9, 15].

Define the full and admitted quantum BV totalizations by

$$\mathrm{Tot}_{\mathcal{U}}^{\mathrm{BV}} := \mathbb{R}\lim_{c \in \mathsf{C}_{\mathcal{U}}} \mathfrak{A}_{\mathrm{BV}}(c), \qquad \mathrm{Tot}_{\mathsf{W}}^{\mathrm{BV}} := \mathbb{R}\lim_{w \in \mathsf{W}} \mathfrak{A}_{\mathrm{BV}}(j(w)),$$

and let

$$\rho_{\mathsf{W}}^{\mathrm{BV}} \colon \mathrm{Tot}_{\mathcal{U}}^{\mathrm{BV}} \longrightarrow \mathrm{Tot}_{\mathsf{W}}^{\mathrm{BV}}$$

be restriction. Its relative Ward-descent complex is

$$K_{\mathrm{BV},\mathsf{W}} := \mathrm{hofib}(\rho_{\mathsf{W}}^{\mathrm{BV}}) \simeq \mathrm{Cone}(\rho_{\mathsf{W}}^{\mathrm{BV}})[-1].$$

Theorem 7.1 (Ward-compatible descent)**.** *Assume the local quantum master equations, the coherent chain-map conditions above, and convergence of the complete derived totalizations. If j is homotopy initial, then*

$$\rho_{\mathsf{W}}^{\mathrm{BV}} \colon \mathrm{Tot}_{\mathcal{U}}^{\mathrm{BV}} \xrightarrow{\simeq_{\mathrm{qis}}} \mathrm{Tot}_{\mathsf{W}}^{\mathrm{BV}}.$$

Consequently $K_{\mathrm{BV},\mathsf{W}}$ is acyclic and the admitted witnesses reconstruct quantum BV cohomology as a graded object. If, in addition, the local diagram and both derived totalizations are formed in a multiplicative derived category—for example complete filtered dg algebras with multiplicative coherent restrictions—then the induced degree-zero isomorphism preserves the Ward-compatible observable-algebra structure.

Proof. Homotopy-initial restriction preserves derived limits. Applying this to the diagram $\mathfrak{A}_{\mathrm{BV}}$ gives the displayed quasi-isomorphism. Its homotopy fiber is therefore zero in the derived category. Quasi-isomorphic complexes have isomorphic cohomology in every degree. Under the additional multiplicative lift, the comparison is a quasi-isomorphism of multiplicative objects, so the induced degree-zero isomorphism is an algebra isomorphism. □

Without the hypothesis, the exact triangle

$$K_{\mathrm{BV},\mathsf{W}} \longrightarrow \mathrm{Tot}^{\mathrm{BV}}_{\mathcal{U}} \longrightarrow \mathrm{Tot}^{\mathrm{BV}}_{\mathsf{W}} \longrightarrow K_{\mathrm{BV},\mathsf{W}}[1]$$

gives the long exact sequence

$$\cdots \to H^m(K_{\mathrm{BV},\mathsf{W}}) \to H^m(\mathrm{Tot}^{\mathrm{BV}}_{\mathcal{U}}) \to H^m(\mathrm{Tot}^{\mathrm{BV}}_{\mathsf{W}}) \to H^{m+1}(K_{\mathrm{BV},\mathsf{W}}) \to \cdots . \tag{8}$$

Thus the relative totalization defect measures missing or ambiguous Ward classes without identifying them with a local anomaly.

7.2 Why local H^0 data are not enough

The theorem reconstructs the entire derived complex before taking cohomology. A weaker claim based only on the diagram of local observable algebras $H^0(\mathfrak{A}(U_c))$ requires additional control.

Proposition 7.2 (Ward hypercohomology audit)**.** *Let $\mathcal{A}$ be a Grothendieck abelian category and suppose, for this proposition, that $\mathsf{C}_{\mathcal{U}}$ and W are nerves of ordinary small categories of finite nerve dimension. Model each derived inverse limit by the normalized cosimplicial replacement of an objectwise K-injective replacement of the diagram $\mathfrak{A}_{\mathrm{BV}} \colon \mathsf{C}_{\mathcal{U}} \to \mathrm{Ch}(\mathcal{A})$. Thus the resulting double complexes have Čech degree $0 \le p \le N$ for some finite N, and use the direct-sum totalization, which here agrees degreewise with the product totalization. Then there are natural, convergent comparison spectral sequences*

$$E_2^{p,q}(\mathcal{U}) = \mathbb{R}^p \lim_{\mathsf{C}_{\mathcal{U}}} H^q(\mathfrak{A}_{\mathrm{BV}}) \Longrightarrow H^{p+q}(\mathrm{Tot}^{\mathrm{BV}}_{\mathcal{U}})$$

and

$$E_2^{p,q}(\mathsf{W}) = \mathbb{R}^p \lim_{\mathsf{W}} H^q(\mathfrak{A}_{\mathrm{BV}} \circ j) \Longrightarrow H^{p+q}(\mathrm{Tot}^{\mathrm{BV}}_{\mathsf{W}}).$$

The row $q = 0$ alone does not determine total degree zero: terms with $p > 0$ and $q = -p$, together with differentials entering or leaving them, may contribute. A sufficient condition for degree-zero reconstruction from local H^0 data is that, for every q and every $p > 0$, the groups $\mathbb{R}^p \lim H^q$ vanish on both sides and

$$\lim_{\mathsf{C}_{\mathcal{U}}} H^0(\mathfrak{A}_{\mathrm{BV}}) \xrightarrow{\simeq} \lim_{\mathsf{W}} H^0(\mathfrak{A}_{\mathrm{BV}} \circ j).$$

Proof. Filter each finite double complex by Čech degree. The bounded horizontal filtration gives a convergent spectral sequence. Vertical cohomology followed by horizontal cohomology is precisely the derived inverse limit of the cohomology diagram, giving the displayed E_2 page. Total degree is $p + q$, so negative ghost or antifield cohomology can feed degree zero through positive Čech degree. Under the stated vanishing, both spectral sequences collapse to their ordinary-limit rows, and the displayed map gives the degree-zero isomorphism. This is the standard hypercohomology/derived-limit construction [1, 17]. □

Remark 7.3 (Complete-filtered extension)*.* The same E_2 description may be used for unbounded or genuinely ∞-categorical indexing diagrams only after specifying a compatible t-structure or tower filtration and proving completeness and strong convergence. For example, it is sufficient to work in a stable presentable ∞-category with a left-complete compatible t-structure, provided the filtered derived limits are complete and the relevant Boardman convergence conditions hold. No such extension is needed for the finite certificate in Proposition 7.2.

7.3 Three distinct failures

Proposition 7.4 (Anomaly, transport, and descent)**.** *The following failures occur at different logical levels.*

(i) *A* local quantum anomaly *is a nonzero obstruction to $\widehat{s}_c^2 = 0$; then the local object is not a quantum BV complex.*

(ii) *A* transport anomaly *is the failure of restrictions to intertwine $\widehat{s}$ up to coherent homotopy; then the local complexes do not form the derived diagram $\mathfrak{A}_{\mathrm{BV}}$.*

(iii) *A* witness-descent defect *occurs after the first two conditions hold but $H^\bullet(K_{\mathrm{BV},\mathsf{W}}) \neq 0$.*

No one of these failures implies either of the other two without additional structure relating their obstruction theories.

Proof. The first prevents the objects of the proposed diagram from existing in $\mathrm{Ch}_{\mathrm{comp}}$. The second prevents the morphisms and coherent composition data from defining a functor there. Only after both have vanished is the homotopy fiber $K_{\mathrm{BV},\mathsf{W}}$ defined canonically. The third is then a property of the comparison of two derived limits, proving the asserted separation. $\square$

8 Lorentzian Yang–Mills solution stacks

8.1 Equation descent as a derived equalizer

Let $\mathcal{C}_G = \mathbf{B}G_{\mathrm{conn}}$ be the stack of G-connections on globally hyperbolic Lorentzian spacetimes. Let $p\colon \mathcal{E}_G \to \mathcal{C}_G$ be the equation-bundle stack: over a connection A, its fiber contains the covariant equation-valued fields of the same type as $d_A{*}F_A$. Assume the Euler–Lagrange map and zero section are natural sections

$$\mathrm{EL}, 0\colon \mathcal{C}_G \rightrightarrows \mathcal{E}_G, \qquad p\,\mathrm{EL} = p0 = \mathrm{id}_{\mathcal{C}_G}.$$

Let $\mathcal{X}$ denote the ambient ∞-category of stacks. Regard p as an object of the slice $\mathcal{X}_{/\mathcal{C}_G}$ and $\mathrm{EL}, 0$ as two morphisms from the terminal slice object $\mathrm{id}_{\mathcal{C}_G}$ to p. Define the solution stack by their derived equalizer in that slice,

$$\mathbf{YM}_G := \mathrm{Eq}^h(\mathrm{EL}, 0) \simeq \mathcal{C}_G \times^h_{\mathcal{E}_G \times_{\mathcal{C}_G} \mathcal{E}_G} \mathcal{E}_G, \tag{9}$$

where the first map is $(\mathrm{EL}, 0)$ and the second map is the relative diagonal. Thus every arrow in (9) is typed over $\mathcal{C}_G$. The homotopy pullback retains gauge transformations and infinitesimal stabilizer data of solutions; an ordinary set-theoretic zero locus would not.

Theorem 8.1 (Yang–Mills solution reconstruction)**.** *Assume $\mathcal{X}$ admits the required limits. Apply full and admitted totalization first in the arrow category $\mathrm{Fun}(\Delta^1, \mathcal{X})$ to the local equation bundle $p\colon \mathcal{E}_G \to \mathcal{C}_G$; the resulting arrow is then viewed in the slice over the corresponding totalized connection object. In this typing, the limit functor preserves the finite homotopy limit defining $\mathbf{YM}_G$. Suppose the admitted witness functor reconstructs both the connection stack and the equation-target stack,*

$$\mathcal{C}_G(M) \xrightarrow{\simeq} \mathrm{Rec}_{\mathsf{W}}(\mathcal{C}_G), \qquad \mathcal{E}_G(M) \xrightarrow{\simeq} \mathrm{Rec}_{\mathsf{W}}(\mathcal{E}_G),$$

with the second equivalence lying over the first. Suppose further that they intertwine p, EL*, and* 0 *up to specified coherent homotopy. Then*

$$\mathbf{YM}_G(M) \xrightarrow{\simeq} \mathrm{Rec}_{\mathsf{W}}(\mathbf{YM}_G).$$

Proof. Limits in an ∞-category commute with limits, and limits in the arrow category are computed pointwise. The compatibility hypotheses identify the two totalized arrows and their two sections in the corresponding slices. Therefore admitted totalization commutes with the finite homotopy limit in (9):

$$\begin{aligned}\mathrm{Rec}_{\mathsf{W}}(\mathbf{YM}_G) &\simeq \mathrm{Eq}^h(\mathrm{Rec}_{\mathsf{W}}(\mathcal{C}_G) \rightrightarrows \mathrm{Rec}_{\mathsf{W}}(\mathcal{E}_G)) \\ &\simeq \mathrm{Eq}^h(\mathcal{C}_G(M) \rightrightarrows \mathcal{E}_G(M)) \simeq \mathbf{YM}_G(M).\end{aligned}$$

The middle equivalence uses both reconstruction hypotheses and their coherent compatibility with the two arrows. □

Corollary 8.2 (Connection descent does not suffice)**.** *Reconstruction of $\mathcal{C}_G$ alone does not imply reconstruction of $\mathbf{YM}_G$. A defect in $\mathcal{E}_G$, in the Euler–Lagrange transformation, or in its coherent witness transport may survive after the underlying connection field has been reconstructed.*

Proof. The derived equalizer depends functorially on both objects and both arrows. Equivalence of only one input does not in general induce an equivalence of the homotopy pullbacks. □

Warning 8.3 (No quantization or mass-gap conclusion)*.* Theorem 8.1 is a classical solution-stack statement, while Theorem 7.1 is conditional on already-constructed renormalized BV data. Neither theorem constructs that quantization, cancels anomalies, proves a spectrum condition, or establishes a Yang–Mills mass gap.

9 Adversarial audit: invariants, minimality, and finite cores

The preceding results are deliberately sufficient rather than maximal. This section subjects their most tempting informal strengthenings to explicit counterexamples. The purpose is not to add another reconstruction framework, but to identify exactly which conclusions survive a change of presentation and which do not.

9.1 Reconstruction equivalence is invariant; witness count is not

Fix a descent diagram $D\colon \mathsf{C} \to \mathcal{X}$. Two witness systems over C are *D-reconstruction equivalent* when their admitted limits are equivalent compatibly with the two restriction maps from $\lim_{\mathsf{C}} D$. This is the diagram-specific equivalence relation that preserves the reconstruction problem.

Proposition 9.1 (Presentation invariance and the cardinality trap)**.** *D-reconstruction-equivalent witness systems have equivalent reconstruction fibers over corresponding witness data. Their numbers of objects, arrows, and displayed coherence cells need not agree and are not reconstruction invariants.*

Proof. The first assertion is invariance of homotopy pullbacks under equivalence, as in Proposition 3.9 and theorem 6.2. For the second, take $\mathsf{C} = *$ and any object $X = D(*)$. The terminal witness category $*$ and every finite ordinal $[m]$ map uniquely to C. Since $[m]$ has an initial object,

$$\lim_{[m]} \mathrm{const}(X) \simeq X \simeq \lim_{*} D.$$

Thus these presentations have the same reconstruction map and fiber although their object and arrow counts grow with m. □

Consequently a phrase such as "the minimal witness system" has no invariant meaning until one fixes a presentation class, a notion of allowed elementary moves, and a cost functional. Homotopy initiality and contractibility of the reconstruction fiber are invariant statements; raw witness cardinality is not.

9.2 Universal finite cores versus diagram-specific finite cores

Definition 9.2 (Finite reconstruction cores)**.** A *universal finite n-core* of an indexing category C is a finite category W with an n-limit-initial functor $j\colon \mathsf{W} \to \mathsf{C}$. For a fixed diagram D, a *D-specific finite core* is a finite witness system for which

$$\lim_{\mathsf{C}} D \longrightarrow \lim_{\mathsf{W}} D \circ j$$

is an equivalence.

Proposition 9.3 (A universal finite core need not exist)**.** *Let C be an infinite discrete category. It has no universal finite n-core for any $n \geq -1$. Nevertheless the terminal diagram $D\colon \mathsf{C} \to \mathcal{X}$, $D(c) = *$, has a one-object D-specific finite core.*

Proof. For any functor from a finite category $j\colon \mathsf{W} \to \mathsf{C}$, some $c \in \mathsf{C}$ is not in its image. Because C is discrete, the comma category $(j \downarrow c)$ is empty. It is not even 0-connective, so j cannot be n-limit-initial. On the other hand, the limit of a terminal diagram over any category is terminal. Choosing any $c_0 \in \mathsf{C}$ gives $\lim_{\mathsf{C}} D \simeq * \simeq D(c_0)$. □

This separates a property of the indexing presentation from a property of one physical diagram. The finite-core question relevant to applications is often diagram-specific; promoting such a result to a universal statement requires a separate cofinality proof.

9.3 Existence of nonabelian fillers does not imply reconstruction

Proposition 9.4 (Existence does not suffice)**.** *In Theorem 5.3, nonemptiness—even connectedness—of every filler space cannot replace contractibility. For any nontrivial group G, the one-stage map*

$$BG \longrightarrow *$$

has a nonempty connected fiber over the unique point, but it is not an equivalence: the filler has relative automorphism group $\pi_1(BG) \cong G$.

Proof. The homotopy fiber of $BG \to *$ is BG. It is nonempty and connected, while its fundamental group is G, so it is not contractible when G is nontrivial. The example is already a completion tower of length one and therefore defeats any stagewise criterion that records only existence or components. □

The same adversarial pattern explains the safeguards in the later sections. Local H^0 Ward data omit higher-derived-limit contributions to total degree zero; connection descent omits the equation bundle and its two sections; and a local anomaly or transport anomaly occurs before the witness-descent complex is even defined. These are not cosmetic qualifications. Each blocks a distinct invalid strengthening of the reconstruction theorem.

10 Conclusions and current interface work

10.1 What has and has not been proved

The standard inputs to this paper are higher-stack descent, truncated homotopy-initiality, mapping-cone obstruction theory in the abelian case, Postnikov obstruction theory for crossed modules, and derived BRST/BV totalization. None is being renamed as a new general theory. The contribution is the relative reconstruction problem placed between full descent and a declared witness diagram, together with a unified defect organization and finite calculations of the resulting fiber.

The principal conclusions can therefore be stated without interface rhetoric. For an admitted witness functor $j\colon \mathsf{W} \to \mathsf{C}_{\mathcal{U}}$:

(i) truncated homotopy-initiality is a diagram-independent sufficient criterion for reconstruction;

(ii) the homotopy fiber of the restriction map separates nonexistence, inequivalent globalization, and witness-invisible symmetry;

(iii) in the abelian gerbe model, one shifted cone computes all three layers;

(iv) in the nonabelian crossed-module model, successive filler spaces retain transported actions that a single abelianized defect group can lose;

(v) witnesswise-complete refinement preserves the relative reconstruction problem; and

(vi) Ward-compatible and Yang–Mills equation descent require reconstruction of the relevant differentials, equation bundles, and coherent arrows, not only of the underlying geometric fields.

The most concrete paper-specific results are the sharp tetrahedral obstruction, the dominated-star chain contraction, the transported-action $\mathbb{RP}^3$ certificate, and the subdivision product recovering one relative obstruction from 24 refined coherence tests. These calculations show, respectively, how one omitted coherence can obstruct globalization, how an apparently large deletion can be homotopically redundant, how monodromy can survive a trivial Postnikov class, and how refinement redistributes rather than duplicates an obstruction.

The negative conclusions are equally structural. Reconstruction is not controlled by witness count, universal finite cores need not exist, and stagewise existence does not eliminate nonabelian automorphisms. These boundaries explain why the paper stops at conditional BV and classical solution-stack statements.

10.2 Open reconstruction problems

The following problems form the current interface-work ledger. They are deliberately posed inside higher gauge theory and AQFT; no broader emergence claim is assumed.

Q1. Diagram-specific finite cores. For a geometrically defined descent diagram $D_{\mathcal{U}}\colon \mathsf{C}_{\mathcal{U}} \to \mathcal{X}_{\leq n}$, find verifiable conditions for the existence of a finite witness system W such that

$$\lim_{\mathsf{C}_{\mathcal{U}}} D_{\mathcal{U}} \xrightarrow{\simeq} \lim_{\mathsf{W}} D_{\mathcal{U}} \circ j.$$

The criterion should distinguish diagram-specific compression from universal n-limit-initiality and should be stable under witnesswise-complete refinement.

Q2. Presentation-invariant reconstruction complexity. Localize witness presentations at D-reconstruction equivalences. Is there a homotopy-invariant cost or complexity on the resulting localization whose minimizers admit finite certificates? Raw numbers of objects and arrows do not descend to this localization by Proposition 9.1.

Q3. A nonabelian relative defect object. Determine when the completion tower of Theorem 5.3 can be represented by a functorial relative object whose homotopy groups recover existence, ambiguity, and relative symmetry without discarding the transported π_1-action on higher coefficients. A satisfactory construction must specialize to $\mathrm{Cone}(r_{\mathsf{W}})[-1]$ in the abelian case and reproduce the twisted $\mathbb{RP}^3$ certificate of Proposition 5.6.

Q4. Ward defect versus anomaly transgression. Under additional renormalization and locality hypotheses, construct—or rule out—a natural comparison from local anomaly classes to $H^\bullet(K_{\mathrm{BV},\mathsf{W}})$. Such a comparison must preserve the logical separation in Proposition 7.4: an anomaly, a transport failure, and a witness-descent defect are not identical merely because all are expressed cohomologically.

Q5. From equation descent to observable descent. Find conditions under which reconstruction of the Yang–Mills solution stack, including the equation-bundle diagram, induces reconstruction of a causal local observable theory. The required hypotheses should make explicit where causal support, the time-slice property, gauge reduction, and derived critical-locus information enter; Theorem 8.1 alone does not supply them.

These questions share a common mathematical form: determine which finite or restricted diagram retains enough coherent structure for a specified global object. The answer is necessarily coefficient-, transport-, and diagram-dependent. That dependence is not a defect of the method; it is the content measured by the reconstruction fiber.

Author and AI-use disclosure

OpenAI ChatGPT and Codex were used during August–September 2026 as research-support tools for literature organization, structural testing, adversarial review, drafting, and editorial revision. The conceptual direction, selection of the interface problem, definitions, theorem claims, examples, physical interpretation, and final mathematical judgments are those of the author. The author reviewed and approved the mathematical statements, proof dependencies, citations, and final wording and accepts full responsibility for the manuscript. No AI system is listed as an author.

A Indexing orientation and truncated initiality audit

This appendix freezes the orientation and connectivity conventions used in Theorems 3.2 and 3.4.

A.1 Left cofinality is the limit convention

Following the left/right terminology used in [12, 7], a functor $j\colon \mathsf{W} \to \mathsf{C}$ is left cofinal when restriction along j preserves limits of arbitrary space-valued diagrams. Joyal's ∞-categorical Quillen–A criterion tests this by the groupoid completions

$$\left| \mathsf{W} \times_{\mathsf{C}} \mathsf{C}_{/c} \right|,$$

whose objects contain arrows $j(w) \to c$. Right cofinality uses $\mathsf{C}_{c/}$, hence arrows $c \to j(w)$, and preserves colimits.

The elementary category $[1] = (0 \to 1)$ fixes the direction. The inclusion $\{0\} \hookrightarrow [1]$ computes limits: for every diagram $X_0 \to X_1$,

$$\lim_{[1]}(X_0 \to X_1) \simeq X_0.$$

For $c = 1$, the relevant comma category is nonempty because it contains $0 \to 1$. The comma construction based on arrows $1 \to 0$ would be empty and therefore cannot be the limit criterion.

A.2 Connectivity and the shift

Our convention is

$$K \text{ is } r\text{-connective} \quad \iff \quad \tau_{\leq r-1} K \simeq *.$$

Equivalently, for $r \geq 1$, the space is nonempty and $\pi_i(K) = 0$ for $0 \leq i < r$. Some literature calls the same condition "$(r-1)$-connected." Therefore the hypothesis in Theorem 3.2 may be written in either of the equivalent forms

$$|(j \downarrow c)| \text{ is } (n+1)\text{-connective} \quad \iff \quad |(j \downarrow c)| \text{ is } n\text{-connected}.$$

The manuscript uses the first formulation to avoid ambiguity at degrees -1 and 0.

A.3 Why an n-truncated stack gives an $(n+1, 1)$-category

An object $Y \in \mathcal{X}$ is n-truncated precisely when $\mathrm{Map}_{\mathcal{X}}(T, Y)$ is an n-truncated space for every T. Hence the full subcategory $\mathcal{X}_{\leq n}$ of n-truncated objects has n-truncated mapping spaces. In the standard higher-categorical convention it is therefore an $(n+1, 1)$-category. Truncated cofinality for limits in this subcategory requires left $(n+1)$-cofinality, which the truncated Theorem–A criterion expresses by $(n+1)$-connectivity of the comma spaces [7]. This is the source of the apparent shift.

A.4 Sanity checks

For set-valued diagrams ($n = 0$), the condition becomes connectedness of each ordinary comma category, the classical initial-functor criterion. For groupoid-valued diagrams ($n = 1$), it becomes simple connectedness of each comma space. For unrestricted ∞-groupoid-valued diagrams, every comma space must be weakly contractible, recovering ordinary homotopy initiality. Finally, Proposition 3.3 shows that lowering $(n+1)$-connectivity to n-connectivity fails already for a constant Eilenberg–Mac Lane diagram.

B Total-complex signs and cone shifts

This appendix records the sign calculation used in Theorem 4.2 and proposition 4.4. The Čech degree is p and the Deligne degree is q. For $x \in C^{p,q}_{\mathcal{U}}$ set

$$Dx = \delta x + (-1)^p d_{\mathcal{D}(2)} x.$$

Since δ commutes with the internal differential before totalization, the mixed terms cancel:

$$D^2 x = (-1)^{p+1} d_{\mathcal{D}(2)} \delta x + (-1)^p \delta d_{\mathcal{D}(2)} x = 0.$$

In total degree two this convention gives

$$D(B,A,g) = (\delta B - dA,\ \delta A + d\log g,\ \delta g).$$

Changing the normalization of $d\log$ or replacing one component by its negative changes the displayed local formulas but not the quasi-isomorphism class of the complex.

For a cochain map $r\colon A^\bullet \to B^\bullet$, we define

$$\mathrm{Cone}(r)[-1]^m = A^m \oplus B^{m-1}, \qquad d(a,b) = (d_A a, r(a) - d_B b).$$

The identity $d^2 = 0$ follows immediately from $d_B r = r d_A$. If $y \in Z^2(B)$, then

$$d(0,y) = (0, -d_B y) = 0,$$

so $(0,y)$ represents the connecting class in $H^3(\mathrm{Cone}(r)[-1])$. Moreover,

$$(0,y) = d(a,b) \quad\Longleftrightarrow\quad d_A a = 0 \ \text{ and } \ r(a) - d_B b = y.$$

This direct calculation removes any dependence on an implicit convention for the shifted cone. The alternative differential $(d_A a, -r(a) - d_B b)$ is isomorphic to ours by $(a,b) \mapsto (a,-b)$.

Finally, the prism operator in Proposition 4.4 has bidegree $(-1,0)$. Because it commutes with $d_{\mathcal{D}(2)}$, its internal cross terms in $Dh + hD$ cancel:

$$(-1)^{p-1} d_{\mathcal{D}(2)} h + (-1)^p h d_{\mathcal{D}(2)} = 0.$$

Thus a Čech chain homotopy is automatically a homotopy of the total Čech–Deligne complexes with the convention used here.

For the Ward-compatible complex there is a third, independent grading. Write p for Čech degree, q for higher-gauge or Deligne degree, and r for BRST/BV ghost degree. For $x \in C^{p,q,r}$ our total differential is

$$D_{\mathrm{tot}} x = \delta x + (-1)^p d_{\mathrm{hg}} x + (-1)^{p+q} \widehat{s} x. \tag{10}$$

Assume first that the three constituent differentials square to zero and commute pairwise before the Koszul signs are inserted. The mixed δ–d_{hg} terms have coefficients $(-1)^{p+1} + (-1)^p = 0$, while the mixed δ–$\widehat{s}$ terms have coefficients $(-1)^{p+q+1} + (-1)^{p+q} = 0$. For the remaining pair, applying d_{hg} before $\widehat{s}$ gives the coefficient

$$(-1)^p(-1)^{p+q+1} = (-1)^{q+1},$$

whereas applying $\widehat{s}$ before d_{hg} gives

$$(-1)^{p+q}(-1)^p = (-1)^q.$$

Those terms cancel as well, and hence $D_{\mathrm{tot}}^2 = 0$. If restriction and the constituent differentials commute only up to coherent homotopy, the derived totalization contains the corresponding higher homotopies; the strict calculation above is its normalized sign model. In particular, ghost degree r is not being identified with Deligne degree q.

C Finite cellular calculations

C.1 The tetrahedral pair

Give Δ^3 the orientation [0123]. The relative cellular cochain complex for $(\Delta^3, \partial\Delta^3)$ with constant coefficients in an abelian group A has exactly one generator, in degree three:

$$C^q(\Delta^3, \partial\Delta^3; A) \cong \begin{cases} A, & q = 3, \\ 0, & q \neq 3. \end{cases}$$

Consequently

$$H^q(\Delta^3, \partial\Delta^3; A) \cong \begin{cases} A, & q = 3, \\ 0, & q \neq 3. \end{cases}$$

For $A = U(1)$, the boundary of an oriented multiplicative 2-cochain is

$$(\delta g)_{0123} = g_{123} g_{023}^{-1} g_{013} g_{012}^{-1}.$$

This gives the certificate in Proposition 4.3 without appealing to a choice of differential-form representatives; compare the standard relative cellular complex in [11].

C.2 The dominated-vertex prism

Let f and g be contiguous simplicial maps. On normalized chains their standard prism is

$$P[v_0, \ldots, v_p] = \sum_{r=0}^{p} (-1)^r [f(v_0), \ldots, f(v_r), g(v_r), \ldots, g(v_p)],$$

where degenerate summands are omitted. It satisfies

$$\partial P + P\partial = g_\# - f_\#.$$

Dualizing yields the cochain homotopy h used in Proposition 4.4; see [11] for the prism operator. For the inclusion of the undominated subcomplex and the collapse $p(*) = 0$, the prism may be taken relative to that subcomplex; then $rh = 0$. With $rs = 1$ and $1 - sr = Dh + hD$, the shifted-cone contraction is not merely abstract: it is the explicit operator

$$H(a, b) = (ha + sb, 0).$$

Indeed,

$$\begin{aligned} d_K H(a, b) &= (Dha + sDb,\, b), \\ H d_K(a, b) &= (hDa + sra - sDb,\, 0), \end{aligned}$$

and their sum is (a, b). This calculation is the finite reconstruction certificate: it supplies a filler and all higher uniqueness homotopies at once.

C.3 The twisted projective-space differential

Let t generate the deck group of $S^3 \to \mathbb{RP}^3$. With a compatible choice of lifted cells, the cellular chain complex of the universal cover is free of rank one over $\mathbb{Z}[\mathbb{Z}_2]$ in degrees zero through three, with

$$\partial_q = 1 + (-1)^q t.$$

This is the standard cellular resolution with local coefficients [6, 11]. Thus the differentials alternate

$$1-t, \qquad 1+t, \qquad 1-t.$$

For the local coefficient system $U(1)_\rho$ on which t acts by inversion, the multiplicative cellular cochain complex is

$$U(1) \xrightarrow{u\mapsto u^2} U(1) \xrightarrow{u\mapsto 1} U(1) \xrightarrow{u\mapsto u^2} U(1).$$

The last arrow is absent after restriction to $\mathbb{RP}^2$. Hence a degree-two witness u on the retained subcomplex has relative obstruction u^2 on the unique omitted 3-cell. When the cell is restored, the global degree-two cohomology is

$$\ker(u \mapsto u^2)/\operatorname{im}(u \mapsto 1) = \{1, -1\},$$

which verifies both Proposition 5.6 and corollary 5.7.

C.4 The barycentric subdivision chain

The nondegenerate 3-simplices of $\operatorname{sd}\Delta^3$ are the maximal flags of nonempty faces

$$\{\sigma(0)\} \subset \{\sigma(0), \sigma(1)\} \subset \{\sigma(0), \sigma(1), \sigma(2)\} \subset \{0, 1, 2, 3\},$$

one for each permutation $\sigma \in S_4$. Orienting a flag by its displayed order gives the subdivision operator (6). In its boundary, every interior 2-face occurs twice with opposite signs. The uncancelled terms are exactly the barycentric subdivision of $\partial[0123]$. Hence

$$\partial s[0123] = s\partial[0123],$$

so $s[0123]$ is a relative cycle for $(\operatorname{sd}\Delta^3, \operatorname{sd}\partial\Delta^3)$.

The last-vertex simplicial approximation induces a chain-homotopy inverse to s, also relative to the boundary [11]. This proves directly that subdivision is a chain equivalence of the two pairs. If a multiplicative relative 3-cochain d is changed by a coboundary δa, then

$$\langle \delta a, s[0123]\rangle = \langle a, \partial s[0123]\rangle = 1$$

in the relative complex. Therefore

$$\prod_{\sigma\in S_4} d_\sigma^{\operatorname{sgn}(\sigma)}$$

is independent of all intermediate filler choices. The 24 refined tetrahedra are local representatives of one obstruction generator, not 24 independent generators.

References

[1] A. K. Bousfield and D. M. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Mathematics, vol. 304, Springer, Berlin, 1972.

[2] J. C. Baez and A. D. Lauda, *Higher-dimensional algebra V: 2-groups*, Theory Appl. Categ. **12** (2004), 423–491; arXiv:math/0307200.

[3] M. Benini and A. Schenkel, *Quantum field theories on categories fibered in groupoids*, Commun. Math. Phys. **356** (2017), 19–64; arXiv:1610.06071.

[4] M. Benini, A. Schenkel, and U. Schreiber, *The stack of Yang–Mills fields on Lorentzian manifolds*, Commun. Math. Phys. **359** (2018), 765–820; arXiv:1704.01378.

[5] M. Benini, A. Schenkel, and L. Woike, *Homotopy theory of algebraic quantum field theories*, Lett. Math. Phys. **109** (2019), 1487–1532; arXiv:1805.08795.

[6] K. S. Brown, *Cohomology of Groups*, Graduate Texts in Mathematics, vol. 87, Springer, New York, 1982.

[7] P. Du, *Reshaping limit diagrams and cofinality in higher category theory*, arXiv:2308.01798.

[8] D. Fiorenza, U. Schreiber, and J. Stasheff, *Čech cocycles for differential characteristic classes: an ∞-Lie theoretic construction*, Adv. Theor. Math. Phys. **16** (2012), 149–250; arXiv:1011.4735.

[9] K. Fredenhagen and K. Rejzner, *Batalin–Vilkovisky formalism in perturbative algebraic quantum field theory*, Commun. Math. Phys. **317** (2013), 697–725; arXiv:1110.5232.

[10] P. G. Goerss and J. F. Jardine, *Simplicial Homotopy Theory*, Progress in Mathematics, vol. 174, Birkhäuser, Basel, 1999.

[11] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002; author's online edition.

[12] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, 2009.

[13] T. Nikolaus, U. Schreiber, and D. Stevenson, *Principal ∞-bundles: general theory*, J. Homotopy Relat. Struct. **10** (2015), 749–801; arXiv:1207.0248.

[14] T. Nikolaus and K. Waldorf, *Four equivalent versions of non-abelian gerbes*, Pacific J. Math. **264** (2013), 355–420; arXiv:1103.4815.

[15] K. Rejzner, *BV quantization in perturbative algebraic QFT: fundamental concepts and perspectives*, arXiv:2004.14272.

[16] U. Schreiber, *Differential cohomology in a cohesive ∞-topos*, arXiv:1310.7930.

[17] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.