Renormalization as Admissible Transport: Ground-Lineage Compatibility, Dirichlet Covariance, and Refinement Descent

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A coarse-graining map is not yet a renormalization map. Even a completely positive, nuclear, state-preserving, and compositional map may fail to transport the carrier selected by the theory. We formulate the resulting interface problem for scale-indexed represented operator systems equipped with distinguished carrier vectors and closed Dirichlet forms. For each raw admissible refinement, an observable coarse-graining map and a bounded selected carrier map are required to satisfy ground-lineage compatibility, so that they describe the same represented scale arrow. Certification is imposed independently by requiring the carrier map to intertwine the Markov semigroups generated by the scale-dependent Dirichlet forms. This Dirichlet covariance preserves the corresponding fixed sectors and, in particular, transports distinguished zero modes. If the certified carrier maps compose on raw refinement presentations and agree on common-refinement-equivalent presentations, the raw transport functor descends to a presentation-independent scale category. A finite noncommutative realization is provided by nested finite von Neumann algebras with trace-preserving conditional expectations and compatible depolarizing Dirichlet semigroups. Scale-nuclear coarse-graining may be imposed separately to provide additional analytic control of the certified arrows. The result separates carrier identification, invariant-sector certification, analytic scale regularity, and refinement descent, so that compatibility is established at the level of scale morphisms before any continuum realization is invoked.

I. INTRODUCTION

A. The scale-arrow problem

A scale map does not become a renormalization map merely because it goes in the right direction.
Let

$$r : \mu \to \lambda$$

be a raw admissible refinement with observable coarse-graining

$$\Phi_r : \mathscr{O}_\mu^{0,\mathrm{tr}} \longrightarrow \mathscr{O}_\lambda .$$

Such maps arise from conditional expectations, quantum channels, block transformations, effective Hamiltonians, integration over unresolved variables, and operator-algebraic scaling constructions. Rigorous operator-algebraic renormalization already provides scaling maps, compatible state families, GNS carrier maps, inductive systems, and continuum scaling constructions[2,9,10]. The problem considered here lies between the construction of a coarse-graining map and the construction of its continuum endpoint.

Suppose the two scale presentations carry distinguished vectors

$$\Omega_\mu \in \mathscr{H}_\mu, \qquad \Omega_\lambda \in \mathscr{H}_\lambda,$$

and that the theory independently selects a bounded carrier map

$$T_r : \mathscr{H}_\mu \longrightarrow \mathscr{H}_\lambda .$$

There are now two descriptions of scale change:

$$X \longmapsto \Phi_r(X)$$

and

$$X\Omega_\mu \longmapsto T_r X \Omega_\mu .$$

The first question is whether these are the same represented transport. The second is whether that carrier map preserves the invariant structure selected by the analytic theory. The third is whether the answer depends on which finite refinement presentation was used.

Those are the three questions addressed here.

The word *admissible* is used throughout relative to the specified Dirichlet interface. A certified arrow preserves the Dirichlet fixed sector selected by that interface. No stronger notion of physical admissibility is implicit.

B. Ground-lineage compatibility

We call

$$\boxed{T_r(X\Omega_\mu) = \Phi_r(X)\Omega_\lambda} \tag{I.1}$$

ground-lineage compatibility (GLC).

Equation (I.1) says that the observable map and the selected carrier map are two representations of the same scale arrow. It is not a positivity condition. A UCP map can fail (I.1). Nor is it merely a statement about expectation values. Equation (I.1) identifies the represented vector, not one scalar functional of it.

Under GLC,

$$X\Omega_\mu \longmapsto \Phi_r(X)\Omega_\lambda$$

is simply T_r on the declared observable carrier. The scale problem has therefore moved from the observable map to the selected carrier map. Nothing has yet been certified.

If the coarse-graining maps are unital and the transport core contains 1, then (I.1) with $X = 1$ gives

$$T_r\Omega_\mu = \Omega_\lambda.$$

We retain preservation of the distinguished vector explicitly in the abstract theorem so that nonunital variants remain covered.

C. Dirichlet certification

For each scale λ, let $\mathscr{E}_\lambda$ be a closed symmetric Dirichlet form associated with a specified ambient GNS or standard L^2-realization containing $\mathscr{O}_\lambda$, with nonnegative self-adjoint generator L_λ and Markov semigroup

$$S_\lambda(t) = e^{-tL_\lambda}.$$

We use the standard noncommutative Dirichlet-form framework[1,7,8].

Set

$$K_\lambda = \ker L_\lambda = \operatorname{Fix} S_\lambda.$$

The distinguished vector satisfies

$$\boxed{L_\lambda\Omega_\lambda = 0.} \tag{I.2}$$

Thus Ω_λ lies in the Dirichlet fixed sector.

Nothing here says that $X\Omega_\lambda$ is a zero mode for arbitrary X. In a nontrivial theory one generally expects the opposite.

The relevant scale covariance is

$$\boxed{T_r S_\mu(t) = S_\lambda(t) T_r.} \tag{I.3}$$

Because T_r is bounded, (I.3) implies

$$T_r \operatorname{Dom}(L_\mu) \subseteq \operatorname{Dom}(L_\lambda), \qquad L_\lambda T_r\xi = T_r L_\mu \xi.$$

Hence

$$\xi \in K_\mu \quad \Longrightarrow \quad T_r\xi \in K_\lambda.$$

In particular, (I.2) at scale μ and

$$T_r\Omega_\mu = \Omega_\lambda$$

give $L_\lambda\Omega_\lambda = 0$.

GLC identifies the carrier arrow. Dirichlet covariance certifies its invariant sector. They are different statements.

D. Closedness

Closedness enters at exactly one place. Since the Dirichlet form is closed, L_λ is closed and

$$K_\lambda = \ker L_\lambda$$

is a closed subspace. Thus

$$\xi_n \in K_\lambda, \qquad \xi_n \to \xi$$

implies

$$\xi \in K_\lambda.$$

The invariant sector cannot leak merely because the carrier is completed.

Closedness does not identify the scale arrow. It does not produce Dirichlet covariance. It does not remove refinement dependence.

E. Raw refinements

Let $\mathsf{Ref}^{s,\mathrm{adm}}$ denote the category of raw admissible refinement presentations. Suppose

$$r : \nu \to \mu, \qquad s : \mu \to \lambda.$$

If

$$T_{s\circ r} = T_s T_r,$$

then Dirichlet covariance is stable under composition:

$$T_s T_r S_\nu(t) = T_s S_\mu(t) T_r = S_\lambda(t) T_s T_r.$$

Hence the certified carrier maps define a functor

$$T^{\mathrm{raw}} : \mathsf{Ref}^{s,\mathrm{adm}} \longrightarrow \mathbf{Hilb}_{\mathrm{Dir}}.$$

At this stage the functor still remembers the raw refinement presentation.

F. Common refinement

Let $r_1, r_2 : \mu \to \lambda$ be raw refinement presentations. Write

$$r_1 \sim_{\mathrm{cr}} r_2$$

when they are related by admitted common-refinement moves. We assume $\sim_{\mathrm{cr}}$ preserves source and target and is a categorical congruence: it is stable under pre- and postcomposition whenever the relevant composites exist.

The coherence condition is

$$\boxed{r_1 \sim_{\mathrm{cr}} r_2 \quad \Longrightarrow \quad T_{r_1} = T_{r_2}.} \tag{I.4}$$

Define

$$\Lambda^{s,\mathrm{adm}} = \mathsf{Ref}^{s,\mathrm{adm}} / \sim_{\mathrm{cr}}.$$

Then T^{raw} descends uniquely to

$$\mathfrak{T} : \Lambda^{s,\mathrm{adm}} \longrightarrow \mathbf{Hilb}_{\mathrm{Dir}}.$$

This is the renormalization morphism system.

G. Main theorem

The global result is Theorem V.4. In brief, ground-lineage compatibility identifies the represented carrier arrow, Dirichlet covariance preserves its fixed sector, raw composition gives a functor on refinement presentations, and common-refinement invariance makes that functor descend to

$$\mathfrak{T} : \Lambda^{s,\mathrm{adm}} \longrightarrow \mathbf{Hilb}_{\mathrm{Dir}}.$$

The body of the paper keeps these four steps separate so that none is hidden inside the word "renormalization."

H. A model in which all hypotheses hold

The assumptions occur simultaneously in a familiar noncommutative setting. Let

$$\mathscr{M}_\lambda \subseteq \mathscr{M}_\mu$$

be nested finite von Neumann algebras with faithful normal trace τ, and let

$$E_{\mu\to\lambda} : \mathscr{M}_\mu \to \mathscr{M}_\lambda$$

be the τ-preserving conditional expectation. Its L^2-implementation is the orthogonal projection

$$T_{\mu\to\lambda} : L^2(\mathscr{M}_\mu, \tau) \to L^2(\mathscr{M}_\lambda, \tau)$$

[11].

Then

$$T_{\mu\to\lambda}\Lambda_\mu(X) = \Lambda_\lambda\left(E_{\mu\to\lambda}(X)\right),$$

so GLC holds. For

$$L_\lambda = I - P_\lambda,$$

with P_λ the projection onto the scalar vector,

$$S_\lambda(t) = P_\lambda + e^{-t}(I - P_\lambda),$$

and trace preservation gives

$$T_{\mu\to\lambda}S_\mu(t) = S_\lambda(t)T_{\mu\to\lambda}.$$

The tower property of conditional expectations gives composition and common-refinement independence.

I. SNC

Scale-nuclear coarse-graining is not the certification theorem. It is extra analytic structure on the same scale arrows. When SNC holds, the certified morphisms may inherit nuclearity, compactness, finite-resolution control, collar estimates, or the corresponding analytic bounds supplied by the SNC result being used. The proof of Theorem V.4 does not use SNC.

Nuclearity and split-type conditions occupy this analytic phase-space side of AQFT[3–6].

J. Position of the result

The paper does not claim novelty for coarse-graining maps, GNS carrier maps, conditional expectations, Dirichlet forms, nuclearity, or functorial renormalization separately. The contribution is the interface theorem that asks these structures to agree on one scale arrow:

$$\text{observable map} \longleftrightarrow \text{selected carrier} \longleftrightarrow \text{Dirichlet fixed structure},$$

followed by descent through refinement equivalence.

The point is not another RG prescription. It is a criterion for when a proposed scale system already constitutes admissible renormalization transport.

K. Continuum realization

The theorem does not construct a continuum endpoint. It provides the scale-morphism system that such an endpoint must realize. One may subsequently ask for a scaling limit, an inductive or projective construction, a local algebraic realization, or another appropriate completion.

In this framework, compatibility has already been settled before that question is asked.

L. Local extension

For region-indexed systems

$$O \longmapsto \mathscr{O}_\lambda(O),$$

a raw scale arrow may transport localization through an order-preserving map

$$\rho_r : \mathsf{Reg} \to \mathsf{Reg}.$$

The local GLC condition is

$$T_r(X\Omega_\mu) = \Phi_r^O(X)\Omega_\lambda,$$

with

$$\Phi_r^O(X) \in \mathscr{O}_\lambda(\rho_r(O)).$$

Dirichlet covariance is imposed regionwise. Under compatibility with isotony and refinement composition, the same descent argument produces certified morphisms of local interfaces.

M. Scope

The theorem does not assert

$$\text{CP} \Longrightarrow \text{GLC}, \qquad \text{GLC} \Longrightarrow \text{Dirichlet covariance},$$

or

$$\text{Dirichlet covariance} \Longrightarrow \text{common-refinement descent}.$$

It does not claim that SNC implies certification, and it does not construct a continuum theory.

Its claim is narrower: when these interfaces close on the same scale arrows, they define a presentation-independent certified transport system.

N. Organization

Section 2 fixes the represented scale interface and GLC. Section 3 develops Dirichlet certification. Section 4 defines certified scale arrows. Section 5 proves composition and common-refinement descent. Section 6 gives the conditional-expectation model and the obstruction examples. Section 7 gives the local-net/collar formulation. Section 8 positions the theorem relative to standard renormalization structures and continuum realization.

II. SCALE-INDEXED REPRESENTED INTERFACES

Let $\mathsf{Ref}^{s,\mathrm{adm}}$ be the category of raw admissible refinement presentations. For each scale λ, let

$$\mathscr{O}_\lambda \subseteq B(\mathscr{H}_\lambda)$$

be a represented operator system with distinguished unit vector

$$\Omega_\lambda \in \mathscr{H}_\lambda.$$

Fix a self-adjoint transport core

$$\mathscr{O}_\lambda^{0,\mathrm{tr}} \subseteq \mathscr{O}_\lambda$$

and set

$$\mathscr{D}_\lambda = \operatorname{span}\left(\{\Omega_\lambda\} \cup \{X\Omega_\lambda : X \in \mathscr{O}_\lambda^{0,\mathrm{tr}}\}\right). \tag{2.1}$$

For a raw refinement $r : \mu \to \lambda$, let

$$\Phi_r : \mathscr{O}_\mu^{0,\mathrm{tr}} \longrightarrow \mathscr{O}_\lambda$$

be the observable coarse-graining map and

$$T_r : \mathscr{H}_\mu \longrightarrow \mathscr{H}_\lambda$$

a bounded selected carrier map. We require

$$T_r\Omega_\mu = \Omega_\lambda.$$

Definition II.1 (Ground-lineage compatibility)**.** The arrow r satisfies *ground-lineage compatibility* if

$$\boxed{T_r(X\Omega_\mu) = \Phi_r(X)\Omega_\lambda} \tag{2.2}$$

for every $X \in \mathscr{O}_\mu^{0,\mathrm{tr}}$.

Equivalently, with $q_\lambda(X) = X\Omega_\lambda$,

$$q_\lambda \Phi_r = T_r q_\mu. \tag{2.3}$$

Proposition II.2 (Carrier realization of GLC)**.** *Assume GLC. Then*

$$X\Omega_\mu = Y\Omega_\mu$$

implies

$$\Phi_r(X)\Omega_\lambda = \Phi_r(Y)\Omega_\lambda.$$

Hence

$$T_r^0(X\Omega_\mu) := \Phi_r(X)\Omega_\lambda$$

is well defined on $\mathscr{D}_\mu$, with

$$T_r^0 = T_r|_{\mathscr{D}_\mu}.$$

Proof. By GLC,

$$\Phi_r(X)\Omega_\lambda = T_r X\Omega_\mu = T_r Y\Omega_\mu = \Phi_r(Y)\Omega_\lambda.$$

The final identity is immediate. □

For iterative observable transport we assume

$$\Phi_r(\mathscr{O}_\mu^{0,\mathrm{tr}}) \subseteq \mathscr{O}_\lambda^{0,\mathrm{tr}}. \tag{2.4}$$

Then

$$T_r\mathscr{D}_\mu \subseteq \mathscr{D}_\lambda. \tag{2.5}$$

III. DIRICHLET CERTIFICATION

For each scale λ, let $\mathscr{E}_\lambda$ be a closed symmetric Dirichlet form on the specified ambient GNS or standard L^2-realization, with nonnegative self-adjoint generator L_λ and semigroup

$$S_\lambda(t) = e^{-tL_\lambda}, \qquad t \geq 0. \tag{3.1}$$

Write

$$K_\lambda = \ker L_\lambda = \operatorname{Fix} S_\lambda. \tag{3.2}$$

Assume

$$L_\lambda \Omega_\lambda = 0. \tag{3.3}$$

Definition III.1 (Dirichlet covariance)**.** A bounded carrier map

$$T_r : \mathscr{H}_\mu \to \mathscr{H}_\lambda$$

is *Dirichlet-covariant* if

$$\boxed{T_r S_\mu(t) = S_\lambda(t) T_r, \qquad t \geq 0.} \tag{3.4}$$

By the generator characterization of strongly continuous semigroups,

$$T_r \operatorname{Dom}(L_\mu) \subseteq \operatorname{Dom}(L_\lambda) \tag{3.5}$$

and

$$L_\lambda T_r \xi = T_r L_\mu \xi, \qquad \xi \in \operatorname{Dom}(L_\mu). \tag{3.6}$$

Theorem III.2 (Fixed-sector certification)**.** *If T_r is Dirichlet-covariant, then*

$$\boxed{T_r K_\mu \subseteq K_\lambda.} \tag{3.7}$$

Proof. For $\xi \in K_\mu$,

$$S_\mu(t)\xi = \xi.$$

Therefore

$$S_\lambda(t) T_r \xi = T_r S_\mu(t) \xi = T_r \xi.$$

Hence $T_r \xi \in K_\lambda$. □

Corollary III.3 (Ground certification)**.** *If*

$$L_\mu \Omega_\mu = 0$$

and

$$T_r \Omega_\mu = \Omega_\lambda,$$

then

$$L_\lambda \Omega_\lambda = 0.$$

Since L_λ is self-adjoint, K_λ is closed. Certification of the fixed sector therefore survives Hilbert completion. No assertion is made that

$$X\Omega_\lambda \in K_\lambda$$

for generic X.

IV. CERTIFIED SCALE ARROWS

Definition IV.1 (Certified raw arrow)**.** A raw refinement $r : \mu \to \lambda$ is *certified* if

$$T_r(X\Omega_\mu) = \Phi_r(X)\Omega_\lambda$$

on $\mathscr{O}_\mu^{0,\mathrm{tr}}$,

$$T_r\Omega_\mu = \Omega_\lambda,$$

and

$$T_r S_\mu(t) = S_\lambda(t) T_r \qquad (t \geq 0).$$

When the observable cores are iterated, we also impose

$$\Phi_r(\mathscr{O}_\mu^{0,\mathrm{tr}}) \subseteq \mathscr{O}_\lambda^{0,\mathrm{tr}}.$$

Theorem IV.2 (Certified admissible transport)**.** *If r is certified, then*

$$T_r(X\Omega_\mu) = \Phi_r(X)\Omega_\lambda$$

on the declared carrier and

$$\boxed{T_r K_\mu \subseteq K_\lambda.}$$

In particular,

$$L_\mu \Omega_\mu = 0 \quad \Longrightarrow \quad L_\lambda \Omega_\lambda = 0.$$

Proof. The first identity is GLC; fixed-sector preservation is Theorem III.2. The final statement follows from preservation of Ω. □

Certification refers to preservation of the specified Dirichlet fixed sector. It does not identify the full represented carrier with that sector.

A. Dirichlet carrier category

Let $\mathbf{Hilb}_{\mathrm{Dir}}$ be the category whose objects are triples

$$(\mathscr{H}, \Omega, S)$$

arising from the specified closed Dirichlet structures, with

$$S(t)\Omega = \Omega,$$

and whose morphisms are bounded maps

$$T : (\mathscr{H}_1, \Omega_1, S_1) \to (\mathscr{H}_2, \Omega_2, S_2)$$

satisfying

$$T\Omega_1 = \Omega_2, \qquad TS_1(t) = S_2(t)T.$$

The generator is determined by the semigroup and is suppressed from the categorical notation.

V. COMPOSITION AND REFINEMENT DESCENT

Let

$$r:\nu\to\mu,\qquad s:\mu\to\lambda$$

be certified raw arrows. Assume

$$T_{s\circ r}=T_sT_r. \tag{5.1}$$

Proposition V.1 (Stability under composition)**.** *The composite T_sT_r preserves the distinguished vectors and intertwines the Dirichlet semigroups:*

$$T_sT_r\Omega_\nu=\Omega_\lambda,$$

and

$$T_sT_rS_\nu(t)=S_\lambda(t)T_sT_r.$$

If additionally

$$\Phi_{s\circ r}=\Phi_s\Phi_r$$

on the transport core, then the composite satisfies GLC.

Proof. Ground-vector preservation is immediate. Further,

$$T_sT_rS_\nu(t)=T_sS_\mu(t)T_r=S_\lambda(t)T_sT_r.$$

If the observable maps compose, then

$$T_sT_r(X\Omega_\nu)=\Phi_s(\Phi_r(X))\Omega_\lambda=\Phi_{s\circ r}(X)\Omega_\lambda.$$

□

If GLC for $s\circ r$ is supplied independently, observable-level composition is unnecessary.

A. Raw transport functor

Assume every arrow of $\mathsf{Ref}^{s,\mathrm{adm}}$ is certified and

$$T_{\mathrm{id}_\lambda}=I_{\mathscr{H}_\lambda},\qquad T_{s\circ r}=T_sT_r. \tag{5.2}$$

Theorem V.2 (Raw certified transport)**.** *The assignments*

$$\lambda\longmapsto(\mathscr{H}_\lambda,\Omega_\lambda,S_\lambda),\qquad r\longmapsto T_r$$

define a functor

$$\boxed{T^{\mathrm{raw}}:\mathsf{Ref}^{s,\mathrm{adm}}\longrightarrow\mathbf{Hilb}_{\mathrm{Dir}}.} \tag{5.3}$$

Proof. Each T_r is a morphism of $\mathbf{Hilb}_{\mathrm{Dir}}$. The displayed identity and composition laws give the functor laws. □

B. Common-refinement equivalence

Let $\sim_{\mathrm{cr}}$ be a source- and target-preserving categorical congruence generated by admitted common-refinement moves. Assume

$$r_1\sim_{\mathrm{cr}}r_2\quad\Longrightarrow\quad T_{r_1}=T_{r_2}. \tag{5.4}$$

Define

$$\boxed{\Lambda^{s,\mathrm{adm}}=\mathsf{Ref}^{s,\mathrm{adm}}/\sim_{\mathrm{cr}}.} \tag{5.5}$$

Let

$$Q:\mathsf{Ref}^{s,\mathrm{adm}}\longrightarrow\Lambda^{s,\mathrm{adm}}$$

be the quotient functor.

Theorem V.3 (Common-refinement descent)**.** *There is a unique functor*

$$\boxed{\mathfrak{T} : \Lambda^{s,\mathrm{adm}} \longrightarrow \mathbf{Hilb}_{\mathrm{Dir}}} \tag{5.6}$$

such that

$$T^{\mathrm{raw}} = \mathfrak{T} \circ Q. \tag{5.7}$$

Explicitly,

$$\mathfrak{T}([r]) = T_r.$$

Proof. Common-refinement invariance makes $\mathfrak{T}([r]) = T_r$ well defined. For composable classes,

$$\mathfrak{T}([s][r]) = \mathfrak{T}([s \circ r]) = T_{s \circ r} = T_s T_r = \mathfrak{T}([s])\mathfrak{T}([r]).$$

Identity preservation is immediate. Uniqueness follows from the quotient property. □

Theorem V.4 (Renormalization as admissible transport)**.** *Suppose every raw admissible refinement $r : \mu \to \lambda$ carries an observable map Φ_r, a bounded carrier map T_r, and closed Dirichlet data*

$$(\mathscr{E}_\lambda, L_\lambda, S_\lambda)$$

such that

$$T_r(X\Omega_\mu) = \Phi_r(X)\Omega_\lambda,$$

$$T_r\Omega_\mu = \Omega_\lambda,$$

$$T_r S_\mu(t) = S_\lambda(t) T_r,$$

the carrier maps preserve identities and raw composition, and

$$r_1 \sim_{\mathrm{cr}} r_2 \quad \Longrightarrow \quad T_{r_1} = T_{r_2}.$$

Then

$$T_r K_\mu \subseteq K_\lambda,$$

the raw scale system defines

$$\boxed{T^{\mathrm{raw}} : \mathsf{Ref}^{s,\mathrm{adm}} \to \mathbf{Hilb}_{\mathrm{Dir}},}$$

and this functor descends uniquely to

$$\boxed{\mathfrak{T} : \Lambda^{s,\mathrm{adm}} \to \mathbf{Hilb}_{\mathrm{Dir}}.}$$

If the same arrows additionally satisfy an independently established SNC condition stable under raw composition and common-refinement equivalence, the descended functor carries the corresponding analytic scale regularity.

Proof. Fixed-sector preservation is Theorem III.2; functoriality is Theorem V.2; descent is Theorem V.3. □

The proof uses exactly three structural inputs: GLC identifies the represented carrier arrow, Dirichlet covariance certifies its invariant sector, and common refinement removes presentation dependence.

VI. A FINITE MODEL AND THREE OBSTRUCTIONS

A. Conditional-expectation model

Let

$$\mathscr{M}_\lambda \subseteq \mathscr{M}_\mu \subseteq \mathscr{M}$$

be finite von Neumann algebras with faithful normal trace τ. Set

$$\mathscr{H}_\lambda = L^2(\mathscr{M}_\lambda, \tau), \qquad \Omega_\lambda = \Lambda_\lambda(1).$$

Let

$$E_{\mu\to\lambda} : \mathscr{M}_\mu \longrightarrow \mathscr{M}_\lambda \tag{6.1}$$

be the τ-preserving conditional expectation and

$$T_{\mu\to\lambda} : L^2(\mathscr{M}_\mu, \tau) \longrightarrow L^2(\mathscr{M}_\lambda, \tau) \tag{6.2}$$

its L^2-implementation. Take

$$\Phi_{\mu\to\lambda} = E_{\mu\to\lambda}.$$

Then

$$T_{\mu\to\lambda}\Lambda_\mu(X) = \Lambda_\lambda\big(\Phi_{\mu\to\lambda}(X)\big), \tag{6.3}$$

and

$$T_{\mu\to\lambda}\Omega_\mu = \Omega_\lambda. \tag{6.4}$$

Let P_λ be the orthogonal projection onto $\mathbb{C}\Omega_\lambda$ and define

$$L_\lambda = I - P_\lambda, \qquad S_\lambda(t) = P_\lambda + e^{-t}(I - P_\lambda). \tag{6.5}$$

Then

$$K_\lambda = \mathbb{C}\Omega_\lambda. \tag{6.6}$$

Since $E_{\mu\to\lambda}$ preserves τ and 1,

$$T_{\mu\to\lambda}P_\mu = P_\lambda T_{\mu\to\lambda}, \tag{6.7}$$

hence

$$T_{\mu\to\lambda}S_\mu(t) = S_\lambda(t)T_{\mu\to\lambda}. \tag{6.8}$$

For nested algebras

$$\mathscr{M}_\kappa \subseteq \mathscr{M}_\lambda \subseteq \mathscr{M}_\mu,$$

the tower property gives

$$E_{\lambda\to\kappa}E_{\mu\to\lambda} = E_{\mu\to\kappa}, \tag{6.9}$$

and therefore

$$T_{\lambda\to\kappa}T_{\mu\to\lambda} = T_{\mu\to\kappa}. \tag{6.10}$$

If

$$\mathscr{M}_\kappa \subseteq \mathscr{M}_{\lambda_1}, \mathscr{M}_{\lambda_2} \subseteq \mathscr{M}_\mu,$$

then

$$T_{\lambda_1\to\kappa}T_{\mu\to\lambda_1} = T_{\lambda_2\to\kappa}T_{\mu\to\lambda_2}. \tag{6.11}$$

Proposition VI.1 (Conditional-expectation realization)**.** *The system above satisfies GLC, Dirichlet covariance, raw composition, and common-refinement invariance. Hence it defines*

$$\mathfrak{T} : \Lambda^{s,\mathrm{adm}} \longrightarrow \mathbf{Hilb}_{\mathrm{Dir}}.$$

B. Complete positivity does not imply GLC

Let

$$\mathscr{M} = M_2(\mathbb{C})$$

with normalized trace, and set

$$\mathscr{H} = L^2(\mathscr{M}, \tau).$$

For

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

define

$$\Phi(X) = ZXZ.$$

Then Φ is a unital trace-preserving $*$-automorphism. Choose the selected carrier map

$$T = I.$$

For

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

one has $\Phi(X) = -X$. Hence

$$T\Lambda(X) = \Lambda(X) \neq -\Lambda(X) = \Lambda(\Phi(X)).$$

Proposition VI.2 (Channel/lineage separation)**.** *A UCP, trace-preserving observable map need not satisfy GLC relative to a separately selected carrier map:*

$$\text{UCP} \not\Longrightarrow \text{GLC}.$$

C. GLC does not imply Dirichlet covariance

On the same

$$\mathscr{H} = L^2(M_2(\mathbb{C}), \tau),$$

take

$$\Phi = I, \qquad T = I.$$

GLC is automatic.

Let P project onto the scalar vector and choose

$$L_\mu = I - P, \qquad L_\lambda = 2(I - P).$$

Then

$$S_\mu(t) = P + e^{-t}(I - P), \qquad S_\lambda(t) = P + e^{-2t}(I - P).$$

Both fix Ω, but

$$TS_\mu(t) = S_\mu(t) \neq S_\lambda(t) = S_\lambda(t)T$$

for $t > 0$.

Proposition VI.3 (Lineage/certification separation)**.** *GLC and preservation of the distinguished vector do not imply Dirichlet covariance:*

$$\text{GLC} \not\Longrightarrow \textit{Dirichlet covariance}.$$

D. Certified arrows do not imply refinement descent

Again let

$$\mathscr{H} = L^2(M_2(\mathbb{C}), \tau)$$

with

$$S(t) = P + e^{-t}(I - P).$$

Define

$$U\Lambda(X) = \Lambda(ZXZ).$$

Then $U\Omega = \Omega$, $UP = PU$, and

$$US(t) = S(t)U.$$

Let

$$r_1, r_2 : \mu \to \lambda$$

be raw refinement presentations with

$$r_1 \sim_{\mathrm{cr}} r_2,$$

and assign

$$T_{r_1} = I, \qquad T_{r_2} = U.$$

Take

$$\Phi_{r_1} = I, \qquad \Phi_{r_2}(X) = ZXZ.$$

Each raw arrow satisfies GLC and Dirichlet covariance, but

$$T_{r_1} \neq T_{r_2}.$$

Proposition VI.4 (Certification/descent separation)**.** *Certified raw arrows need not be invariant under common-refinement presentation:*

$$\textit{certification} \not\Longrightarrow \textit{refinement descent}.$$

The three conditions in Theorem V.4 are therefore independent:

$$\text{GLC}, \qquad \text{Dirichlet covariance}, \qquad \text{common-refinement invariance}.$$

VII. LOCAL NETS AND COLLAR TRANSPORT

Let Reg be a poset of admissible regions. For each scale λ, let

$$O \longmapsto \mathscr{O}_\lambda(O)$$

be an isotone represented operator-system net. Assume

$$1 \in \mathscr{O}_\lambda^{0,\mathrm{tr}}(O)$$

for every admissible region O.

For a raw refinement

$$r : \mu \to \lambda,$$

let

$$\rho_r : \mathsf{Reg} \longrightarrow \mathsf{Reg} \tag{7.1}$$

be order preserving and

$$\Phi_r^O : \mathscr{O}_\mu^{0,\mathrm{tr}}(O) \longrightarrow \mathscr{O}_\lambda^{0,\mathrm{tr}}(\rho_r(O)). \tag{7.2}$$

Exact localization is $\rho_r(O) = O$. A collar formulation allows

$$O \subseteq \rho_r(O) \subseteq O^{+\varepsilon_r}. \tag{7.3}$$

For each region define

$$\mathscr{H}_\lambda(O) = \overline{\mathscr{O}_\lambda^{0,\mathrm{tr}}(O)\Omega_\lambda}. \tag{7.4}$$

Assume

$$T_r \mathscr{H}_\mu(O) \subseteq \mathscr{H}_\lambda(\rho_r(O)). \tag{7.5}$$

Definition VII.1 (Local GLC)**.** The raw arrow satisfies local GLC on O if

$$\boxed{T_r(X\Omega_\mu) = \Phi_r^O(X)\Omega_\lambda} \tag{7.6}$$

for $X \in \mathscr{O}_\mu^{0,\mathrm{tr}}(O)$.

Assume compatibility with isotony:

$$\Phi_r^{O_2}\big|_{\mathscr{O}_\mu^{0,\mathrm{tr}}(O_1)} = j_{\rho_r(O_1),\rho_r(O_2)}\Phi_r^{O_1} \tag{7.7}$$

whenever

$$O_1 \subseteq O_2.$$

For each pair (λ, O), let

$$S_{\lambda,O}(t) = e^{-tL_{\lambda,O}}$$

be the semigroup associated with a closed Dirichlet form on $\mathscr{H}_\lambda(O)$, with

$$L_{\lambda,O}\Omega_\lambda = 0. \tag{7.8}$$

Local Dirichlet covariance means

$$\boxed{T_r S_{\mu,O}(t) = S_{\lambda,\rho_r(O)}(t) T_r} \tag{7.9}$$

on $\mathscr{H}_\mu(O)$. Hence

$$T_r(\ker L_{\mu,O}) \subseteq \ker L_{\lambda,\rho_r(O)}. \tag{7.10}$$

Theorem VII.2 (Local renormalization interface)**.** *Assume for every raw refinement r and region O:*

$$T_r(X\Omega_\mu) = \Phi_r^O(X)\Omega_\lambda,$$

$$T_r S_{\mu,O}(t) = S_{\lambda,\rho_r(O)}(t) T_r,$$

the observable maps commute with isotony, and

$$\rho_{s\circ r} = \rho_s \circ \rho_r.$$

Then T_r transports the local Dirichlet fixed sectors and respects regional inclusion.

If additionally

$$T_{s\circ r} = T_s T_r$$

and common-refinement-equivalent raw arrows agree after passage to the corresponding common target region, the local scale system descends through

$$\mathsf{Ref}^{s,\mathrm{adm}} \longrightarrow \Lambda^{s,\mathrm{adm}}.$$

Proof. Fixed-sector preservation follows from Theorem III.2 applied regionwise. Isotony gives regional naturality. Composition follows from the stated identities, and common-refinement descent is the argument of Theorem V.3 after the target inclusions. □

This is an interface theorem, not a Haag–Kastler reconstruction. Locality, covariance, time-slice, duality, and multiplicative realization remain separate questions. If local SNC is available, its analytic estimates may be imposed on these same arrows.

VIII. DISCUSSION AND CONCLUSION

The theorem does not introduce another RG prescription. It gives a criterion for when a proposed scale-arrow system already carries certified renormalization transport.

GLC identifies the represented carrier map. Dirichlet covariance preserves its invariant sector. Common-refinement invariance removes dependence on refinement presentation. The result is

$$\boxed{\mathfrak{T} : \Lambda^{s,\mathrm{adm}} \longrightarrow \mathbf{Hilb}_{\mathrm{Dir}}.} \tag{8.1}$$

A. Relation to standard coarse-graining

Complete positivity, conditional expectations, state consistency, and RG composition are compatible with the theorem but do not replace its interface conditions. A UCP map may fail GLC. A GLC-compatible carrier may fail Dirichlet covariance. Certified raw arrows may fail common-refinement descent. The conditional-expectation model shows that the conditions can also hold simultaneously.

Scaling algebras and operator-algebraic real-space RG provide closely neighboring scale frameworks[2,9,10]. The present theorem isolates the additional carrier-identification, Dirichlet-certification, and refinement-descent conditions on a given family of scale arrows.

B. Dirichlet structure

The fixed sector

$$K_\lambda = \ker L_\lambda$$

is the invariant sector specified by the chosen Dirichlet interface. Certification means

$$T_r K_\mu \subseteq K_\lambda .$$

It does not mean

$$X\Omega_\lambda \in K_\lambda$$

for generic observables. The generator remains nontrivial away from the fixed sector.

C. SNC

The proof nowhere uses SNC. When available, SNC is extra analytic structure on the already certified arrows. It may supply nuclearity, compactness, finite-resolution bounds, collar control, or related phase-space estimates. Nuclearity and split properties have an established AQFT literature[3–6]. Any theorem relating SNC to the certification mechanism here would be additional.

D. Local systems

The local extension adds the localization map ρ_r. The same carrier transport is required to respect the represented local interfaces and their Dirichlet structures, with support allowed to enlarge through a controlled collar.

No full AQFT reconstruction is claimed. A later C^*- or W^*-net realization must establish its own locality, covariance, additivity, duality, and time-slice properties.

E. Continuum realization

The theorem does not construct an endpoint. It provides the scale-morphism system that an endpoint must realize.

A scaling limit, inductive construction, projective construction, or local algebraic completion may still be required. Those problems can remain difficult. What changes is their input.

Before the endpoint is invoked, one already has

$$\mathfrak{T} : \Lambda^{s,\mathrm{adm}} \to \mathbf{Hilb}_{\mathrm{Dir}}.$$

The carrier has been identified. Its invariant structure has been certified. The auxiliary refinement presentation has been removed.

The continuum problem begins there.

F. Conclusion

A scale-dependent theory contains several distinct objects:

$$\Phi_r, \qquad T_r, \qquad S_\lambda(t), \qquad r.$$

GLC identifies the observable and carrier descriptions of the scale arrow. Dirichlet covariance preserves the specified invariant sector. Composition organizes the certified arrows on raw refinements. Common-refinement invariance removes the raw presentation.

What remains is

$$\boxed{\mathfrak{T} : \Lambda^{s,\mathrm{adm}} \longrightarrow \mathbf{Hilb}_{\mathrm{Dir}}.}$$

This is renormalization as admissible transport.

A continuum realization may come later. It is not required to manufacture the compatibility already present in the scale morphisms:

$$\boxed{\text{renormalization compatibility is a morphism problem before it is a limit problem.}}$$

AUTHOR DECLARATIONS

Conflict of Interest

The author has no conflicts to disclose.

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Author Contributions

J. Scott Little: Conceptualization (lead); Formal analysis (lead); Investigation (lead); Methodology (lead); Writing – original draft (lead); Writing – review & editing (lead).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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